Optimal investment under multiple defaults and recursive system of BSDEs
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Indifference pricing

Single default model

Let $M_t = 1_{\tau < t} - \int_0^t \hat{\lambda}_s ds$ with $\hat{\lambda}_s = 1_{s < \tau} \lambda_s$ be the compensated martingale of the default process.

We assume that the price process follows

$$dS_t = S_t (\nu dt + \sigma dW_t + \varphi dM_t), \quad S_0 > 0$$

Let

$$H = H_T^1 1_{\tau < T} + H_T^0 1_{T < \tau}$$

where $H_T^1 = H_T^1(\tau)$ with $H_T^1(u) \in \mathcal{F}_T$ and $H^0(T) \in \mathcal{F}_T$. 
**Exponential models** Assuming immersion property.

The value function is \( e^{X_t} - Z_t \) where

\[
dZ_t = \left( \frac{\nu^2}{2\sigma^2} - \frac{\nu}{\sigma} \tilde{z}_t + (e^{\tilde{z}_t} - \tilde{z}_t - 1)\hat{\lambda}_t \right) dt - \tilde{z}_t dW_t - \tilde{z}_t dM_t, \quad Z_T = H
\]

and one can solve this equation in two parts

After the default

\[
dZ^1_t = \left( \frac{\nu^2}{2\sigma^2} - \frac{\nu}{\sigma} \tilde{z}_t \right) dt - \tilde{z}_t dW_t, \quad Z^1_T = H^1_T
\]

This leads to a solution \( Z^1_t(\tau) \), hence to a family of processes \( Z^1_t(u) \). and, since \( \tilde{z}_t = jump = Z^1_t(t) - Z^0_t \)

\[
dZ^0_t = \left( \frac{\nu^2}{2\sigma^2} - \frac{\nu}{\sigma} \tilde{z}_t + (e^{Z^0_t - Z^1_t(t)}) - 1 \right) dt - \tilde{z}_t dW_t, \quad Z^0_T = H^0_T
\]
In the case of deterministic coefficients, deterministic target $H^0$ and deterministic functions $H_1(\cdot)$, the problem can be explicitly solved.

No density process involved: this is due to immersion property

General case: write $W$ as a semimartingale in the enlarged filtration. This decomposition involves the density process
Mean Variance Hedging

The value function is of the form $V_t x^2 + U_t x + A_t$ where

$$dV_t = \left( \frac{(\nu V_t + \tilde{v}_t \sigma + \lambda_t \varphi \tilde{v}_t)^2}{V_t \sigma^2 + \varphi^2 \lambda_t (\tilde{v}_t + V_t)} \right) dt + \tilde{v}_t dW_t + \tilde{v}_t dM_t$$

$V_T = 1.$

The solution is

$$V_t = V^0_t \mathbb{1}_{t<\tau} + V^1_t \mathbb{1}_{\tau \leq t}$$

where

$$V^1_t = e^{\nu \sigma^2 (T-t)}$$

and

$$dV^0_t = \frac{(\nu V^0_t + \tilde{v}_t \sigma + \lambda_t \varphi (V^0_t - V^1_t))^2}{V^0_t \sigma^2 + \varphi^2 \lambda_t V^1_t} dt + \tilde{v}_t dW_t + (V^1_t - V^0_t) \lambda dt$$

$V^0_T = 1.$

In the case of deterministic coefficients, one has to solve

\[
\begin{align*}
dV_t^0 &= \left( \frac{\nu V_t^0 + \lambda_t \varphi (V_t^0 - V_t^1)^2}{V_t^0 \sigma^2 + \varphi^2 \lambda_t V_t^1} + (V_t^1 - V_t^0) \lambda_t \right) dt \\
V_T^0 &= 1.
\end{align*}
\]