Random G-Expectations

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New advances in Backward SDEs for financial engineering applications
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Outline

1. Random G-Expectations

2. Axiomatic Framework and Superhedging (*joint work with Mete Soner*)

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2. Axiomatic Framework and Superhedging (*joint work with Mete Soner*)

Peng's G-Expectation

- Intuition: Volatility is uncertain, but prescribed to lie in a fixed interval $D = [a, b]$.
- $\mathcal{E}^G(X)$ is the worst-case expectation of a random variable X over all these scenarios for the volatility.
- Ω: canonical space of continuous paths on $[0, T]$.
- B: canonical process.
- $\mathcal{P}^G$ = martingale laws under which $d\langle B\rangle_t/dt \in D$, then

$$\mathcal{E}_0^G(X) = \sup_{P\in\mathcal{P}^G} E^P[X], \quad X \in L^0(\mathcal{F}_T^\circ) \text{ regular enough.}$$

- $\mathcal{E}_0^G$ is a sublinear functional.

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Conditional G-Expectation

- Extension to a conditional G-expectation $\mathcal{E}_t^G$ for $t > 0$.
- Nontrivial as measures in $\mathcal{P}^G$ are singular.
- Peng's approach: for $X = f(B_T)$ with f Lipschitz, define $\mathcal{E}_t^G(X) = u(t, B_t)$ where

$$-u_t - G(u_{xx}) = 0, \quad u(T, x) = f; \quad G(x) := \frac{1}{2} \sup_{y \in D} xy.$$

- Extend to $X = f(B_{t_1}, \dots, B_{t_n})$ and pass to completion.
- Time-consistency property: $\mathcal{E}_s^G \circ \mathcal{E}_t^G = \mathcal{E}_s^G$ for $s \leq t$.

Short (and Incomplete) History

- Uncertain volatility model: Avellaneda, Levy, Parás (95), T. Lyons (95).
- BSDEs and g-expectations: Pardoux, Peng (90), ...
- Capacity-based analysis of volatility uncertainty: Denis, Martini (06).
- G-expectation and related calculus were introduced by Peng (07).
- Dual description via $\mathcal{P}^G$ is due to Denis, M. Hu, Peng (10).

- Constraint on law of B_T Galichon, Henry-Labordère, Touzi (1?).
- Relations to 2BSDEs Cheridito, Soner, Touzi, Victoir (07), Soner, Touzi, J. Zhang (10).
- Bion-Nadal, Kervarec (10).

Random G-Expectation

- Allow for updates of the volatility bounds.
- Take into account historical volatility: path-dependence.

- Replace $D = [a, b]$ by a stochastic interval $\mathbf{D}_t(\omega) = [a_t(\omega), b_t(\omega)]$. (In general: a progressive set-valued process.)
- This corresponds to a random function G (possibly infinite).
- At $t = 0$ define $\mathcal{E}_0(X) = \sup_{P \in \mathcal{P}} E^P[X]$ for corresponding set $\mathcal{P}$.

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Our Approach

- At $t > 0$: want $\mathcal{E}_t(X)$ " $=$ " $\sup_{P' \in \mathcal{P}} E^{P'}[X|\mathcal{F}_t^\circ]$.
- We shall construct $\mathcal{E}_t(X)$ such that

$$\mathcal{E}_t(X) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(t,P)}{}^{P} E^{P'}\big[X\big|\mathcal{F}_t^\circ\big] \quad P\text{-a.s. for all } P \in \mathcal{P},$$

where $\mathcal{P}(t,P) := \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_t^\circ\}$.

- Non-Markov problem: PDE approach not suitable.
- Pathwise definition: condition $\mathbf{D}, \mathcal{P}, X$ on ω up to t:

$$\mathcal{E}_t(X)(\omega) := \sup_{P \in \mathcal{P}(t,\omega)} E^P[X^{t,\omega}], \quad \omega \in \Omega.$$

- Benefits: Control arguments, $\mathbf{D}$ need not be bounded.
- Time consistency corresponds to dynamic programming principle.
- Approach follows Soner, Touzi, Zhang (10).
- Here $\mathcal{P}(t,\omega)$ is path-dependent. Regularity of $\omega \mapsto \mathcal{P}(t,\omega)$ is needed.

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Strong Formulation of Volatility Uncertainty

- P_0 Wiener measure, $\mathbb{F}^\circ$ raw filtration of B.
- $\overline{\mathcal{P}}_S = \big\{P^\alpha := P_0 \circ (\int \alpha^{1/2} dB)^{-1},\ \alpha > 0,\ \int_0^T \alpha\, dt < \infty\big\}$.
- Define $\langle B \rangle$ and $\hat{a} = d\langle B \rangle / dt$ simultaneously under all $P \in \overline{\mathcal{P}}_S$. (E.g. by Föllmer's (81) pathwise calculus.)
- $\mathcal{P} := \big\{P \in \overline{\mathcal{P}}_S : \hat{a} \in \mathrm{Int}^\delta\, \mathbf{D}\ ds \times P\text{-a.e. for some } \delta > 0\big\}$, where $\mathrm{Int}^\delta\, \mathbf{D} := [a + \delta, b - \delta]$ for $\delta > 0$.

