Stochastic Target Problems with Controlled Loss in Jump Diffusion Models

Séminaire Bachelier - Journée des Doctorants

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We want to study the problem

\[ \nu(t, x, p) := \inf \{ y \geq -\kappa : \mathbb{E} \left[ \Psi(X^\nu_t, x(T), Y^\nu_t, y) \right] \geq p \text{ for some } \nu \} \]

introduced by Bouchard, Elie, Touzi (2009) for Brownian controlled SDEs, in the case of jump diffusion processes \( X^\nu \) and \( Y^\nu \).

\[
\begin{align*}
    dX &= \mu_X(X, \nu)ds + \sigma_X(X, \nu)dW + \int_E \beta_X(X, \nu, e)J(de, ds) \\
    dY &= \mu_Y(Z, \nu)ds + \sigma_Y(Z, \nu)dW + \int_E \beta_Y(Z, \nu, e)J(de, ds)
\end{align*}
\]

where \( Z \) stands for \((X, Y)\).

**Notations**: The controls \( \nu \) are in \( \mathcal{U} \) and take values in \( U \).
Examples
Financial Market

$X^\nu$: Stocks (possibly influenced by a large investor strategy $\nu$)
$Y^\nu$: Portfolio process of the large investor

The market is incomplete
We do not treat the dual problem, but directly the primal
Examples

Insurance Market

$X^\nu$: Sources of risks

$Y^\nu$: Portfolio process of the insurance
Examples
Superhedging

\[ \Psi(x, y) := \mathbb{1}_{\{y \geq g(x)\}} , \]

\[ v(t, x, 1) = \inf \left\{ y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P} \left[ Y_{t,x,y}^\nu(T) \geq g \left( X_{t,x}^\nu(T) \right) \right] = 1 \right\} . \]

Remark

- For \( \mathcal{U} \) compact and no jumps, Soner and Touzi (2002)
- For \( \mathcal{U} \) compact and bounded jumps, Bouchard (2002)
- For the American case, Bouchard and Vu (2009)

Example If \( g(x) = (x - K)^+ \), then

\[ v(t, x, 1) := \inf \left\{ y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \left( X_{t,x}^\nu(T) - K \right)^+ \mathbb{P}\text{-a.s.} \right\} . \]

Hedging a European call option with finite credit line.
Examples
Quantile Hedging

Ψ(x, y) := 1_{y \geq g(x)},

υ(t, x, p) = \inf \left\{ y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P} \left[ Y_{t,x,y}^{\nu}(T) \geq g \left( X_{t,x}^{\nu}(T) \right) \right] \geq p \right\}.

Remark

- In "standard" financial models, Follmer and Leukert (1999)
- In general settings, but no jumps, Bouchard, Elie and Touzi (2009)
Examples

Loss Function

\[ \Psi(x, y) := -\rho \left( (y - g(x))^+ \right), \] 
with \( \rho \) convex non-decreasing

\[ v(t, x, p) = \inf \left\{ y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E} \left[ \rho \left( (Y_{t,x,y}^{\nu}(T) - g(X_{t,x}^{\nu}(T))^+) \right) \right] \leq p \right\}. \]

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Examples

Success Ratio

\[ \Psi(x, y) := \mathbb{1}_{\{g(x) \leq y\}} + \frac{y}{g(x)} \mathbb{1}_{\{g(x) > y\}}, \text{ for } y \geq 0, \]

\[ v(t, x, p) = \inf \left\{ y \geq 0 : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E} \left[ \frac{Y^\nu_{t,x,y}(T)}{g(X^\nu_{t,x}(T))} \land 1 \right] \geq p \right\}. \]

