A Discrete-time approximation for reflected BSDEs related to "switching problem"

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Reflected BSDEs

- For $(b, \sigma) : \mathbb{R}^d \to \mathbb{R}^d \times M^d$ Lipschitz (σ may be degenerate) :

$$X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$$

- 'Simply' reflected BSDEs on a boundary $l(X)$:

$$Y_t = g(X_T) + \int_t^T f(X_t, Y_t, Z_t)\mathrm{d}t - \int_t^T (Z_t)'\mathrm{d}W_t + \int_t^T \mathrm{d}K_t$$

(C1) $Y_t \geq l(X_t)$ (constrained value process)

(C2) $\int_0^T \Big(Y_t - l(X_t)\Big)\mathrm{d}K_t = 0$ ("optimality" of K)

- Extension: doubly reflected BSDEs, reflected BSDEs in convex domain $\hookrightarrow$ *normal reflection*

Geometric framework

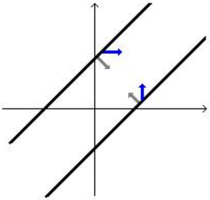
- Multidimensional value process constrained in a domain $\mathcal{C}$ ($d \geq 2$)

  $$\mathcal{C} = \{y \in \mathbb{R}^d | y^i \geq \mathcal{P}^i(y) := \max_j (y_j - c_{ij})\}$$

  with $c_{ii} = 0$, $\inf_{i \neq j} c_{ij} > 0$, $c_{ij} + c_{jk} > c_{ik}$

  $\hookrightarrow \mathcal{P}$ (oblique projection) is L-lipschitz with $L > 1$ (euclidean norm)

- example $d = 2$, oblique direction of reflection



Obliquely reflected BSDEs

- System of reflected BSDEs: for $1 \leq i \leq d$,

$$Y^i_t = g^i(X_T) + \int_t^T f^i(X_u, Y^i_u, Z^i_u)\mathrm{d}u - \int_t^T (Z^i_u)'\mathrm{d}W_u + K^i_T - K^i_t$$

(C1) $Y_t \in \mathcal{C}$ (constrained by K)

(C2) $\int_0^T \Big(Y^i_t - \mathcal{P}^i(Y_t)\Big)\mathrm{d}K^i_t = 0$ ('optimality' of K)

- Hu and Tang 07, Hamadene and Zhang 08

Starting and Stopping problem (1)

Hamadene and Jeanblanc (01):

- Consider e.g. a power station producing electricity whose price is given by a diffusion process X: $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$
- Two modes for the power station:
  mode $\mathbf{1}$: operating, profit is then $f^1(X_t)\mathrm{d}t$
  mode $\mathbf{2}$: closed, profit is then $f^2(X_t)\mathrm{d}t$
  $\hookrightarrow$ switching from one mode to another has a cost: $c > 0$
- Management decide to produce electricity only when it is profitable enough.
- The management strategy is (θ_j, α_j) : θ_j is a sequence of stopping times representing switching times from mode α_{j-1} to α_j.

  $(a_t)_{0 \le t \le T}$ is the state process (the management strategy).

Starting and Stopping problem (2)

- Following a strategy a from t up to T, gives

$$J(a,t) = \int_t^T f^{a_s}(X_s)\mathrm{d}s - \sum_{j\geq 0} c\mathbf{1}_{\{t\leq\theta_j\leq T\}}$$

- The optimization problem is then (at $t=0$, for $\alpha_0 = 1$)

$$Y_0^1 := \sup_a \mathbb{E}[J(a,0)]$$

At any date $t\in[0,T]$ in state $i\in\{1,2\}$, the value function is Y_t^i.

Solution

- Y is solution of a coupled optimal stopping problem

$$Y_t^1 = \mathrm{ess}\sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^\tau f(1, X_s)\mathrm{d}s + (Y_\tau^2 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

$$Y_t^2 = \mathrm{ess}\sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^\tau f(2, X_s)\mathrm{d}s + (Y_\tau^1 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

- The optimal strategy (θ_j^*, α_j^*) is given by

$$\theta_{j+1}^* := \inf\{s \ge \theta_j^* \mid Y_s^{\alpha_j^*} = \max_{i \in \{1,2\}} Y_s^i - c\}$$

$$\alpha_{j+1}^* := \mathbf{1} \text{ if } \alpha_j^* = 2\,, \ \textbf{or } \mathbf{2} \text{ if } \alpha_j^* = 1\,.$$

