Probabilistic Representation of Weak Solutions of PDEs with Polynomial Growth Coefficients via Corresponding Backward SDEs

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Outline

PDEs with polynomial growth coefficients

We study

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\mathscr{L}u(t,x) - f\big(t,x,u(t,x),(\sigma^*\nabla u)(s,x)\big), \ \ t\in[0,T], \\ u(T,x) = h(x). \end{cases}$$

where $\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$.

$$|f(s,x,y,z)| \leq C(|f_0(s,x)| + |y|^p + |z|),$$

where $p \geq 1$ and $\int_0^T \int_{\mathbb{R}^d} |f_0(s,x)|^{2p}\rho^{-1}(x)dxds < \infty$.

Time Variables Transformation

If f is independent of t, defining $v(t,\cdot) \triangleq u(T-t,\cdot)$, we have

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \mathscr{L}v(t,x) + f\big(x, v(t,x), (\sigma^*\nabla v)(s,x)\big), \ \ t \in [0,T], \\ v(0,x) = h(x), \end{cases}$$

Definition for Weak Solutions of PDEs

Banach space $L_\rho^m(\mathbb{R}^d;\mathbb{R}^1)$ is the ρ-weighted $L^m(\mathbb{R}^d;\mathbb{R}^1)$ space with the norm $\left(\int_{\mathbb{R}^d}|u(x)|^m\rho^{-1}(x)dx\right)^{\frac{1}{m}}$, where $\rho(x)=(1+|x|)^q$, $q>d$.

We call u a weak solution if $(u,\sigma^*\nabla u)\in L^{2p}([0,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))$ $\otimes L^2([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and for $\forall\varphi\in C_c^\infty(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}u(t,x)\varphi(x)dx-\int_{\mathbb{R}^d}u(T,x)\varphi(x)dx\\
&-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}\left((\sigma^*\nabla u)(s,x)\right)^*(\sigma^*\nabla\varphi)(x)dxds\\
&-\int_t^T\int_{\mathbb{R}^d}u(s,x)div\left((b-\tilde{A})\varphi\right)(x)dxds\\
=&\int_t^T\int_{\mathbb{R}^d}f\left(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\right)\varphi(x)dxds.
\end{aligned}$$

Corresponding Backward SDEs

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \langle Z_r^{t,x}, dW_r\rangle.$$

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad s \geq t.$$

Definition

We call $(Y_s^{t,x}, Z_s^{t,x})$ a solution of corresponding BSDE if $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2p}([t,T]; L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and $(Y_s^{t,x}, Z_s^{t,x})$ satisfies the form of corresponding BSDE for a.e. x, with probability one.

Solution Space of BSDE

$(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2p}([t,T]; L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$
means

- $(Y_s^{t,\cdot}, Z_s^{t,\cdot})$ is adapted to the filtration generated by W
- $Y_s^{t,\cdot}$ is continuous w.r.t. s in $L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1)$
- $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ satisfies

$$\Big(E[\sup_{s\in[t,T]} \|Y_s^{t,\cdot}\|^{2p}_{L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1)}]\Big)^{\frac{1}{2p}} + \Big(E[\int_t^T \|Z_s^{t,\cdot}\|^2_{L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)} ds]\Big)^{\frac{1}{2}} < \infty.$$

Nonlinear Feynman-Kac formula of weak solutions of PDEs with polynomial growth coefficients

Function $u(t,\cdot) \triangleq Y_t^{t,\cdot}$ is the unique weak solution of PDE with polynomial growth coefficient. Moreover,

$$u(s, X_s^{t,\cdot}) = Y_s^{t,\cdot},\ (\sigma^* \nabla u)(s, X_s^{t,\cdot}) = Z_s^{t,\cdot} \ \textit{for a.e. } s \in [t, T] \ a.s.$$

Some Existing Correspondence Results in the Sense of Classical Solutions or Viscosity Solutions of PDEs

- Peng 1991 (Stochastics)
  PDEs/BSDEs with smooth coefficients (classical solutions)
- Pardoux & Peng 1992
  PDEs/BSDEs with Lipschitz coefficients (viscosity solutions)
- Pardoux & Tang 1999 (PTRF)
  PDEs/BSDEs with linear growth coefficients
- Pardoux 1999
  PDEs/BSDEs with continuous increasing coefficients
- Kobylanski 2000 (AOP), Briand & Hu 2006 (PTRF)
  PDEs/BSDEs with quadratic growth coefficients on z