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Examples

Utility indifference Price in incomplete Markets

\[ \Psi(x, y) := U(y - g(x)) \], with \( U \) concave non-decreasing,

\[ v(t, x, p) = \inf \{ y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E} \left[ U(Y_{t,x,y_0+y(T)} - g(X_{t,x}(T))) \right] \geq p \}. \]

\[ \left( p := \sup_{\nu \in \mathcal{U}} \mathbb{E} \left[ U(Y_{t,x,y_0}(T)) \right] \right) \]
Geometric Dynamic Programming Principle
(Soner Touzi (2002), Bouchard Vu (2009))

Fix \((t, x)\) and \(\{\theta^\nu, \nu \in \mathcal{U}\}\) a family of \([t, T]\)-valued stopping times,

\[(\text{GDP1})\; :\; y > v(t, x, 1) \Rightarrow \exists \nu \in \mathcal{U} \text{ s.t.} \]

\[Y_{t, x, y}^\nu (\theta^\nu) \geq v(\theta^\nu, X_{t, x}^\nu (\theta^\nu), 1).\]

\[(\text{GDP2})\; :\; \text{For every } -\kappa \leq y < v(t, x, 1), \nu \in \mathcal{U}\]

\[\mathbb{P} \left[ Y_{t, x, y}^\nu (\theta^\nu) > v(\theta^\nu, X_{t, x}^\nu (\theta^\nu), 1) \right] < 1.\]
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\]
Formal GDPP with Jumps

(Bouchard Elie Touzi (2009) , Bouchard Vu (2009))

\[ y > v(t, x, p) \Rightarrow \exists \nu \in \mathcal{U} \text{ s.t. } Y_{t,x,y}(\theta^\nu) \geq v(\theta^\nu, X_{t,x}(\theta^\nu), p), \]

but

\[ y > v(t, x, p) \Rightarrow \exists \nu \in \mathcal{U} \text{ s.t. } Y_{t,x,y}(\theta^\nu) \geq v(\theta^\nu, X_{t,x}(\theta^\nu), P) \]

where \( P := \mathbb{E} \left[ \Psi (X_{t,x}(T), Y_{t,x,y}(T)) \middle| \mathcal{F}_t \right], \) and \( \mathbb{E} [P] = p, \) i.e.

\[ P := p + \int_t^T \alpha_s \cdot dW_s + \int_t^T \int_E \chi_s(e) \tilde{J}(de, ds). \]
Formal GDPP with Jumps
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\[ P := p + \int_t^\cdot \alpha_s \cdot dW_s + \int_t^\cdot \int_E \chi_s(e) \tilde{J}(de, ds). \]
Main difficulties in Bouchard Elie Touzi (2009):

- $\alpha$ possibly unbounded $\Rightarrow$ unbounded controls

$\Rightarrow$ Local relaxation.
Formal GDPP with Jumps
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Main difficulties here:

- $\alpha$ and $\chi$ possibly unbounded $\Rightarrow$ unbounded controls and unbounded jumps

$\Rightarrow$ Non-local Relaxation.

- The control $\chi$ is a measurable function
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Main difficulties here:

- $\alpha$ and $\chi$ possibly unbounded $\Rightarrow$ unbounded controls and unbounded jumps
  $\Rightarrow$ Non-local Relaxation.

- The control $\chi$ is a measurable function
Reduction of the Problem

We then reduce to the problem:

\[ v(t, x, p) = \inf \left\{ y \geq -\kappa : \exists (\nu, \alpha, \chi) \in \mathcal{U} \times L^2 \times H^2 \lambda \text{ s.t.} \right. \]

\[ \Psi \left( X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T) \right) \geq P^\alpha,\chi_{t,p}(T) \left. \right\} \]

where \( H^2 \lambda \) denotes the set of maps \( \chi : \Omega \times [0, T] \times E \to \mathbb{R} \) s.t.

\[ \mathbb{E} \left[ \int_0^T \int_E (\chi_t(e))^2 \lambda(de)dt \right] < \infty, \]

and \( \lambda(de)dt \) is the intensity of \( J(de, dt) \).
Geometric Dynamic Programming Principle

