Optimal multiple stopping time problems of jumps diffusion processes

Imène BEN LATIFA

ENIT-LAMSIN

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Outline

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In this talk we study the optimal multiple stopping time problem which consists on computing the essential supremum of the expectation of the pay-off over multiple stopping times, which is defined for each stopping time S by

$$v(S) = \operatorname{ess} \sup_{\tau_1,\dots,\tau_d \geq S} E[\psi(\tau_1, \dots, \tau_d) | \mathcal{F}_S].$$

We will specify this set later.

Example: **Swing options**

A **swing option** is an option with many exercise rights of American type.

Swing options are usually embedded in energy contracts.

Pricing Swing options are important (the energy market will be deregulated and energy contacts will be priced according to their financial risk).

In the energy market, the consumption is very complex. it depends on exogenous parameters (Weather, temperature). The consumption could increases sharply and prices follow.

Such problems are studied by Davison and Anderson (2003) and also Carmona and Touzi (2008).

In their paper Carmona and Touzi showed that pricing Swing option is related to optimal multiple stopping time problems. They studied the Black Scholes case.

In this paper we extend their results in a market where jumps are permitted.

Outline

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space, where $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ a filtration which satisfies the usual conditions.

- Let $T \in (0, \infty)$ be the option's maturity time i.e. the time of expiration of our right to stop the process or exercise.
- $\mathcal{S}$ the set of $\mathbb{F}$-stopping times with values in $[0, T]$.
- $\mathcal{S}_\sigma = \{\tau \in \mathcal{S} \; ; \; \tau \geq \sigma\}$ for every $\sigma \in \mathcal{S}$.
- $\delta > 0$ the refracting period which separates two successive exercises.
- We also fix $\ell \geq 1$ the number of rights we can exercise.
- $$\mathcal{S}_\sigma^{(\ell)} := \left\{ \begin{array}{l} (\tau_1, \dots, \tau_\ell) \in \mathcal{S}^\ell, \; \tau_1 \in \mathcal{S}_\sigma, \; \tau_i - \tau_{i-1} \geq \delta \textit{ on } \{\tau_{i-1} + \delta \leq T\} \textit{ a.s}, \\ \tau_i = T \textit{ on } \{\tau_{i-1} + \delta > T\} \textit{ a.s}, \forall \; i = 2, \dots, \ell \end{array} \right\}$$

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Let $X = \{X_t\}_{t\geq 0}$ be a non-negative càdlàg $\mathbb{F}$-adapted process. We assume that X satisfies the integrability condition :

$$E\left[\bar{X}^p\right] < \infty \quad \text{for some} \quad p > 1, \quad \text{where} \quad \bar{X} = \sup_{0\leq t\leq T} X_t. \qquad (2)$$

We introduce the following optimal multiple stopping time problem :

$$Z_0^{(\ell)} \quad := \quad \sup_{(\tau_1,\ldots,\tau_\ell)\in\mathcal{S}_0^{(\ell)}} E\left[\sum_{i=1}^{\ell} X_{\tau_i}\right]. \qquad (3)$$

The optimal multiple stopping time problem consist in computing the maximum expected reward $Z_0^{(\ell)}$ and finding the optimal exercise strategy $(\tau_1, ..., \tau_\ell) \in \mathcal{S}_0^{(\ell)}$ at which the supremum in (3) is attained, if such a strategy exist.

Carmona and Touzi (2008) have related the optimal multiple stopping time problems to a cascade of ordinary stopping time problems.
Let us define $Y^{(i)}$ as the Snell envelop of the reward process $X^{(i)}$ which is defined as follows

$$Y^{(0)} = 0 \quad \text{and} \quad Y_t^{(i)} = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} E\left[X_\tau^{(i)} | \mathcal{F}_t\right], \quad \forall\ t \geq 0, \quad \forall\ i = 1, ..., \ell,$$

where the i-th exercise reward process $X^{(i)}$ is given by :

$$X_t^{(i)} \;=\; X_t + E\left[Y_{t+\delta}^{(i-1)} | \mathcal{F}_t\right] \quad \text{for } 0 \leq t \leq T - \delta \qquad (4)$$

and

$$X_t^{(i)} \;=\; X_t \qquad \text{for} \quad t > T - \delta$$

Now, we shall prove that $Z_0^{(\ell)}$ can be computed by solving inductively ℓ single optimal stopping time problems sequentially. This result is proved in the paper of Carmona and Touzi [1] under the assumption that the process X is continuous a.s.
In this work we prove this result for càdlàg processes.