System of reflected BSDEs

Y is the solution of the following system of reflected BSDEs:

$$Y_t^i = \int_t^T f(i, X_s)\mathrm{d}s - \int_t^T (Z_s^i)'\mathrm{d}W_s + \int_t^T \mathrm{d}K_s^i \; , \; i \in \{1, 2\} \; ,$$

with (the coupling...)

$$Y_t^1 \geq Y_t^2 - c \text{ and } Y_t^2 \geq Y_t^1 - c \; , \; \forall t \in [0, T]$$

and ('optimality' of K)

$$\int_0^T \Big(Y_s^1 - (Y_s^2 - c)\Big)\mathrm{d}K_s^1 = 0 \text{ and } \int_0^T \Big(Y_s^2 - (Y_s^1 - c)\Big)\mathrm{d}K_s^2 = 0$$

Remark: related obstacle problem

- On $\mathbb{R} \times [0, T)$

$$\min\Big(-\partial_t u^1 - \mathcal{L}u^1 - f^1, u^1 - u^2 + c \Big) = 0$$
$$\min\Big(-\partial_t u^2 - \mathcal{L}u^2 - f^2, u^2 - u^1 + c \Big) = 0$$
$$u^1 \geq u^2 - c \text{ and } u^2 \geq u^1 - c$$

- Terminal condition

$$u(T, .) = 0$$

- Link via

$$Y_t^1 = u^1(t, X_t) \text{ and } Y_t^2 = u^2(t, X_t)$$

"Switching" problem - "switched" BSDEs

- "Switching" strategy $a = (\alpha_j, \theta_j)_j$ starting at (i, t)
  $N^a = \#\{k \in \mathbb{N}^* | \theta_k \le T\}$
- State process - cost process
  $a_s = \alpha_0 \mathbf{1}_{0 \le s \le \theta_0} + \sum_{j=1}^{N^a} \alpha_{j-1} \mathbf{1}_{\theta_{j-1} < s \le \theta_j}$, $A^a_s := \sum_{j=1}^{N^a} c_{\alpha_{j-1}, \alpha_j} \mathbf{1}_{\theta_j \le s \le T}$
- "Switched" BSDE (following the strategy a)

$$U^a_t = g^{a_T}(X_T) + \int_t^T f^{a_s}(X_s, U^a_s, V^a_s) \mathrm{d}s - \int_t^T V^a_s \mathrm{d}W_s - A^a_T + A^a_t$$

- Representation (a^*: optimal strategy)

$$Y^i_t = \operatorname*{esssup}_a U^a_t = U^{a*}_t$$

Approximation of the forward SDE

- For $(b,\sigma) : \mathbb{R}^d \to \mathbb{R}^d \times M^d$ Lipschitz :

$$X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$$

- Euler scheme X with $\pi = \{0 = t_0 < ... < t_n < ... < t_N = T\}$:

$$\begin{cases} X_0^\pi &= X_0 \\ X_t^\pi &= X_{t_n}^\pi + b(X_{t_n}^\pi)(t - t_n) + \sigma(X_{t_n}^\pi)(W_t - W_{t_n}),\ t \in (t_n, t_{n+1}] \end{cases}$$

- Error (b, σ Lipschitz)

$$\mathcal{E}rr(X, X^\pi) := \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - X_t^\pi|^2\right]^{\frac{1}{2}} \leq \frac{C}{\sqrt{N}}$$

$(\max_n |t_{n+1} - t_n| \leq \frac{C}{N})$

A scheme for the RBSDE: an example

- RBSDE, Snell envelop of $l(X_t)$:
  $Y_t = l(X_T) - \int_t^T (Z_u)'\mathrm{d}W_u + \int_t^T \mathrm{d}K_s \ , \ Y_t \geq l(X_t)$
- Discrete Snell envelop of $(l(X^\pi)_{t_n})_n$:

$$\widetilde{Y}^\pi_{t_n} := \mathbb{E}\left[Y^\pi_{t_{n+1}} \mid \mathcal{F}_{t_n}\right]$$
$$Y^\pi_{t_n} := \widetilde{Y}^\pi_{t_n} \vee l(X^\pi_{t_n})$$

$\hookrightarrow$ terminal condition $Y^\pi_T := l(X^\pi_T)$.