Some Existing Correspondence Results in the Sense of Weak Solutions of PDEs

- Barles & Lesigne 1997, Bally & Matoussi 2001 (JTP)
  PDEs/BSDEs, SPDEs/BDSDEs with Lipschitz coefficients
- Zhang & Zhao 2010 (JDE)
  SPDEs/BDSDEs with linear growth coefficients
- Matoussi & Xu 2008 (EJP)
  PDEs with obstacle

  They used a different method to get weak convergence for a fixed x and the choice of subsequence depends on x.

Outline

Conditions

(H.1). For a given $p \geq 1$, $\int_{\mathbb{R}^d} |h(x)|^{2p} \rho^{-1}(x) dx < \infty$.

(H.2). There exist $C \geq 0$ and a function f_0 with $\int_0^T \int_{\mathbb{R}^d} |f_0(s,x)|^{2p} \rho^{-1}(x) dxds < \infty$ s.t.

$$|f(s,x,y,z)| \leq C(|f_0(s,x)| + |y|^p + |z|).$$

(H.3). There exists $\mu \in \mathbb{R}^1$ s.t. for any $s \in [0,T]$, $y_1, y_2 \in \mathbb{R}^1$, $x, z \in \mathbb{R}^d$,

$$(y_1 - y_2)\big(f(s,x,y_1,z) - f(s,x,y_2,z)\big) \leq \mu |y_1 - y_2|^2.$$

(H.4). Function $y \to f(s,x,y,z)$ is continuous and $z \to f(s,x,y,z)$ is globally Lipschitz continuous.

(H.5). Coefficients $b \in C_b^2(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^3(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$ and σ satisfies the uniform ellipticity condition.

Simplify Condition (H.3)

For a.e. $x \in \mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x})$ solves the studied BSDE if and only if $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x}) = (\mathrm{e}^{\mu s} Y_s^{t,x}, \mathrm{e}^{\mu s} Z_s^{t,x})$ solves the following BSDE:

$$\tilde{Y}_s^{t,x} = \mathrm{e}^{\mu T} h(X_T^{t,x}) + \int_s^T \tilde{f}(r, X_r^{t,x}, \tilde{Y}_r^{t,x}, \tilde{Z}_r^{t,x}) dr - \int_s^T \langle \tilde{Z}_r^{t,x}, dW_r \rangle,$$

where $\tilde{f}(r,x,y,z) = \mathrm{e}^{\mu r} f(r, x, \mathrm{e}^{-\mu r} y, \mathrm{e}^{-\mu r} z) - \mu y$ and $\tilde{f}$ satisfies

$$(y_1 - y_2)\big(\tilde{f}(s,x,y_1,z) - \tilde{f}(s,x,y_2,z)\big) \leq 0.$$

We can replace Condition (H.3) by

(H.3)*. For any $s \in [0, T]$, $y_1, y_2 \in \mathbb{R}^1$, $x, z \in \mathbb{R}^d$,

$$(y_1 - y_2)\big(f(s,x,y_1,z) - f(s,x,y_2,z)\big) \leq 0.$$

Generalized Equivalence of Norm Principle

Lemma

If $s \in [t, T]$, $\varphi : \Omega \otimes \mathbb{R}^d \to \mathbb{R}^1$ *is independent of* $\mathscr{F}^W_{t,s}$ *and* $\varphi\rho^{-1} \in L^1(\Omega \otimes \mathbb{R}^d; \mathbb{R}^1)$, *then there exist* $c, C > 0$ *s.t.*

$$\begin{aligned} cE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx] &\leq E[\int_{\mathbb{R}^d} |\varphi(X^{t,x}_s)|\rho^{-1}(x)dx] \\ &\leq CE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx]. \end{aligned}$$