We recall the result of Nicole EL KAROUI (1981) to prove the existence of optimal stopping time for each ordinary stopping time problem.
So we need to define

$$\tau_1^* \quad = \quad \inf\{t \geq 0 \; ; \; Y_t^{(\ell)} = X_t^{(\ell)}\}$$

and for $i = 2, ..., \ell$

$$\tau_i^* \quad = \quad \inf\{t \geq \delta + \tau_{i-1}^* \; ; \; Y_t^{(\ell-i+1)} = X_t^{(\ell-i+1)}\} \mathbf{1}_{\{\delta+\tau_{i-1}^* \leq T\}} + T\mathbf{1}_{\{\delta+\tau_{i-1}^* > T\}}.$$

We recall then the result of Nicole EL KAROUI (1981)

Theorem 1

(Existence of optimal stopping time)
For all $i = 1, ..., \ell$ *let* $(X_t^{(\ell-i+1)})_t$, *be a non negative càdlàg process that satisfies the integrability condition*
$E[\sup_{t\in[0,T]} X_t^{(\ell-i+1)}] < \infty$.
Then, for each $t \geq 0$ *there exists an optimal stopping time for* $Y_t^{(\ell-i+1)}$. *Moreover* τ_i^* *is the minimal optimal stopping time for* $Y_{\tau_{i-1}^*+\delta}^{(\ell-i+1)}$, *i.e.* $Y_{\tau_{i-1}^*+\delta}^{(\ell-i+1)} = E\left[X_{\tau_i^*}^{(\ell-i+1)} | \mathcal{F}_{\tau_{i-1}^*+\delta}\right]$ *a.s..*

Which show the existence of optimal stopping time for each ordinary stopping time problem.

Our aim now is to prove the existence of optimal stopping time for each ordinary stopping time problem.
So, let us prove that $X^{(i)}$ satisfies the assumptions of the last theorem.

Lemma 2

We have that X is a càdlàg adapted process, then for $i = 1, ..., \ell$, $X^{(i)}$ is a càdlàg adapted process .

Integrability condition:

Lemma 3

Under our assumptions on the state process X we have the following integrability condition of the process $X^{(i)}$ when $p > 1$

$$E\left[\left(\bar{X}^{(i)}\right)^p\right] < \infty \quad \textit{where} \quad \bar{X}^{(i)} = \sup_{0 \le t \le T} X_t^{(i)},$$

Now we are able to relate the optimal multiple stopping time to the cascade of ordinary stopping time problems, and we have this result

Theorem 4

(existence of an optimal multiple stopping time)

Let us assume that the non-negative, càdlàg adapted process X satisfies the integrability condition (2). Then,

$$Z_0^{(\ell)} = Y_0^{(\ell)} = E\left[\sum_{i=1}^{\ell} X_{\tau_i^*}\right]$$

where, $Z_0^{(\ell)}$ is the value function, $Y_0^{(\ell)}$ is the value of the snell envelop of $X^{(\ell)}$ at time 0 when we have ℓ rights of exercise.

Outline

Swing options in the jump diffusion model

Our aim is to characterise the sequence of multiple optimal stopping time.
We study the case of swing options in the jump diffusion model.
We assume that the state process $X = \{X_t\}_{t \geq 0}$ evolves according to the following stochastic differential equation:

$$dX_t \quad = \quad b(t, X_{t-})dt + \sigma(t, X_{t-})dW_t + \int_{\mathbb{R}} \gamma(t, X_{t-}, z)\tilde{v}(dt, dz),$$

where $b, \sigma,$ and γ are continuous functions with respect to (t, x), and Lipschitz with respect to x. W is a standard Brownian motion and v is a homogeneous Poisson random measure with intensity measure $q(dt, dz) = dt \times m(dz)$, where m is the Lévy measure on $\mathbb{R}$ of v and $\tilde{v}(dt, dz) := (v - q)(dt, dz)$ is called the compensated jump martingale random measure of v. The Lévy measure m is a positive σ-finite measure on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} m(dz) \quad < \quad +\infty \tag{5}$$

Formulation of the Multiple Stopping Problem

The previous results holds on the Markovian context. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a Lipschitz payoff function. The value function of the Swing option problem with ℓ exercise rights and refraction time $\delta > 0$ is given by:

$$v^{(\ell)}(X_0) = \sup_{(\tau_1,\ldots,\tau_\ell)\in \mathcal{S}_0^{(\ell)}} E\left[\sum_{i=1}^{\ell} e^{-r\tau_i}\phi(X_{\tau_i})\right], \qquad (6)$$

To solve this problem, we define inductively the sequence of ℓ optimal single stopping time problems given by :

$$v^{(k)}(t, X_t) = \text{ess} \sup_{\tau \in \mathcal{S}_t} E\left[e^{-r(\tau-t)}\phi^{(k)}(\tau, X_\tau)|X_t\right], \ v^{(0)} \equiv 0 \quad (7)$$

where $k = 1, ..., \ell$ and

$$\phi^{(k)}(t, X_t) = \phi(X_t) + e^{-r\delta}E\left[v^{(k-1)}(t+\delta, X_{t+\delta})|X_t\right], \qquad \forall\ 0 \leq t \leq T$$

$$\phi^{(k)}(t, X_t) = \phi(X_t), \qquad \forall\ T - \delta < t \leq T.$$

To apply the previous result on the multiple optimal stopping time problem, we have to show that conditions of section 2 are satisfied. Let us denote the reward process by $Z_t := e^{-rt}\phi(X_t)$.