Set

\[ P_{t,p}^{\alpha,\chi}(\cdot) := p + \int_t \alpha_s \cdot dW_s + \int_0^t \int_E \chi_s(e) \tilde{J}(de, ds). \]

\textbf{(GDP1)} : \( y > v(t, x, p) \Rightarrow \exists (\nu, \alpha, \chi) \in \mathcal{U} \times L^2 \times H^2_\lambda \) s.t.

\[ Y_{t,x,y}^\nu (\theta^\nu) \geq v (\theta^\nu, X_{t,x}^\nu (\theta^\nu), P_{t,p}^{\alpha,\chi} (\theta^\nu)) \]

for all stopping times \( \theta^\nu \).

\textbf{(GDP2)} : \( y < v(t, x, p) \Rightarrow \) for all \( \theta^\nu \leq T, (\nu, \alpha, \chi) \in \mathcal{U} \times L^2 \times H^2_\lambda \)

\[ \mathbb{P} \left[ Y_{t,x,y}^\nu (\theta^\nu) > v (\theta^\nu, X_{t,x}^\nu (\theta^\nu), P_{t,p}^{\alpha,\chi} (\theta^\nu)) \right] < 1. \]
Geometric Dynamic Programming Principle

Set

$$P_{t,p}^{\alpha,\chi}(\cdot) := p + \int_t^\cdot \alpha_s \cdot dW_s + \int_0^t \int_E \chi_s(e) \tilde{J}(de, ds).$$

(GDP1) : $y > v(t, x, p)$ $\Rightarrow$ $\exists (\nu, \alpha, \chi) \in \mathcal{U} \times L^2 \times H^2$ s.t.

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for all stopping times $\theta^\nu$.

(GDP2) : $y < v(t, x, p)$ $\Rightarrow$ for all $\theta^\nu \leq T, (\nu, \alpha, \chi) \in \mathcal{U} \times L^2 \times H^2$

$$\mathbb{P} \left[ Y_{t,x,y}^\nu (\theta^\nu) > v(\theta^\nu, X_{t,x}^\nu (\theta^\nu), P_{t,p}^{\alpha,\chi} (\theta^\nu)) \right] < 1.$$
Geometric Dynamic Programming Principle

Set

\[ P_{t,p}^{\alpha,\chi}(\cdot) := p + \int_t^\infty \alpha_s \cdot dW_s + \int_0^\infty \int_E \chi_s(e) \tilde{J}(de, ds). \]

\text{(GDP1)}: \quad y > v(t, x, p) \Rightarrow \exists (\nu, \alpha, \chi) \in \mathcal{U} \times L^2 \times H^2_\lambda \text{ s.t.}

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for all stopping times \( \theta^\nu \).

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\[ \mathbb{P} \left[ Y_{t,x,y}^\nu (\theta^\nu) > v(\theta^\nu, X_{t,x}^\nu (\theta^\nu), P_{t,p}^{\alpha,\chi} (\theta^\nu)) \right] < 1. \]
Formal PDE Derivation

We hence study the problem

\[ v(t, x) := \inf \left\{ y \geq -\kappa : \hat{\Psi} \left( X_{t,x}^\nu(T), Y_{t,x,y}^\nu(T) \right) \geq 0 \text{ for some } \nu \in \mathcal{U} \right\} \]

with

\[ dX = \mu_X(X, \nu)ds + \sigma_X(X, \nu)dW + \int_E \beta_X(X, \nu, e)J(de, ds) \]

\[ dY = \mu_Y(Z, \nu)ds + \sigma_Y(Z, \nu)dW + \int_E \beta_Y(Z, \nu, e)J(de, ds) \]

where \( Z \) stands for \((X, Y)\).