Truncated BSDEs

We define for each $n \in N$

$$f_n(s,x,y,z) \triangleq f\left(s,x,\Pi_n(y),z\right),$$

where $\Pi_n(y) = \frac{\inf(n,|y|)}{|y|}y$. Then f_n satisfies

(H.2)′. For any $s \in [0,T]$, $y \in \mathbb{R}^1$, $x,z \in \mathbb{R}^d$ and C given in (H.2),

$$|f_n(s,x,y,z)| \leq C(|f_0(s,x)| + |n|^{p-1}|y| + |z|).$$

(H.3)′. For any $s \in [0,T]$, $y_1,y_2 \in \mathbb{R}^1$, $x \in \mathbb{R}^d$,

$$(y_1 - y_2)\big(f_n(s,x,y_1,z) - f_n(s,x,y_2,z)\big) \leq 0.$$

(H.4)′. Function $y \rightarrow f_n(s,x,y,z)$ is continuous and $z \rightarrow f_n(s,x,y,z)$ is globally Lipschitz continuous.

Results on BSDEs with linear growth coefficients

BSDEs with linear growth coefficient f_n:

$$Y_s^{t,x,n} = h(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})dr - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle.$$

Theorem

The truncated BSDE has a unique solution $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. $u_n(t,x) \triangleq Y_t^{t,x,n}$ *is the unique weak solution of truncated PDE:*

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) = -\mathscr{L} u_n(t,x) - f_n\big(t, x, u_n(t,x), (\sigma^* \nabla u)(t,x)\big), \\ u_n(T,x) = h(x). \end{cases}$$

Moreover, for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ *a.s.* $\omega \in \Omega$,

$$u_n(s, X_s^{t,x}) = Y_s^{t,x,n}, \ (\sigma^* \nabla u_n)(s, X_s^{t,x}) = Z_s^{t,x,n}.$$

Lemma

Under Conditions (H.1), (H.2), (H.3)*, (H.4) and (H.5), if $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ are the solutions of truncated BSDEs, then we have

$$E\Big[\int_t^T \sup_n \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^{2p} \rho^{-1}(x)dxds\Big]$$
$$+\sup_n E\Big[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^{2p-2} |Z_s^{t,x,n}|^2 \rho^{-1}(x)dxds\Big] < \infty.$$

Weak Convergence Limit of BSDEs

Define $U_s^{t,x,n} \triangleq f_n(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n})$, $s \geq t$, then

$$\sup_n E[\int_t^T \int_{\mathbb{R}^d} (|Y_s^{t,x,n}|^2 + |Z_s^{t,x,n}|^2 + |U_s^{t,x,n}|^2)\rho^{-1}(x)dxds] < \infty.$$

Therefore by Alaoglu lemma, there exists a subsequence, still denoted by $(Y_s^{t,x,n}, Z_s^{t,x,n}, U_s^{t,x,n})$, s.t.

$$(Y_s^{t,x,n}, Z_s^{t,x,n}, U_s^{t,x,n}) \rightharpoonup (Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}) \ weakly$$

in $L^2(\Omega \otimes [t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^d) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$.

Taking the weak limit in $L^2(\Omega \otimes [t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$ to truncated BSDEs, we know that $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x})$ satisfies

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T U_r^{t,x} dr - \int_s^T \langle Z_r^{t,x}, dW_r \rangle.$$

Rellich-Kondrachov Compactness Theorem

The following theorem comes from Robinson 2001.

Theorem

Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Here $X \subset\subset H$ means X is compactly embedded in H. Suppose that u_n is a sequence which is uniformly bounded in $L^2([0,T];X)$, and du_n/dt is uniformly bounded in $L^p([0,T];Y)$, for some $p>1$. Then there is a subsequence which converges strongly in $L^2([0,T];H)$.

In our case: Note that $H^1_\rho(U_i;\mathbb{R}^1) \subset\subset L^2_\rho(U_i;\mathbb{R}^1)$, where $U_i = \{x \in \mathbb{R}^d : |x| \leq i\}$ are closed balls in $\mathbb{R}^d$. We take

$$X = H^1_\rho(U_i;\mathbb{R}^1),\ H = L^2_\rho(U_i;\mathbb{R}^1) \text{ and } Y = H^{1^*}_\rho(U_i;\mathbb{R}^1).$$

To Derive a Strongly Convergent Subsequence

Lemma

Let $(Y_s^{t,x,n}, Z_s^{t,x,n})$ *be the solutions of truncated BSDEs and* $Y_s^{t,x}$ *is its weak limit in* $L^2(\Omega\otimes[t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, *then there is a subsequence of* $Y_s^{t,x,n}$, *still denoted by* $Y_s^{t,x,n}$, *converges strongly to* $Y_s^{t,x}$ *in* $L^2(\Omega\otimes[t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$.

