The Risk-Sensitive Switching Problem Under Knightian Uncertainty

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Let $B := (B_t)_{t\leq T}$ a BM on a probability space $(\Omega, \mathcal{F}, P)$; $(F_t)_{t\leq T}$ the completed natural filtration of B.

A switching problem is a stochastic control where the decision maker moves among m states or modes when she decides according to the best profitability.

There are several works on the switching problem (H.-Jeanblanc, Djehiche-H.-Popier, H.-Zhang, Hu-Tang, Zervos (s.p.), Ly Vath-Pham, Ly Vath-Pham-XYZ , Zervos,
Porchet-Touzi, Carmona-Ludkowski,...).

Examples of a switching problems

• In financial markets when a trader invests his/her money between several assets (economies) according their profitability

• In the energy market when a manager of a power plant puts it in the mode which occurs the best profitability. in assets investor puts his money in in the case

A strategy of switching has two components (when $m \geq 3$):

- $(\tau_n)_{n\geq 0}$ an increasing sequence of stopping times: they are the times when the decision maker decides to switch.

- a sequence $(\xi_n)_{n\geq 0})$ of r.v. with values in $\mathcal{J} := \{1, ..., m\}$ (the different states) such that ξ_n is F_{τ_n}-measurable which stands for the state to which the system is switched at τ_n from its current one.

Remark: When $m = 2$, a strategy has only one component, i.e., stopping times.

With a strategy $(\delta, \xi) = ((\tau_n)_{n\geq 0}, (\xi_n)_{n\geq 0})$ is associated an indicator of the state of the system which is $(u_t)_{t\leq T}$ given by:

$$u_0 = 1 \text{ and } u_t = \xi_n \text{ if } t \in]\tau_n, \tau_{n+1}] \ (n \geq 0).$$

When a strategy (δ, ξ) is implemented usually the yield is given by:

$$J(\delta, \xi) := E[\int_0^T \psi_{u_s}(s)ds - \sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n < T]}]$$

where

- $\psi_k(t, \omega)$ is the instantaneous profit in the state k

- $\ell_{kl}(t, \omega) \geq c > 0$ is the switching cost from state k to state l at t.

The problem is to focus on

$$J^* := \sup_{(\delta,\xi)} J(\delta, \xi).$$

This problem is linked to systems of Reflected BSDEs with inter-connected obstacles or oblique reflection of the following type: for $i \in \mathcal{J} := \{1, ..., m\}$,

$$\begin{cases} Y_t^i = \int_t^T \psi_i(u)du - \int_t^T Z_u^i dB_u + K_T^i - K_t^i \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(t) + Y_t^j\}, \\ \int_0^T (Y_u^i - \max\limits_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(u) + Y_u^j\}) dK_u^i = 0. \end{cases} \tag{1}$$

where K^i are continuous and non-decreasing and $\mathcal{J}^{-i} := \mathcal{J} - \{i\}$.

The solution of (6) provides the optimal strategy (δ^*, ξ^*) and $J_1^* = Y_0^1$ (Djehiche, H., Popier, 07).

Knightian uncertainty: means that the probability of the future is not fixed and a family of probabilities P^u are likewise.

Risk-sensitiveness: means that the criterion is of type

$$E[e^{\theta\zeta}]$$

where θ is related to risk attitude of the controller.

So let us set:

$J(\delta, \xi; u) :=$

$E^u[\exp\{\int_0^T (\psi_{u_s}(s, X_s) + h(s, X_s, u_s))ds - A_T^\delta\}]$

where

- X verifies

$$dX_t = \varrho(t, X_t)dt + \sigma(t, X_t)dB_t, t \le T$$

are factors which determine prices in the market and

$$\forall \lambda > 0, E[e^{\lambda \sup_{t \le T} |X_t|}] < \infty.$$

- $u := (u_t)_{t \le T}$ is a stochastic process valued in U (not bounded)

- P^u is a probability such that:

$$\frac{dP^u}{dP} = \mathcal{E}_T(\int_0^\cdot b(t, X_t)dB_t)$$

- $A_T^\delta := \sum_{n \ge 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n, X_{\tau_n}) 1\!\!1_{[\tau_n < T]}$

- h is a premium which satisfies:

$$l(u) \le h(t, x, u) \le C(1 + |x| + l(u))$$

with $l(u) \to \infty$ as $|u| \to \infty$.