Proposition 0.1

Z satisfies the following conditions :

$$Z \text{ is càdlàg}, \tag{8}$$

$$E\left[\bar{Z}^2\right] < \infty \text{ where } \quad \bar{Z} = \sup_{0 \le t \le T} Z_t, \tag{9}$$

We obtain that

$$v^{(\ell)}(X_0) = E\left[\sum_{i=1}^{\ell} e^{-r\theta_i^*}\phi(X_{\theta_i^*})\right]$$

where

$$\theta_1^* = \inf\{t \geq 0 \; ; \; v^{(\ell)}(t, X_t) = \phi^{(\ell)}(t, X_t)\}$$

For $2 \leq k \leq \ell$, we define

$$\theta_k^* = \inf\{t \geq \delta + \theta_{k-1}^*; v^{(\ell-k+1)}(t, X_t) = \phi^{(\ell-k+1)}(t, X_t)\}\mathbf{1}_{\{\delta+\theta_{k-1}^* \leq T\}} + T\mathbf{1}_{\{\delta+\theta_{k-1}^* > T\}}. \tag{10}$$

Outline

Our next result is related to the regularity of the sequence of value functions
We have proved that for all $k = 1, ..., \ell$, we have that the value function $v^{(k)}$ and $\phi^{(k)}$ have the Lipschitz property with respect to x.

Lemma 5

for all $k = 1, ..., \ell$, there exist $K > 0$ such that for all $t \in [0, T)$, $x, y \in \mathbb{R}$

$$|v^{(k)}(t,x) - v^{(k)}(t,y)| \leq K|x-y| \quad \textit{and} \quad |\phi^{(k)}(t,x) - \phi^{(k)}(t,y)| \leq K|x-y|$$

Theorem 6

For $k = 1, ..., \ell$ and for all $t < s \in [0, T]$, $x \in \mathbb{R}$, there exist a constant $C > 0$ such that

$$\left| v^{(k)}(s,x) - v^{(k)}(t,x) \right| \leq C(1+|x|)\sqrt{s-t} \tag{11}$$

and

$$\left| \phi^{(k)}(t,x) - \phi^{(k)}(s,x) \right| \leq C(1+|x|)\sqrt{s-t} \tag{12}$$

Then we obtain the continuity of each value function.

Now we relate the cascade of optimal ordinary stopping time problems to a sequence of HJB variational inequalities:

$$\min\{rV^{(k)}(t,x) - \frac{\partial V^{(k)}}{\partial t}(t,x) - A(t,x,\frac{\partial V^{(k)}}{\partial x}(t,x),\frac{\partial^2 V^{(k)}}{\partial x^2}(t,x)) - B(t,x,V^{(k)}); \; V^{(k)}(t,x) - \phi^{(k)}(t,x)\} = 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R} \tag{13}$$

$$V^{(k)}(T,x) = \phi(x), \quad \forall x \in \mathbb{R}$$

where A is the generator associated to the diffusion term, it is defined as follows

$$A(t,x,p,M) = \frac{1}{2}\sigma^2(t,x)M + b(t,x)p, \text{ for all } t \in [0,T], x \in \mathbb{R}, p \in \mathbb{R}, M \in \mathbb{R} \tag{14}$$

and for $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$, the operator B is related to the jumps of the state process and is defined as follows

$$B(t,x,\varphi) = \int_{\mathbb{R}} [\varphi(t,x+\gamma(t,x,z)) - \varphi(t,x) - \gamma(t,x,z)\frac{\partial \varphi}{\partial x}(t,x)]m(dz). \tag{15}$$

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Proposition 0.2

For all $k = 1, \ldots, \ell$, the value function $v^{(k)}$ is a viscosity solution of the HJB-VI (13) on $[0, T) \times \mathbb{R}$.

Uniqueness of viscosity solution

Theorem 7

(Comparison Theorem)
Assume that b, σ and γ are Lipschitz in x, X have a moment of second order and the Lipschitz continuity of ϕ hold. Let $u^{(k)}$ (resp. $v^{(k)}$), $k = 1, \ldots, \ell$, be a viscosity subsolution (resp. supersolution) of (13). Assume also that $u^{(k)}$ and $v^{(k)}$ are Lipschitz, have a linear growth in x and holder in t. If

$$u^{(k)}(T, x) \leq v^{(k)}(T, x) \qquad \forall x \in \mathbb{R}, \tag{16}$$

then

$$u^{(k)}(t, x) \leq v^{(k)}(t, x) \qquad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{17}$$

Further research

- We will consider a numerical scheme of the HJB-VI, based on finite difference method to approximate the derivatives and Monte Carlo method to approximate the expectation which appear in the definition of $\phi^{(k)}$.
- The numerical scheme should verify the stability, consistency and monotonicity conditions. We know by the results of Barles and Souganidis (1991) that such scheme is convergent to the solution of the HJB-VI.
- An important question is: what is the rate of convergence of such scheme?

Main references :

- C. Carmona and N. Touzi (2008). Optimal Multiple Stopping and Valuation of Swing Options, *Mathematical Finance, 18 239-268. MR2395575*.
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Thanks for your attention.