Moreover,

$$E[\int_t^T\int_{\mathbb{R}^d}|Y_s^{t,x}|^{2p}\rho^{-1}(x)dxds]<\infty.$$

Proof. Let $u_n(s,x) \triangleq Y_s^{s,x,n}$, then by Zhang & Zhao 2010, $u_n(s,X_s^{t,x}) = Y_s^{t,x,n}$, $(\sigma^*\nabla u_n)(s,X_s^{t,x}) = Z_s^{t,x,n}$ for a.e. $s \in [t,T]$, $x \in \mathbb{R}^d$ a.s. and $u_n(s,x)$, $0 \le s \le T$, satisfies

$$du_n(s,x)/ds = -\mathscr{L}u_n(s,x) - f_n\big(s,x,u_n(s,x),(\sigma^*\nabla u_n)(s,x)\big).$$

We verify u_n are uniformly bounded in $L^2([0,T];H^1_\rho(\mathbb{R}^d;\mathbb{R}^1))$ as follows:

$$\begin{aligned}
& \sup_n \|u_n\|^2_{L^2([0,T];H^1_\rho(\mathbb{R}^d;\mathbb{R}^1))} \\
= & \sup_n \int_0^T \int_{\mathbb{R}^d} (|u_n(s,x)|^2 + |\nabla u_n(s,x)|^2)\rho^{-1}(x)dxds \\
\le & C_p \sup_n \int_0^T \int_{\mathbb{R}^d} (|u_n(s,x)|^2 + |(\sigma^*\nabla u_n)(s,x)|^2)\rho^{-1}(x)dxds \\
\le & C_p \sup_n E[\int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x,n}|^2 + |Z_s^{0,x,n}|^2)\rho^{-1}(x)dxds] < \infty.
\end{aligned}$$

Also du_n/ds are uniformly bounded in $L^2([0,T];H_\rho^{1^*}(\mathbb{R}^d;\mathbb{R}^1))$ since

$$\sup_n \|\mathscr{L}u_n\|^2_{L^2([0,T];H_\rho^{1^*}(\mathbb{R}^d;\mathbb{R}^1))} < \infty$$

and

$$\sup_n \|f_n\|^2_{L^2([0,T];H_\rho^{1^*}(\mathbb{R}^d;\mathbb{R}^1))} < \infty.$$

Apply Rellich-Kondrachov compactness theorem to U_i and pick up the diagonal subsequence of u_n. The diagonal subsequence converges strongly in all $L^2([0,T];L^2_\rho(U_i;\mathbb{R}^1))$, $i\in\mathbb{N}$.

Lemma

Let $u_n(t,x)$ be the weak solutions of truncated PDEs, then

$$\lim_{i\to\infty}\sup_n\int_0^T\int_{\mathbb{R}^d}|u_n(s,x)|^2 I_{U_i{}^c}(x)\rho^{-1}(x)dxds=0.$$

Then, u_n converges strongly in $L^2([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ and $Y_s^{t,x,n}$ converges strongly in $L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ to its weak limit $Y_s^{t,x}$.

If we define $u(s,x) \triangleq Y_s^{s,x}$, then by equivalence of norm principle we further have

$$u_n(s,x) \to u(s,x) \; strongly \;\; in \; L^2([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$$

and

$$Y_s^{t,x} = u(s, X_s^{t,x}) \;\; for \; a.e. \; s \in [t,T], \;\; x \in \mathbb{R}^d \;\; a.s.$$

Moreover, we can prove $E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x}|^{2p}\rho^{-1}(x)dxds] < \infty$. For this, we only need to prove $\int_0^T \int_{\mathbb{R}^d} |u(s,x)|^{2p}\rho^{-1}(x)dxds < \infty$.

