Stochastic Target Problems with Controlled Loss in Jump Diffusion Models[1]

Séminaire Bachelier - Journée des Doctorants

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Introduction

We want to study the problem

$$v(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}\right)\right] \geq p \text{ for some } \nu\right\}$$

introduced by Bouchard, Elie, Touzi (2009) for Brownian controlled SDEs, **in the case of jump diffusion processes $\mathbf{X}^{\nu}$ and $\mathbf{Y}^{\nu}$**.

$$dX = \mu_X(X,\nu)ds + \sigma_X(X,\nu)dW + \int_E \beta_X(X,\nu,e)J(de,ds)$$

$$dY = \mu_Y(Z,\nu)ds + \sigma_Y(Z,\nu)dW + \int_E \beta_Y(Z,\nu,e)J(de,ds)$$

where Z stands for (X,Y).

Notations : The controls ν are in $\mathcal{U}$ and take values in U.

Examples

Financial Market

X^ν : Stocks (possibly influenced by a large investor strategy ν)
Y^ν : Portfolio process of the large investor
The market is incomplete
We do not treat the dual problem, but directly the primal

Examples

Insurance Market

X^{ν} : Sources of risks
Y^{ν} : Portfolio process of the insurance

Examples

Superhedging

$\Psi(x, y) := \mathbb{1}_{\{y \geq g(x)\}}$,

$$v(t, x, 1) = \inf \left\{ y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T) \right) \right] = 1 \right\}.$$

Remark

- *For U compact and no jumps, Soner and Touzi (2002)*
- *For U compact and bounded jumps, Bouchard (2002)*
- *For the American case, Bouchard and Vu (2009)*

Example *If $g(x) = (x - K)^+$, then*

$$v(t, x, 1) := \inf \left\{ y \geq -\kappa : \exists\ \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(T) \geq \left(X^{\nu}_{t,x}(T) - K \right)^+ \mathbb{P}\text{-a.s.} \right\}.$$

Hedging a European call option with finite credit line.

Examples

Quantile Hedging

$$\Psi(x,y) := \mathbb{1}_{\{y \geq g(x)\}},$$

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right)\right] \geq p\right\}.$$

Remark

- *In "standard" financial models, Follmer and Leukert (1999)*
- *In general settings, but no jumps, Bouchard, Elie and Touzi (2009)*

Examples

Loss Function

$\Psi(x,y) := -\rho\left((y-g(x))^-\right)$, with ρ convex non-decreasing

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\rho\left(\left(Y^{\nu}_{t,x,y}(T) - g(X^{\nu}_{t,x}(T))\right)^-\right)\right] \leq p\right\}.$$

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Examples

Success Ratio

$\Psi(x,y) := \mathbb{1}_{\{g(x)\le y\}} + \frac{y}{g(x)}\mathbb{1}_{\{g(x)>y\}}$, for $y \ge 0$,

$$v(t,x,p) = \inf\left\{ y \ge 0 : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\frac{Y^{\nu}_{t,x,y}(T)}{g(X^{\nu}_{t,x}(T))} \wedge 1\right] \ge p \right\}.$$

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Examples

Utility indifference Price in incomplete Markets

$\Psi(x, y) := U(y - g(x))$, with U concave non-decreasing,

$$v(t, x, p) = \inf\left\{y \geq -\kappa : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[U\left(Y^{\nu}_{t,x,y_0+y}(T) - g(X^{\nu}_{t,x}(T))\right)\right] \geq p\right\}.$$

$$\left(p := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[U\left(Y^{\nu}_{t,x,y_0}(T)\right)\right]\right)$$

Geometric Dynamic Programming Principle

(Soner Touzi (2002) , Bouchard Vu (2009))

Fix (t, x) and $\{\theta^\nu, \nu \in \mathcal{U}\}$ a family of $[t, T]$-valued stopping times,

(GDP1) : $y > v(t, x, 1) \Rightarrow \exists\ \nu \in \mathcal{U}$ s.t.

$$Y^\nu_{t,x,y}(\theta^\nu) \geq v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu), 1\right).$$

(GDP2) : For every $-\kappa \leq y < v(t, x, 1), \nu \in \mathcal{U}$

$$\mathbb{P}\left[Y^\nu_{t,x,y}(\theta^\nu) > v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu), 1\right)\right] < 1.$$