**Notations**: The controls \( \nu \) are in \( \mathcal{U} \) and take values in \( \mathcal{U} \)…
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**Notations:** The controls \( \nu \) are in \( \mathcal{U} \) and take values in \( \mathcal{U} \ldots \) is a space of unbounded measurable functions
Formal PDE Derivation

\[ dY_{t,x,y}^\nu = \mu_Y(X, Y, \nu)ds + \sigma_Y(X, Y, \nu)dW_s + \int_E \beta_Y(X, Y, \nu(e), e)J(de, ds) \]

\[ \geq dv(s, X(s)) \]

\[ = \mathcal{L}^\nu v(\cdot)ds + D_xv(\cdot)\sigma_X(\cdot)dW_s + \int_E [v(\cdot + \beta_X(\cdot)) - v(\cdot)] J(de, ds) \]

which leads to

\[ \sup_{u \in \mathcal{N}_0} \{ \min \{ \mu_Y(x, y, u) - \mathcal{L}^u v(t, x), \mathcal{G}^u v(t, x) \} \} = 0 \]

where

\[ \mathcal{G}^u v(t, x) := \inf_{e \in E} \{ \beta_Y(\cdot, v(\cdot), e) - v(\cdot + \beta_X(\cdot, e)) + v(\cdot) \} \]

and

\[ \mathcal{N}_\varepsilon := \{ u \in U \text{ s.t. } |\sigma_Y(x, y, u) - Dv(t, x)\sigma_X(x, u)| \leq \varepsilon \} . \]
Formal PDE Derivation

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which leads to

\[ \sup_{\nu \in N_0} \left\{ \min \{\mu_Y(x, y, u) - \mathcal{L}^u v(t, x), \mathcal{G}^u v(t, x)\} \right\} = 0 \]

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\[ N_\varepsilon := \{u \in U \text{ s.t. } |\sigma_Y(x, y, u) - Dv(t, x)\sigma_X(x, u)| \leq \varepsilon\} . \]
The Relaxation of Bouchard Elie Touzi (2009)

\[ H^*(\Theta) = \limsup_{\epsilon \downarrow 0, \Theta' \to \Theta} H_\epsilon(\Theta') \quad H_*(\Theta) = \liminf_{\epsilon \downarrow 0, \Theta' \to \Theta} H_\epsilon(\Theta'), \]

with \( \Theta' = (t', x', y, k, q, A) \), \( \Theta = (\cdot, v(\cdot), \partial_t v(\cdot), Dv(\cdot), D^2 v(\cdot)) (t, x) \)

and

\[ H_\epsilon(\Theta) = \sup_{u \in \mathcal{N}_\epsilon} \left\{ \mu_Y(z, u) - k - \mu_X(x, u) \cdot q - \frac{1}{2} \text{Tr} \left[ \sigma_X \sigma_X^T(x, u) A \right] \right\} \]

and

\[ \mathcal{N}_\epsilon(x, y, q) := \{ u \in U \ s.t. \ |\sigma_Y(x, y, u) - q\sigma_X(x, u)| \leq \epsilon \} . \]
Our Relaxation

The relaxation of is no longer sufficient to ensure the upper (resp. lower) semi-
continuity of $H^*$ (resp. $H_*$) in the non-local term $G^u v(t, x, p)$.

$$H^*(\Theta, \varphi) = \limsup_{\varepsilon \downarrow 0, \Theta' \to \Theta, \psi \to \varphi} H_\varepsilon(\Theta', \psi)$$

$$H_*(\Theta, \varphi) = \limsup_{\varepsilon \downarrow 0, \Theta' \to \Theta, \psi \to \varphi} H_\varepsilon(\Theta', \psi),$$

with $\Theta' = (t', x', y, k, q, A)$, $\Theta = (\cdot, v(\cdot), \partial_t v(\cdot), Dv(\cdot), D^2v(\cdot)) (t, x)$ and

$$H_\varepsilon(\Theta, \psi) = \sup_{u \in N_\varepsilon} \left\{ \min \left\{ \mu_Y(z, u) - k - \mu_X(x, u) \cdot q - \frac{1}{2} \text{Tr} \left[ \sigma_X \sigma_X^T(x, u) A \right] , \right. \right.$$

$$\left. \left. \inf_{e \in E} \left\{ \beta_Y(x, y, u, e) - \psi(t, x + \beta_X(x, u, e)) + \psi(t, x) \right\} \right\} \right\}$$

where $\psi \to \varphi$ has to be understood in the sense that $\psi$ converges
uniformly on compact sets towards $\varphi$, and