Problem: Characterization, properties and computation of

$$J^* = \sup_{\delta} \inf_{u} J(\delta, \xi; u).$$

Does an optimal strategy (δ^*, u^*) exist?

So let H be the hamiltonian of the problem,

$$H(t,x,z,u) := zb(t,x,u) + h(t,x,u)$$

and

$$H^*(t,x,z) := \inf_{u \in U} H(t,x,z,u).$$

Assume hereafter $m = 2$.

The system of reflected BSDEs associated with the problem is:

$$\begin{cases} \bullet Y_t^1 = \int_t^T [\psi_1(s, X_s) + H^*(s, X_s, Z_s^1) + \\ \qquad \frac{1}{2}|Z_s^1|^2] ds - \int_t^T Z_s^1 dB_s + K_T^1 - K_t^1; \\ \\ \bullet Y_t^2 = \int_t^T [\psi_2(s, X_s) + H^*(s, X_s, Z_s^2) + \\ \qquad \frac{1}{2}|Z_s^2|^2] ds - \int_t^T Z_s^2 dB_s + K_T^2 - K_t^2; \\ \\ \bullet Y_t^1 \geq Y_t^2 - \ell_{12}(t, X_t); \\ \qquad [Y_t^1 - Y_t^2 + \ell_{12}(t, X_t)] dK_t^1 = 0; \\ \\ \bullet Y_t^2 \geq Y_t^1 - \ell_{21}(t, X_t); \\ \qquad [Y_t^2 - Y_t^1 + \ell_{21}(t, X_t)] dK_t^2 = 0. \end{cases} \tag{2}$$

Verification theorem: If there exist two triplets of processes (Y^i, Z^i, K^i), $i = 1, 2$ which satisfy (2) then we have:

$$\exp\{Y_0^1\} = \sup_{\delta \in \mathcal{D}} \inf_{u \in \mathcal{U}} J(\delta, u)$$

and the optimal strategy (δ^*, u^*) is given by

$\tau_0^* := 0$ and for $n = 0, \cdots,$

$$\begin{array}{rcl} \tau_{2n+1}^* & := & \inf\{t \ge \tau_{2n}^* : Y_t^1 = Y_t^2 - \ell_{12}(t, X_t)\} \\ \tau_{2n+2}^* & := & \inf\{t \ge \tau_{2n+1}^* : Y_t^2 = Y_t^1 - \ell_{21}(t, X_t)\}. \end{array}$$

and

$$u_t^* := \sum_{n \ge 0} [u^*(t, X_t, Z_t^1) 1_{[\tau_{2n}^*, \tau_{2n+1}^*)}(t) + \\ u^*(t, X_t, Z_t^2) 1_{[\tau_{2n+1}^*, \tau_{2n+2}^*)}(t)].$$

Sketch of the proof: the problems are related to the lack of integrability and of regularity of the data of the problem.

Step 1: Expression of the payoffs via BSDEs

Let (δ, u) admissible. Then there exists a unique pair of $\mathcal{P}$-measurable processes $(Y^{\delta,u}, Z^{\delta,u})$ such that P-a.s, $\int_0^T |Z_s^{\delta,u}|^2 ds < \infty$, the process $(L_t^u e^{Y_t^{\delta,u} + \int_0^t h(s, X_s, u_s) ds})_{t \le T}$ is of class [D] and for any $t \le T$,

$$Y_t^{\delta,u} = -A_T^\delta + \int_t^T (\psi^\delta(s, X_s) + H(s, X_s, u_s, Z_s^{\delta,u}) + \tfrac{1}{2}|Z_s^{\delta,u}|^2)ds - \int_t^T Z_s^{\delta,u} dB_s. \tag{3}$$