- More general domain/reflection:

$$Y^\pi_{t_n} = \widetilde{Y}^\pi_{t_n} \vee l(X^\pi_{t_n}) \rightarrow Y^\pi_{t_n} = \mathcal{P}(\widetilde{Y}^\pi_{t_n})$$

Moonwalk scheme for the RBSDE

- Implicit Euler scheme for the "BSDE part" :

$$\widetilde{Y}^{\pi}_{t_n} := \mathbb{E}\left[Y^{\pi}_{t_{n+1}} \mid \mathcal{F}_{t_n}\right] + (t_{n+1} - t_n) f(X^{\pi}_{t_n}, Y^{\pi}_{t_n}, \bar{Z}^{\pi}_{t_n})$$
$$\bar{Z}^{\pi}_{t_n} := (t_{n+1} - t_n)^{-1}\, \mathbb{E}\left[(W_{t_{n+1}} - W_{t_n})(Y^{\pi}_{t_{n+1}})' \mid \mathcal{F}_{t_n}\right]$$

- Taking into account the reflection

$$Y^{\pi}_{t_n} := \widetilde{Y}^{\pi}_{t_n} \mathbf{1}_{t_n \notin \Re} + \mathcal{P}(\widetilde{Y}^{\pi}_{t_n}) \mathbf{1}_{t_n \in \Re}$$

  $\Re \subset \pi$ is the reflection grid with κ dates.
- and terminal condition $Y^{\pi}_T = \widetilde{Y}^{\pi}_T := g(X^{\pi}_T)$.

Continuous version

- Piecewise continuous version of the scheme $(Y^\pi, \widetilde{Y}^\pi, Z^\pi)$:

$$Y^\pi_{t_{i+1}} = \mathbb{E}_{t_n}\big[Y^\pi_{t_{n+1}}\big] + \int_{t_n}^{t_{n+1}} (Z^\pi_u)'\mathrm{d}W_u$$

$$\widetilde{Y}^\pi_t = Y^\pi_{t_{n+1}} + (t_{n+1} - t) f(X^\pi_{t_n}, Y^\pi_{t_n}, \bar{Z}^\pi_{t_n}) - \int_t^T (Z^\pi_u)'\mathrm{d}W_u\,.$$

- Observe that : $\bar{Z}^\pi_{t_n} = \frac{1}{t_{n+1}-t_n}\mathbb{E}_{t_n}\Big[\int_{t_n}^{t_{n+1}} (Z^\pi_u)'\mathrm{d}u\Big]$
- is an approximation of $\bar{Z}$: $\bar{Z}_{t_n} := \frac{1}{t_{n+1}-t_n}\mathbb{E}_{t_n}\Big[\int_{t_n}^{t_{n+1}} (Z_u)'\mathrm{d}u\Big]$ which is a “proxy” for Z...

Error to control

- Error of interest :

$$\mathcal{E}rr(Y,\widetilde{Y}^\pi) := \sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}^\pi_t|^2\Big]^{\frac{1}{2}} \text{ or } \max_n \mathbb{E}\Big[|Y_{t_n} - \widetilde{Y}^\pi_{t_n}|^2\Big]^{\frac{1}{2}}$$

$$\mathcal{E}rr(Z,\bar{Z}^\pi) := \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_n}^{t_{n+1}} |Z_t - \bar{Z}^\pi_{t_n}|^2 \mathrm{d}t\right]^{\frac{1}{2}} = ||Z - \bar{Z}^\pi||_{\mathcal{H}^2}$$

- "Regularity" term

$$\mathcal{R}eg(Y) := \max_n \sup_{t\in[t_n,t_{n+1}]} \mathbb{E}\Big[|Y_t - Y_{t_n}|^2\Big]$$

$$\mathcal{R}eg(Z) := \mathbb{E}\left[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} |Z_t - \bar{Z}_{t_n}|^2 \mathrm{d}t\right]^{\frac{1}{2}}$$

Stability problem - formal discussion

In the multidimensional case, consider a scheme with pertubation $Y^{\pi *}$, $\widetilde{Y}^{\pi *}$ and the error between the two schemes is given by $\delta \widetilde{Y}^{\pi}$, δY^{π}.

- Using "classical" arguments, we obtain at step n

$$\mathbb{E}\Big[|\delta \widetilde{Y}^{\pi}_{t_n}|^2\Big] \le e^{\frac{C}{N}}(\mathbb{E}\big[|\delta Y^{\pi}_{t_{n+1}}|^2\big] + \text{pertubation terms})$$

- To iterate (at $t_n \in \Re$), remark that:

$$|\delta Y^{\pi}_{t_{n+1}}| := |Y^{\pi}_{t_{n+1}} - Y^{\pi *}_{t_{n+1}}| = |\mathcal{P}(\widetilde{Y}^{\pi}_{t_{n+1}}) - \mathcal{P}(\widetilde{Y}^{\pi *}_{t_{n+1}})| \le L|\delta \widetilde{Y}^{\pi}_{t_{n+1}}|$$

- iteration gives

$$\mathbb{E}\Big[|\delta \widetilde{Y}^{\pi}_{t_n}|^2\Big] \le L^{2\kappa} \times C \sum \text{pertubation terms}$$

normal reflection: $L \le 1$, OK. but oblique reflection... $L > 1$.