Since $\int_0^T \int_{\mathbb{R}^d} |u_n(s,x) - u(s,x)|^2\rho^{-1}(x)dxds \to 0$, we can derive a subsequence $u_n(s,x)$ s.t. for a.e. $s \in [t,T]$, $x \in \mathbb{R}^d$,

$$u_n(s,x) \to u(s,x) \text{ and } \sup_n |u_n(s,x)| < \infty.$$

(Borrowing ideas from Lepeltier & San Martin 1997)

Therefore, by Hölder inequality, for any $\delta > 0$,

$$\lim_{N\to\infty} \sup_n \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p-\delta} I_{\{|u_n(s,x)|^{2p-\delta} > N\}}(s,x)\rho^{-1}(x)dxds = 0.$$

Noticing $u_n(s,x) \to u(s,x)$ for a.e. $s \in [0,T]$, $x \in \mathbb{R}^d$, we have

$$
\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} |u(s,x)|^{2p-\delta} \rho^{-1}(x) dxds \\
= \ & \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p-\delta} \rho^{-1}(x) dxds \\
\leq \ & \sup_n \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p-\delta} \rho^{-1}(x) dxds \\
\leq \ & C_p \Big(\sup_n \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p} \rho^{-1}(x) dxds \Big)^{\frac{2p-\delta}{2p}} \leq C_p,
\end{aligned}
$$

where the last $C_p < \infty$ is a constant independent of n and δ. Then using Fatou lemma to take the limit as $\delta \to 0$ in above inequality, we can get $\int_0^T \int_{\mathbb{R}^d} |u(s,x)|^{2p} \rho^{-1}(x) dxds < \infty$.

By Itô's formula and the strong convergence of the subsequence $Y^{t,\cdot,n}_\cdot \to Y^{t,\cdot}_\cdot$, we can deduce

Corollary

$Z^{t,\cdot,n}_\cdot \to Z^{t,\cdot}_\cdot$ *strongly in* $L^2(\Omega \otimes [t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ *also.*

Corollary

$Y^{t,\cdot,n}_\cdot \to Y^{t,\cdot}_\cdot$ *strongly in* $S^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ *and* $Y^{t,\cdot}_\cdot \in S^{2p}([t,T]; L^{2p}_\rho(\mathbb{R}^d;\mathbb{R}^1))$.

Identification of the Limiting BSDEs

Lemma

The random field U, Y and Z have the following relation:

$$U_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \quad \textit{for a.e. } s \in [t,T],\ x \in \mathbb{R}^d \textit{ a.s.}$$

Proof. Similarly as before, we can find a subsequence $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfying $(Y_s^{t,x,n}, Z_s^{t,x,n}) \to (Y_s^{t,x}, Z_s^{t,x})$ and $\sup_n |Y_\cdot^{t,\cdot,n}| + \sup_n |Z_\cdot^{t,\cdot,n}| < \infty$ for a.e. $s \in [t,T]$, $x \in \mathbb{R}^d$ a.s.

Let $\mathcal{K}$ be a set in $\Omega \otimes [t,T] \otimes \mathbb{R}^d$ s.t.

$$\sup_n |Y_s^{t,x,n}| + \sup_n |Z_s^{t,x,n}| + |f_0(s, X_s^{t,x})| < K.$$

Then as $K \uparrow \infty$, $\mathcal{K} \uparrow \Omega \otimes [t,T] \otimes \mathbb{R}^d$.

Thus, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty} E\Big[\int_t^T \int_{\mathbb{R}^d} |f_n(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}) I_{\mathcal{K}}(s,x) \\
&\qquad\qquad - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) I_{\mathcal{K}}(s,x)|^2 \rho^{-1}(x)\,dxds\Big] \\
\leq\ & 2E\Big[\int_t^T \int_{\mathbb{R}^d} \lim_{n\to\infty} |f_n(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}) \\
&\qquad\qquad - f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n})|^2 I_{\mathcal{K}}(s,x) \rho^{-1}(x)\,dxds\Big] \\
&+2E\Big[\int_t^T \int_{\mathbb{R}^d} \lim_{n\to\infty} |f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}) \\
&\qquad\qquad - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})|^2 I_{\mathcal{K}}(s,x) \rho^{-1}(x)\,dxds\Big].
\end{aligned}$$

Obviously, due to the continuity of $(y,z) \to f(s,x,y,z)$,

$$\lim_{n\to\infty} |f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}) - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})|^2 = 0$$

for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.