Moreover, we have:

$$\begin{aligned} \exp\{Y_0^{\delta,u}\} &= E^u[\exp\{\int_0^T (\psi^\delta(s, X_s) \\ &\qquad + h(s, X_s, u_s))ds - A_T^\delta\}] \\ &= J(\delta, u). \end{aligned} \tag{4}$$

Step 2: Let $\delta \in \mathcal{D}$, then there exists a unique pair of $\mathcal{P}$-measurable processes $(Y^{\delta,*}, Z^{\delta,*})$ such that $(e^{Y_t^{\delta,*}})_{t \le T} \in \mathcal{E} := \bigcap_{p \ge 1} \mathcal{S}^p$, $(e^{Y_t^{\delta,*}} Z_t^{\delta,*})_{t \le T} \in \mathcal{H}^{2,d}$ and for any $t \le T$,

$$Y_t^{\delta,*} = -A_T^\delta + \int_t^T (\psi^\delta(s, X_s) + H^*(s, X_s, Z_s^{\delta,*}) + \tfrac{1}{2}|Z_s^{\delta,*}|^2)ds - \int_t^T Z_s^{\delta,*} dB_s. \tag{5}$$

Moreover, $\forall t \le T$, $\forall \delta \in \mathcal{D}$,

$$Y_t^{\delta,*} = \text{essinf}_{u \in \mathcal{U}} Y_t^{\delta,u}.$$

Step 3: Reduction of the problem

$$\sup_{\delta \in \mathcal{D}} \inf_{u \in \mathcal{U}} J(\delta, u) = \sup_{\delta \in \mathcal{B}} \inf_{u \in \mathcal{U}} J(\delta, u).$$

where

$$\mathcal{B} := \{\delta := (\tau_n)_{n \ge 0} \in \mathcal{D}, \exists K_\delta, \text{ such that } \tau_n = T, \text{ for any } n \ge K_\delta\}.$$

Step 4: end of the proof by induction.

Let $\delta \in \mathcal{B}$ then by a backward induction we have:

$$Y_0^1 \ge Y_0^{\delta,*}.$$

As (in using the system of reflected BSDEs) we have:

$$Y_0^1 = Y_0^{\delta^*,*}$$

therefore

$$Y_0^1 = \sup_{\delta \in \mathcal{D}} Y_0^{\delta,*} = \sup_{\delta \in \mathcal{D}} \inf_{u \in \mathcal{U}} Y_0^{\delta,u}$$

which implies that

$$\exp(Y_0^1) = \sup_{\delta \in \mathcal{D}} \inf_{u \in \mathcal{U}} J(\delta, u) = J(\delta^*, u^*).$$

Therefore the problem turns into solving the system (2).

Theorem: The system of reflected BSDEs with inter-connected obstacles (2) has a unique solution.

Sketch of the proof:

Step 1: Let us consider the following system:

For $i = 1, ..., m$,

$$
\begin{cases}
Y_t^i = \xi_i + \int_t^T f_i(u, Y_u^1, ..., Y_u^m, Z_u^i)du \\
\qquad\qquad - \int_t^T Z_u^i dB_u + K_T^i - K_t^i \\
Y_t^i \geq \max_{j \in \mathcal{J}^{-i}} h_{ij}(\omega, t, Y_t^j) \\
\int_0^T (Y_u^i - \max\limits_{j \in \mathcal{J}^{-i}} h_{ij}(\omega, u, Y_u^j)) dK_u^i = 0.
\end{cases}
\tag{6}
$$

We first extend the result by H.-Zhang (07) to the case of continuous coefficients f_j with linear growth in using inf-convolution techniques.