Method (1/2)

$\hookrightarrow$ known approximation results for reflected BSDEs in the case of : simple reflection, double reflection, reflection in a convex domain with *normal* reflection

$\hookrightarrow$ using the following method:

1. Discretize the reflection: use a discrete grid $\Re$ of the time interval $[0, T]$. New object called "Discretely Reflected BSDE" (DR)

2. Propose an approximation scheme for the DR using the discrete grid π (assuming $\Re \subset \pi$)

Method (2/2)

3. Prove the convergence of the scheme to the DR when π is refined: as in the non-reflected case, need for "regularity"

4. Prove that the DR converges to the RBSDE when $\Re$ is refined.

5. Use the approximation scheme of the DR to approximate the RBSDE (assuming $\Re \subset \pi$) and combine **3.** & **4.** (setting $\Re$ and π in a convenient way) to obtain convergence results.

$\hookrightarrow$ We use the same method here, but proofs are different ! main difficulty: stablity.

Discretely reflected BSDEs

Given a grid $\Re = \{0 = r_0 < ... < r_k < ... < r_\kappa = T\}$.
a triplet $(Y^d, \tilde{Y}^d, Z^d)$ satisfying

$$Y^d_T = \tilde{Y}^d_T := g(X_T)$$

and, for $j \leq \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\begin{cases} \tilde{Y}^d_t &= Y^d_{r_{j+1}} + \int_t^{r_{j+1}} f(X, \tilde{Y}^d, Z^d)\mathrm{d}u - \int_t^{r_{j+1}} (Z^d_u)'\mathrm{d}W_u \quad , \\ Y^d_t &= \tilde{Y}^d_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(\tilde{Y}^d_t)\mathbf{1}_{\{t \in \Re\}}. \end{cases}$$

example: simply reflected on $l(X)$, set $f = 0$, $g = l$... DR is the discrete Snell envelop of $l(X_r)_{r \in \Re}$

Discretely Obliquely Reflected BSDE - Representation

- It can be rewritten

$$\tilde{Y}^d_t = g(X_T) + \int_t^T f(X_s, \tilde{Y}^d_s, Z^d_s)\mathrm{d}s - \int_t^T (Z^d_s)'\mathrm{d}W_s + \tilde{K}^d_T - \tilde{K}^d_T$$

$$\tilde{K}^d_t = \sum_{r \in \Re \setminus \{0\}} \Delta \tilde{K}^d_r 1_{t \geq r}\ , \ \Delta \tilde{K}^d_r = Y^d_r - \tilde{Y}^d_r = \mathcal{P}(\tilde{Y}^d_r) - \tilde{Y}^d_r$$

- Same representation property as the obliquely RBSDE

$$(\tilde{Y}^d_t)^i = \operatorname*{esssup}_a U^a_t = U^{a*}_t\ , \quad \forall i \leq d$$

but switching times take their values in $\Re$!

Regularity results for discretely obliquely RBSDEs

- Stability of DR with respect to the parameter f,b,σ... allows us to regularize them.
- Representation using the optimal strategy a^* $(f = f(x))$

$$(Z^d_t)^{i'} = \mathbb{E}\left[\nabla g^{a^*_T}(X_T)D_tX_T + \int_t^T \nabla f^{a^*_s}(X_s)D_tX_s\mathrm{d}s \mid \mathcal{F}_t\right]$$

- allows us to obtain

$$\mathcal{R}eg(Z^d) \le C\left(\sqrt{\frac{\kappa}{N}} + \frac{1}{N^{\frac{1}{4}}}\right)$$

- Observe : $\mathcal{R}eg(\tilde{Y}^d) \le \frac{C}{\sqrt{N}}$

Error between the scheme and the discretely RBSDE

- Using "classical" arguments : $C > 1$

$$\mathcal{E}rr(\widetilde{Y}^d, \widetilde{Y}^\pi) + \mathcal{E}rr(Z^d, \bar{Z}^\pi) \leq C^\kappa \Big(\mathcal{E}rr(X, X^\pi) + \mathcal{R}eg(\tilde{Y}^d) + \mathcal{R}eg(Z^d) \Big)$$

- Stronger assumption : f does not depend z, two steps

$$(a) \quad \mathcal{E}rr(\tilde{Y}^d, \widetilde{Y}^\pi) + \mathcal{E}rr(Y^d, Y^\pi) \leq \frac{C}{\sqrt{N}}$$

$$(b) \quad \mathcal{E}rr(Z^d, \bar{Z}^\pi) \ \leq \ C\Big(\sqrt{\frac{\kappa}{N}} + \frac{1}{N^{\frac{1}{4}}}\Big)$$

where we used the regularity of $(\tilde{Y}^d, Z^d)$.