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Formal GDPP with Jumps

(Bouchard Elie Touzi (2009) , Bouchard Vu (2009))

$$y > v(t,x,p) \not\Rightarrow \exists\ \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(\theta^{\nu}) \geq v\left(\theta^{\nu}, X^{\nu}_{t,x}(\theta^{\nu}), p\right),$$

but

$$y > v(t,x,p) \Rightarrow \exists\ \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(\theta^{\nu}) \geq v\left(\theta^{\nu}, X^{\nu}_{t,x}(\theta^{\nu}), P\right)$$

where $P := \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\middle|\mathcal{F}_t\right]$, and $\mathbb{E}[P] = p$, i.e.

$$P := p + \int_t^{\cdot} \alpha_s \cdot dW_s + \int_t^{\cdot}\int_E \chi_s(e)\widetilde{J}(de,ds).$$

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Main difficulties in Bouchard Elie Touzi (2009) :

- α possibly unbounded $\Rightarrow$ unbounded controls

$\Rightarrow$ Local relaxation.

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Main difficulties here :

- α and χ possibly unbounded $\Rightarrow$ unbounded controls and **unbounded jumps**

$\Rightarrow$ Non-local Relaxation.

- The control χ is a measurable function

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Reduction of the Problem

We then reduce to the problem :

$$v(t,x,p) = \inf \{ \, y \geq -\kappa : \exists \, (\nu, \alpha, \chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}^2_\lambda \text{ s.t. } \Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq P^{\alpha,\chi}_{t,p}(T) \, \}$$

where $\mathbb{H}^2_\lambda$ denotes the set of maps $\chi : \Omega \times [0,T] \times E \to \mathbb{R}$ s.t.

$$\mathbb{E}\left[\int_0^T \int_E (\chi_t(e))^2 \, \lambda(de)dt\right] < \infty,$$

and $\lambda(de)dt$ is the intensity of $J(de,dt)$.

Geometric Dynamic Programming Principle

Set

$$P^{\alpha,\chi}_{t,p}(\cdot) := p + \int_t^{\cdot} \alpha_s \cdot dW_s + \int_0^{\cdot} \int_E \chi_s(e) \widetilde{J}(de, ds).$$

<u>(GDP1) :</u> $y > v(t, x, p) \Rightarrow \exists\ (\nu, \alpha, \chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}^2_\lambda$ s.t.

$$Y^{\nu}_{t,x,y}(\theta^\nu) \geq v\left(\theta^\nu, X^{\nu}_{t,x}(\theta^\nu), P^{\alpha,\chi}_{t,p}(\theta^\nu)\right)$$

for all stopping times θ^ν.

<u>(GDP2) :</u> $y < v(t, x, p) \Rightarrow$ for all $\theta^\nu \leq T, (\nu, \alpha, \chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}^2_\lambda$

$$\mathbb{P}\left[Y^{\nu}_{t,x,y}(\theta^\nu) > v\left(\theta^\nu, X^{\nu}_{t,x}(\theta^\nu), P^{\alpha,\chi}_{t,p}(\theta^\nu)\right)\right] < 1.$$

Geometric Dynamic Programming Principle

Set

$$P_{t,p}^{\alpha,\chi}(\cdot) := p + \int_t^{\cdot} \alpha_s \cdot dW_s + \int_0^{\cdot} \int_E \chi_s(e) \widetilde{J}(de, ds).$$

(GDP1) : $y > v(t,x,p) \Rightarrow \exists\ (\nu, \alpha, \chi) \in \mathcal{U} \times \mathbb{L}^2 \times \mathbb{H}^2_\lambda$ s.t.

$$Y_{t,x,y}^{\nu}(\theta^\nu) \geq v\left(\theta^\nu, X_{t,x}^{\nu}(\theta^\nu), P_{t,p}^{\alpha,\chi}(\theta^\nu)\right)$$

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Formal PDE Derivation

We hence study the problem

$$v(t,x) := \inf\left\{y \geq -\kappa : \widehat{\Psi}\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\}$$

with

$$dX = \mu_X(X,\nu)ds + \sigma_X(X,\nu)dW + \int_E \beta_X(X,\nu,e)J(de,ds)$$
$$dY = \mu_Y(Z,\nu)ds + \sigma_Y(Z,\nu)dW + \int_E \beta_Y(Z,\nu,e)J(de,ds)$$

where Z stands for (X,Y).