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$$H^*(\Theta, \varphi) = \limsup_{\varepsilon \searrow 0, \Theta' \to \Theta} \limsup_{\psi \rightharpoonup u.c. \varphi} H_\varepsilon(\Theta', \psi)$$

$$H_*(\Theta, \varphi) = \limsup_{\varepsilon \searrow 0, \Theta' \to \Theta} \limsup_{\psi \rightharpoonup u.c. \varphi} H_\varepsilon(\Theta', \psi),$$

with $\Theta' = (t', x', y, k, q, A)$, $\Theta = (\cdot, v(\cdot), \partial_t v(\cdot), Dv(\cdot), D^2v(\cdot))$ $(t, x)$ and

$$H_\varepsilon(\Theta, \psi) = \sup_{u \in \mathcal{N}_\varepsilon} \left\{ \min \left\{ \mu_Y(z, u) - k - \mu_X(x, u) \cdot q - \frac{1}{2} \text{Tr} \left[ \sigma_X \sigma_X^T(x, u) A \right] \right. \right.$$

$$\left. \left. \inf_{e \in E} \{ \beta_Y(x, y, u, e) - \psi(t, x + \beta_X(x, u, e)) + \psi(t, x) \} \right\} \right\}$$

where $\psi \rightharpoonup u.c. \varphi$ has to be understood in the sense that $\psi$ converges
uniformly on compact sets towards $\varphi$, and

$$\mathcal{N}_\varepsilon(x, y, q) := \{ u \in U \text{ s.t. } |\sigma_Y(x, y, u) - q\sigma_X(x, u)| \leq \varepsilon \}.$$
Our main results

Theorem

*The function* $v_*$ *is viscosity supersolution on* $[0, T) \times X$ *of*

$$H^*v_* \geq 0.$$

*Under some extra assumption of regularity of the set* $\mathcal{N}^0$, *the function* $v_*$ *is a viscosity subsolution on* $[0, T) \times X$ *of*

$$\min \{ H_*v^*, v^* + \kappa \} \leq 0.$$
Our main results

Theorem

The function \( \nu_* \) is viscosity supersolution on \([0, T) \times X\) of

\[
H^* \nu_* \geq 0.
\]

Under some extra assumption of regularity of the set \( N^0 \), the function \( \nu^* \) is a viscosity subsolution on \([0, T) \times X\) of

\[
\min \{ H_* \nu^*, \nu^* + \kappa \} \leq 0.
\]
Sketch of the Proof (Supersolution):

Let $\phi$ be a test function, and assume that

$$H^* \phi(t_0, x_0) =: -2 \eta < 0.$$ 

Define

$$\tilde{\phi}(t, x) := \phi(t, x) - \iota |x - x_0|^4 \text{ for } \iota > 0.$$ 

By the definition of $H^*$, we may find $\varepsilon > 0$ and $\iota > 0$ small enough such that

$$\min \left\{ \mu_Y(x, y, u) - \mathcal{L}^u \tilde{\phi}(t, x), \mathcal{G}^u \tilde{\phi}(t, x) \right\} \leq -\eta$$

for all $u \in \mathcal{N}_\varepsilon(x, y, D\tilde{\phi}(t, x))$ and $(t, x, y)$ s.t. $(t, x) \in B_\varepsilon(t_0, x_0)$ and $|y - \tilde{\phi}(t, x)| \leq \varepsilon.$
We then have
\[(v_* - \tilde{\varphi})(t, x) \geq \zeta \wedge \iota \varepsilon^4 =: \xi > 0 \text{ for } (t, x) \in \mathcal{V}_\varepsilon(t_0, x_0),\]
with
\[\mathcal{V}_\varepsilon(t_0, x_0) := \partial_p B_\varepsilon(t_0, x_0) \cup [t_0, t_0 + \varepsilon) \times B_\varepsilon^c(x_0).\]