Sketch of proof (1/2)

1. Recall that $(\widetilde{Y}^d, \widetilde{K}^d)$ has a representation in term of "switched" BSDEs U for an optimal strategy a^*: $(\widetilde{Y}^d_t)^i = U^{a^*}_t$.

2. Remark that $(\widetilde{Y}^\pi, \widetilde{K}^\pi)$ can be seen as a discretely RBSDE, it also has a representation in term of "switched" BSDEs (U^π) for an optimal strategy a^π: $(\widetilde{Y}^\pi_t)^i = U^{\pi,a^\pi}_t$.

3. We introduce another discretely RBSDE $(\widetilde{\check{Y}}, \widetilde{\check{K}})$ with driver $\check{f}_t := f(X_t, \widetilde{Y}^d_t) \vee f(X^\pi_{t_i}, \widetilde{Y}^\pi_{t_i})$ for $t \in [t_i, t_{i+1})$ and terminal condition $\check{g}_T := g(X_T) \vee g(X^\pi_T)$. It also has a representation in term of "switched" BSDEs $(\check{U})$ for an optimal strategy $\check{a}$: $(\widetilde{\check{Y}}_t)^i = \check{U}^{\check{a}}_t$.

Sketch of proof (2/2)

4. Using comparison theorem, we observe that $(\tilde{\check{Y}}_t)^i \geq (\tilde{Y}^d_t)^i \vee (\tilde{Y}^\pi_t)^i$, $\forall(t,i)$.

5. Combining step 1-4, we obtain

$$0 \leq (\tilde{\check{Y}}_t)^i - (\tilde{Y}^d_t)^i \leq \check{U}^{\check{a}}_t - U^{\check{a}}_t \text{ and } 0 \leq (\tilde{\check{Y}}_t)^i - (\tilde{Y}^\pi_t)^i \leq \check{U}^{\check{a}}_t - U^{\pi,\check{a}}_t$$

6. This leads to

$$|(\tilde{Y}^d_t)^i - (\tilde{Y}^\pi_t)^i|^2 \leq 2(|\check{U}^{\check{a}}_t - U^{\pi,\check{a}}_t|^2 + |\check{U}^{\check{a}}_t - U^{\check{a}}_t|^2)$$

where the right-hand side term is very easy to control...

Error between the discretely RBSDE and the obliquely RBSDE (f bounded in z)

Also in two steps :

$$(i) \quad \mathbb{E}\Big[\sup_{r\in\Re}|Y_r - \tilde{Y}^d_r|^2\Big] \leq \frac{C}{\kappa}$$

then applying “classical” arguments and using (i)

$$(ii) \quad \mathcal{E}rr(Y, Y^d) + \mathcal{E}rr(Y, \tilde{Y}^d) + \mathcal{E}rr(Z, Z^d) \leq \frac{C}{\kappa^{\frac{1}{4}}}$$

Convergence results

- Whenever f is bounded in z, the scheme converges... but the bound is $\frac{C^\epsilon}{\log(N)^{\frac{1}{4}-\epsilon}}$, $\forall \epsilon > 0$.
- Combining the previous controls, when f does not depend on z:

$$\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Y, \widetilde{Y}^\pi) \le \frac{C}{N^{\frac{1}{4}}} \quad \text{and} \quad \mathcal{E}rr(Z, \bar{Z}^\pi) \le \frac{C}{N^{\frac{1}{6}}}$$

- And we obtain a better convergence rate at the grid point π

$$\max_{n \le N} \mathbb{E}\big[|Y_{t_n} - Y^\pi_{t_n}|^2\big] \le \frac{C}{\sqrt{N}}$$

Concluding remarks

- Extension: Constant costs$\rightarrow$ C^2 costs: $C\frac{1}{N^\alpha} \rightarrow C_\epsilon \frac{1}{N^{\alpha-\epsilon}}$, $\epsilon > 0$.
  case of Lipschitz costs ?
- Method to obtain stability allows to slightly extend existence and uniqueness for the obliquely RBSDE. $f^i = f(y, z^i)$
- Assumption on f: $f^i = f(y^i, z^i)$ & f bounded in z or $f^i = f(y^i)$.
  Reasonable convergence rate when $f^i = f^i(y)$ or when it depends on z ?