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Formal PDE Derivation

$$dY^{\nu}_{t,x,y} = \mu_Y(X,Y,\nu)ds + \sigma_Y(X,Y,\nu)dW_s + \int_E \beta_Y(X,Y,\nu(e),e)J(de,ds)$$

$$\geq dv(s,X(s))$$

$$= \mathcal{L}^{\nu}v(\cdot)ds + D_x v(\cdot)\sigma_X(\cdot)dW_s + \int_E \left[v\left(\cdot + \beta_X(\cdot)\right) - v(\cdot)\right] J(de,ds)$$

which leads to

$$\sup_{u\in\mathcal{N}_0} \left\{\min\left\{\mu_Y(x,y,u) - \mathcal{L}^u v(t,x), \mathcal{G}^u v(t,x)\right\}\right\} = 0$$

where

$$\mathcal{G}^u v(t,x) := \inf_{e\in E} \left\{\beta_Y(\cdot, v(\cdot), e) - v\left(\cdot + \beta_X(\cdot,e)\right) + v(\cdot)\right\}$$

and

$$\mathcal{N}_\varepsilon := \left\{u \in U \text{ s.t. } |\sigma_Y(x,y,u) - Dv(t,x)\sigma_X(x,u)| \leq \varepsilon\right\}.$$

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The Relaxation of Bouchard Elie Touzi (2009)

$$H^*(\Theta) = \limsup_{\varepsilon \searrow 0, \Theta' \to \Theta} H_\varepsilon(\Theta') \qquad H_*(\Theta) = \liminf_{\varepsilon \searrow 0, \Theta' \to \Theta} H_\varepsilon(\Theta'),$$

with $\Theta' = (t', x', y, k, q, A)$, $\Theta = \left(\cdot, v(\cdot), \partial_t v(\cdot), Dv(\cdot), D^2 v(\cdot)\right)(t, x)$ and

$$H_\varepsilon(\Theta) = \sup_{u \in \mathcal{N}_\varepsilon} \left\{ \mu_Y(z, u) - k - \mu_X(x, u) \cdot q - \frac{1}{2} \mathsf{Tr}\left[\sigma_X \sigma_X^T(x, u) A\right] . \right\}$$

and

$$\mathcal{N}_\varepsilon(x, y, q) := \left\{ u \in U \text{ s.t. } |\sigma_Y(x, y, u) - q\sigma_X(x, u)| \leq \varepsilon \right\}.$$

Our Relaxation

The relaxation of is no longer sufficient to ensure the upper (resp. lower) semi continuity of H^* (resp. H_*) in the non-local term $\mathcal{G}^u v(t,x,p)$.

$$H^*(\Theta,\varphi) = \limsup_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta \\ \psi \xrightarrow[u.c.]{} \varphi}} H_\varepsilon(\Theta',\psi) \qquad H_*(\Theta,\varphi) = \limsup_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta \\ \psi \xrightarrow[u.c.]{} \varphi}} H_\varepsilon(\Theta',\psi),$$

with $\Theta' = (t',x',y,k,q,A)$, $\Theta = \left(\cdot, v(\cdot), \partial_t v(\cdot), Dv(\cdot), D^2 v(\cdot)\right)(t,x)$ and

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where $\psi \xrightarrow[u.c.]{} \varphi$ has to be understood in the sense that ψ converges uniformly on compact sets towards φ, and

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$$\left. \left. \inf_{e \in E} \left\{ \beta_Y(x,y,u,e) - \psi\left(t, x + \beta_X(x,u,e)\right) + \psi(t,x) \right\} \right\} \right\}$$

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Our main results

Theorem

The function v_* is viscosity supersolution on $[0,T) \times \mathbf{X}$ of

$$H^* v_* \geq 0.$$

Under some extra assumption of regularity of the set $\mathcal{N}^0$, the function v^* is a viscosity subsolution on $[0,T) \times \mathbf{X}$ of

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Sketch of the Proof (Supersolution) :