Since $Y_s^{t,x,n} \to Y_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s., there exists a $N(s,x,\omega)$ s.t. when $n \geq N(s,x,\omega)$, $|Y_s^{t,x,n}| \leq |Y_s^{t,x}| + 1$. So taking $n \geq \max\{N(s,x,\omega),\ |Y_s^{t,x}| + 1\}$, we have

$$f_n(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}) = f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}),$$

thus $\lim_{n\to\infty} |f_n(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}) - f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n})|^2 = 0$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. Therefore

$$U_s^{t,x,n} I_{\mathcal{K}}(s,x) \to f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) I_{\mathcal{K}}(s,x) \ \textit{strongly}$$

in $L^2(\Omega \otimes [t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$. On the other hand,

$$U_s^{t,x,n} I_{\mathcal{K}}(s,x) \rightharpoonup U_s^{t,x} I_{\mathcal{K}}(s,x) \ \textit{weakly}$$

in $L^2(\Omega \otimes [t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$. So

$$f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) I_{\mathcal{K}}(s,x) = U_s^{t,x} I_{\mathcal{K}}(s,x) \ \textit{for a.e.}\ r \in [t, T],\ x \in \mathbb{R}^d \ \textit{a.s.}$$

Outline

Existence and Uniqueness Theorem of PDE

Theorem

Define $u(t,x) \triangleq Y_t^{t,x}$, *then* $u(t,x)$ *is the unique weak solution to*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\mathscr{L}u(t,x) - f\big(t,x,u(t,x),(\sigma^*\nabla u)(s,x)\big), \\ u(T,x) = h(x). \end{cases}$$

Moreover,

$$u(s,X_s^{t,x}) = Y_s^{t,x},\ (\sigma^*\nabla u)(s,X_s^{t,x}) = Z_s^{t,x}$$

for a.e. $s \in [t,T]$, $x \in \mathbb{R}^d$ *a.s.*

Existence

Let $u^n(s,x)$ be the weak solutions of truncated PDEs. Then $(u_n, \sigma^*\nabla u_n) \in L^2([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes L^2([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and for $\forall \varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^1)$,

$$
\begin{aligned}
& \int_{\mathbb{R}^d} u_n(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} u_n(T,x)\varphi(x)dx \\
& -\frac{1}{2}\int_t^T \int_{\mathbb{R}^d} \big((\sigma^*\nabla u_n)(s,x)\big)^*(\sigma^*\nabla\varphi)(x)dxds \\
& -\int_t^T \int_{\mathbb{R}^d} u_n(s,x)div\big((b-\tilde{A})\varphi\big)(x)dxds \\
= \; & \int_t^T \int_{\mathbb{R}^d} f_n\big(s,x,u_n(s,x),(\sigma^*\nabla u_n)(s,x)\big)\varphi(x)dxds.
\end{aligned}
$$

We can prove along a subsequence that each term of the above formula converges to the corresponding term of test function form of studied PDE.

Uniqueness

The uniqueness of PDE comes from the uniqueness of BSDE. Let u be a solution. Define $F(s,x) \triangleq f\left(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\right)$, then

$$\int_0^T \int_{\mathbb{R}^d} |F(s,x)|^2 \rho^{-1}(x)dxds < \infty.$$

If we define $Y_s^{t,x} \triangleq u(s,X_s^{t,x})$ and $Z_s^{t,x} \triangleq (\sigma^*\nabla u)(s,X_s^{t,x})$, then by Bally & Matoussi 2001, $(Y_s^{t,x}, Z_s^{t,x})$ is a solution of the following linear BSDE:

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T F(r,X_r^{t,x})dr - \int_s^T \langle Z_r^{t,x}, dW_r\rangle.$$

Noting the definition of $F(s,x)$, $(Y_s^{t,x}, Z_s^{t,x})$ is a solution of corresponding nonlinear BSDE.

Thank You !!

Tunisia Pavilion at EXPO 2010 Shanghai
(picture from www.expo2010.cn)