Step 2: We use an exponential transform for (2) and we obtain:

$\bullet \bar{Y}_t^1 = 1 + \int_t^T (\bar{Y}_s^1)^+[\psi_1(s, X_s)+$
$H^*(s, X_s, \frac{\bar{Z}_s^1}{(\bar{Y}_s^1)^+})]ds - \int_t^T \bar{Z}_s^1 dB_s + \bar{K}_T^1 - \bar{K}_t^1;$

$\bullet \bar{Y}_t^2 = 1 + \int_t^T (\bar{Y}_s^2)^+[\psi_2(s, X_s)+$
$H^*(s, X_s, \frac{\bar{Z}_s^2}{(\bar{Y}_s^2)^+})]ds - \int_t^T \bar{Z}_s^2 dB_s + \bar{K}_T^2 - \bar{K}_t^2;$

$\bullet \bar{Y}_t^1 \geq e^{-g_{12}(t,X_t)}\bar{Y}_t^2; \;\; \bar{Y}_t^2 \geq e^{-g_{21}(t,X_t)}\bar{Y}_t^1$

$\bullet (\bar{Y}_t^1 - e^{-g_{12}(t,X_t)}\bar{Y}_t^2)d\bar{K}_t^1 = 0$ and

$(\bar{Y}_t^2 - e^{-g_{21}(t,X_t)}\bar{Y}_t^1)d\bar{K}_t^2 = 0$

(7)

Finally we show that this system has a solution and we go back to (2).

Dynamic Programming Principle: Y^1 and Y^2 satisfy the following DPP:

$$Y_t^1 = esssup_{\delta=(\tau_n)_{n\geq 0}\in\mathcal{D}_t^1} E[\int_t^{\tau_n} \Phi_{u_s}(s, X_s, Z_s^{u_s})ds$$

$$-\sum_{k=1,n} \ell_{u_{\tau_{k-1}},u_{\tau_k}} 1_{[\tau_k<T]} + Y_{\tau_n}^{u_{\tau_n}} 1_{[\tau_n<T]}|F_t]$$

where

- $\mathcal{D}_t^1$ is the set of admissible strategies such that $\tau_1 \geq t$ and $u_0 = 1$

- $$\Phi_i(t, x, z) = \psi_i(t, x) + H^*(t, x, z) + \frac{1}{2}|z|^2.$$

The same is true for Y^2.

With the help of this DPP we show that:

Theorem: Assume that:

(i) U is compact and h is bounded

(ii) the functions ϱ and σ are jointly continuous

(iii) the functions $\psi_i(t,x)$ and $\Phi_i(t,x,z)$ are continuous.

Then there exists two bounded deterministic functions $v^1(t,x)$ and $v^2(t,x)$ such that $Y_s^{i,t,x} = v^i(s, X_s^{t,x})$ for any $s \in [t,T]$. Moreover (v^1, v^2) is a unique solution in viscosity sense for its associated HJB equation. : $i = 1,2$ $(j \neq i)$,

$$\begin{array}{l}\min\{v^i(t,x) - v^j(t,x) + \ell(t,x); \\ -\partial v^i - \mathcal{L}v^i(t,x) - \Phi_i(t,x,(\nabla v^i)\sigma(t,x))\} = 0\end{array}$$

where $\mathcal{L}$ is the generator associated with X.

The problem is continuity of v^i, $i = 1,2$. Existence is classical.

Step 1: the optimal strategy (τ_n) satisfies

$$P[\tau_n < T] \leq Cn^{-1}, \forall n \geq 1.$$

Then we write

$$v^1(t,x) =$$
$$sup_{\delta=(\tau_n)_{n\geq 0}\in\tilde{\mathcal{D}}^1} E[\int_t^{\tau_n} 1_{[s\geq t]}\Phi_{u_s}(s, X_s^{t,s}, Z_s^{u_s})ds$$

$$-\sum_{k=1,n} \ell_{u_{\tau_{k-1}},u_{\tau_k}}(\tau_k, X_{\tau_k}^{t,x})1_{[\tau_k<T]} + Y_{\tau_n}^{u_{\tau_n}} 1_{[\tau_n<T]}|F_t]$$

Finally we use the results by M.Kobylanski (00) to show that v^i are viscosity solutions. Uniqueness is classical.