Let φ be a test function, and assume that

$$H^*\varphi(t_0, x_0) =: -2\eta < 0.$$

Define

$$\widetilde{\varphi}(t,x) := \varphi(t,x) - \iota\,|x - x_0|^4 \text{ for } \iota > 0.$$

By the definition of H^*, we may find $\varepsilon > 0$ and $\iota > 0$ small enough such that

$$\min\{\mu_Y(x,y,u) - \mathcal{L}^u\widetilde{\varphi}(t,x), \mathcal{G}^u\widetilde{\varphi}(t,x)\} \leq -\eta$$

$$\text{for all } u \in \mathcal{N}_\varepsilon\left(x, y, D\widetilde{\varphi}(t,x)\right)$$

$$\text{and } (t,x,y) \text{ s.t.} (t,x) \in B_\varepsilon(t_0,x_0) \text{ and } |y - \widetilde{\varphi}(t,x)| \leq \varepsilon.$$

We then have

$$(v_* - \widetilde{\varphi})\,(t,x) \geq \zeta \wedge \iota\varepsilon^4 =: \xi > 0 \text{ for } (t,x) \in \mathcal{V}_\varepsilon(t_0,x_0),$$

with

$$\mathcal{V}_\varepsilon(t_0,x_0) := \partial_p B_\varepsilon(t_0,x_0) \cup [t_0, t_0+\varepsilon) \times B^c_\varepsilon(x_0).$$

Let $(t_n,x_n)_{n\geq 1} \to (t_0,x_0)$ s.t. $v(t_n,x_n) \to v_*(t_0,x_0)$ and set $y_n := v(t_n,x_n) + n^{-1}$.

For each $n \geq 1$, $y_n > v(t_n, x_n)$ together with (GDP1) : there exists some $\nu^n \in \mathcal{U}$ s.t.

$$Y^n(t \wedge \theta_n) \geq v\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) \geq \widetilde{\varphi}\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right), \quad t \geq t_n,$$

where

$$\begin{aligned}\theta_n^o &:= \{s \geq t_n : (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\} \\ \theta_n &:= \{s \geq t_n : |Y^n(s) - \widetilde{\varphi}\left(s, X^n(s)\right)| \geq \varepsilon\} \wedge \theta_n^o.\end{aligned}$$

We then have

$$\begin{aligned}Y^n(t \wedge \theta_n) - \widetilde{\varphi}\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) &\geq \left[\varepsilon \mathbb{1}_{\{\theta_n < \theta_n^o\}} + \xi \mathbb{1}_{\{\theta_n = \theta_n^o\}}\right] \mathbb{1}_{\{t \geq \theta_n\}}. \\ &\geq (\varepsilon \wedge \xi) \mathbb{1}_{\{t \geq \theta_n\}} \geq 0.\end{aligned}$$

We conclude by using Itô's lemma, and by making a "change of measure" to obtain a contradiction. □

On the terminal condition (formally)

In the expected loss case

$$v(t,x,p) := \inf\left\{y \geq -\kappa : \exists\ \nu : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p\right\}$$

leads to

$$v(t,x,p) = \inf\left\{y \geq -\kappa : \exists\ \nu, \alpha, \chi : \Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq P^{\alpha,\chi}_{t,p}(T)\right\}.$$

Define

$$\psi(x,p) := \inf\left\{y : \Psi(x,y) \geq p\right\}.$$

We may expect that

$$v(T,x,p) = \psi(x,p).$$

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For the Quantile Hedging (Bouchard Elie Touzi (2009))

$$\Psi(x, y) := \mathbb{1}_{\{y - g(x)\}}$$

leads to

$$\psi(x, p) = g(x)\mathbb{1}_{\{p>0\}}.$$

Discontinuous in p, we hedge or not ! !
$\Rightarrow$ If v is convex in its p-variable

$$v(T, x, p) = \mathsf{Conv}\left(\psi(x, p)\right) = pg(x).$$

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We may generalize it :
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Conclusion

- When the image of Ψ is of the form $[m, M]$, with m and/or M are finite, we proved boundary conditions at $p = m$ and/or $p = M$.
- In the B&S model and a complete market, using the Fenchel-Legendre transform of v with respect to the p-variable in the PDE, Bouchard, Elie and Touzi recover the dual problem, which is a control problem
  In incomplet markets, we recover in the same way a control problem, but we need a comparison theorem to conclude as they do.
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