Let \((t_n, x_n)_{n \geq 1} \rightarrow (t_0, x_0)\) s.t. \(v(t_n, x_n) \rightarrow v_*(t_0, x_0)\) and set
\[y_n := v(t_n, x_n) + n^{-1}.\]
For each $n \geq 1$, $y_n > v(t_n, x_n)$ together with (GDP1) : there exists some $v^n \in U$ s.t.

$$Y^n(t \wedge \theta_n) \geq v(t \wedge \theta_n, X^n(t \wedge \theta_n)) \geq \tilde{\varphi}(t \wedge \theta_n, X^n(t \wedge \theta_n)), \quad t \geq t_n,$$

where

$$\theta^n_0 := \{s \geq t_n : (s, X^n(s)) \notin B_{\varepsilon}(t_0, x_0)\}$$

$$\theta_n := \{s \geq t_n : |Y^n(s) - \tilde{\varphi}(s, X^n(s))| \geq \varepsilon\} \wedge \theta^n_0.$$  

We then have

$$Y^n(t \wedge \theta_n) - \tilde{\varphi}(t \wedge \theta_n, X^n(t \wedge \theta_n)) \geq \left[ \varepsilon 1_{\{\theta_n < \theta^n_0\}} + \xi 1_{\{\theta_n = \theta^n_0\}} \right] 1_{\{t \geq \theta_n\}}.$$  

$$\geq (\varepsilon \wedge \xi) 1_{\{t \geq \theta_n\}} \geq 0.$$  

We conclude by using Itô’s lemma, and by making a ”change of measure” to obtain a contradiction.
On the terminal condition (formally)

In the expected loss case

\[ v(t, x, p) := \inf \left\{ y \geq -\kappa : \exists \nu : \mathbb{E} \left[ \Psi \left( X_{t,x}^{\nu}(T), Y_{t,x,y}^{\nu}(T) \right) \right] \geq p \right\} \]

leads to

\[ v(t, x, p) = \inf \left\{ y \geq -\kappa : \exists \nu, \alpha, \chi : \Psi \left( X_{t,x}^{\nu}(T), Y_{t,x,y}^{\nu}(T) \right) \geq P_{t,p}^{\alpha,\chi}(T) \right\} . \]

Define

\[ \psi(x, p) := \inf \left\{ y : \Psi(x, y) \geq p \right\} . \]

We may expect that

\[ v(T, x, p) = \psi(x, p). \]
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For the Quantile Hedging (Bouchard Elie Touzi (2009))

\[ \Psi(x, y) := 1_{\{y - g(x)\}} \]

leads to

\[ \psi(x, p) = g(x) 1_{\{p > 0\}}. \]

Discontinuous in \( p \), we hedge or not!!

\[ \Rightarrow \text{If } \nu \text{ is convex in its } p \text{-variable} \]

\[ \nu(T, x, p) = \text{Conv}(\psi(x, p)) = pg(x). \]
On the terminal condition (formally)

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⇒ If \(v\) is convex in its \(p\)-variable

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On the terminal condition (formally)

We may generalize it:
If $v$ is convex in its $p$-variable

$$v(T, x, p) = \text{Conv} \left( \psi(x, p) \right).$$
When the image of $\Psi$ is of the form $[m, M]$, with $m$ and/or $M$ are finite, we proved boundary conditions at $p = m$ and/or $p = M$.

In the B&S model and a complete market, using the Fenchel-Legendre transform of $v$ with respect to the $p$-variable in the PDE, Bouchard, Elie and Touzi recover the dual problem, which is a control problem. In incomplete markets, we recover in the same way a control problem, but we need a comparison theorem to conclude as they do.

⇒ Need to specify a model
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Conclusion

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