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Conditioning and Regularity

- To condition X to ω up to t, set $X^{t,\omega}(\cdot) := X(\omega \otimes_t \cdot)$, where $\otimes_t$ is the concatenation at t.
- $X^{t,\omega}$ is an r.v. on the space Ω^t of paths starting at time t.
- On Ω^t we have B^t, P_0^t, $\hat{a}^t$, $\overline{\mathcal{P}}_S^t$, ... as for $t = 0$.
- $\mathcal{P}(t,\omega) := \{P \in \overline{\mathcal{P}}_S^t : \hat{a}^t \in \operatorname{Int}^\delta \mathbf{D}^{t,\omega}\ ds \times P\text{-a.e. on } [t,T] \times \Omega^t, \delta > 0\}$.
- Define $\mathcal{E}_t(X)$ as the value function

$$\mathcal{E}_t(X)(\omega) := \sup_{P \in \mathcal{P}(t,\omega)} E^P[X^{t,\omega}], \quad \omega \in \Omega.$$

- Regularity: $X \in UC_b(\Omega)$ and $\mathbf{D}$ uniformly continuous: for all $\delta > 0$ and $(t,\omega) \in [0,T] \times \Omega$ there exists $\varepsilon = \varepsilon(t,\omega,\delta) > 0$ s.t.

$$\|\omega - \omega'\|_t \leq \varepsilon \quad \Rightarrow \quad \operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) \subseteq \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega}) \quad \forall (s,\tilde{\omega}) \in [t,T] \times \Omega^t.$$

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Consequences of Uniform Continuity

- $\omega \mapsto \mathcal{E}_t(X)(\omega)$ is $\mathcal{F}_t^\circ$-measurable and LSC for $X \in \mathrm{UC}_b(\Omega)$.

Theorem (DPP, time consistency)

Let $X \in \mathrm{UC}_b(\Omega)$ and $0 \leq s \leq t \leq T$. Then

- $\mathcal{E}_s(X)(\omega) = \sup_{P \in \mathcal{P}(s,\omega)} E^P\left[\mathcal{E}_t(X)^{s,\omega}\right]$ *for all* $\omega \in \Omega$,
- $\mathcal{E}_s(X) = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(s,P)} E^{P'}\left[\mathcal{E}_t(X)\middle|\mathcal{F}_s^\circ\right]$ P*-a.s.* *for all* $P \in \mathcal{P}$,

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On the proof:

- Main problem due to stochastic **D**: admissibility of pastings.
- Regularity of $\mathcal{E}_t(X)$ turns out not to be a problem.

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Extension to Completion of $\mathrm{UC}_b(\Omega)$

- $L^1_{\mathcal{P}}$ = space of r.v. X such that $\|X\|_{L^1_{\mathcal{P}}} := \sup_{P\in\mathcal{P}} \|X\|_{L^1(P)} < \infty$.
- $\mathbb{L}^1_{\mathcal{P}}$ = closure of $\mathrm{UC}_b \subset L^1_{\mathcal{P}}$ (can be described explicitly).
- DPP implies that $\mathcal{E}_t$ is 1-Lipschitz wrt. $\|\cdot\|_{L^1_{\mathcal{P}}}$, hence extends to

$$\mathcal{E}_t : \mathbb{L}^1_{\mathcal{P}} \to L^1_{\mathcal{P}}(\mathcal{F}^\circ_t).$$

- **Theorem.** For $X \in \mathbb{L}^1_{\mathcal{P}}$ the DPP holds:

$$\mathcal{E}_s(X) = \operatorname*{ess\,sup}^P_{P'\in\mathcal{P}(s,P)} E^{P'}\left[\mathcal{E}_t(X)\middle|\mathcal{F}^\circ_s\right] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P}.$$

In particular, $\mathcal{E}_s(X)$ is characterized by

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Axiomatic Setup

For the random G-expectations, we had:

- a set $\mathcal{P} \subseteq \overline{\mathcal{P}}_S$ with a time consistency is a property,
- an aggregated r.v. for $\operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(s,P)} E^{P'}\left[X \middle| \mathcal{F}_s^\circ\right]$, $P \in \mathcal{P}$,
- for X in a subspace $\mathbb{L}^1_{\mathcal{P}} \subseteq L^1_{\mathcal{P}}$.

Axiomatic approach:

- start with some set $\mathcal{P} \subseteq \overline{\mathcal{P}}_S$.
- $\mathcal{P}$ is assumed to be stable under $\mathbb{F}^\circ$-pasting ($\approx$ time consistency): for all $P \in \mathcal{P}$ and $P_1, P_2 \in \mathcal{P}(\mathcal{F}_t^\circ, P)$ and $\Lambda \in \mathcal{F}_t^\circ$,

$$\bar{P}(\cdot) := E^P\left[P_1(\,\cdot\,|\mathcal{F}_t^\circ)\mathbf{1}_\Lambda + P_2(\,\cdot\,|\mathcal{F}_t^\circ)\mathbf{1}_{\Lambda^c}\right] \in \mathcal{P}.$$

- aggregated r.v. $\mathcal{E}_s^\circ(X) = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\mathcal{F}_s^\circ, P)} E^{P'}\left[X \middle| \mathcal{F}_s^\circ\right]$ P-a.s., $P \in \mathcal{P}$
- for all X in some subspace $\mathcal{H} \subseteq L^1_{\mathcal{P}}$.

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Getting Path Regularity

• $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_t\}_{0\leq t\leq T}$, where $\hat{\mathcal{F}}_t := \mathcal{F}^\circ_{t+} \vee \mathcal{N}^{\mathcal{P}}$ and $\mathcal{N}^{\mathcal{P}} = \mathcal{P}$-polar sets.

Take right limits of $\{\mathcal{E}^\circ_t(X),\, t \in [0,T]\}$:

Theorem

For $X \in \mathcal{H}$, there exists a unique càdlàg $\hat{\mathbb{F}}$-adapted process Y,

- $Y_t = \mathcal{E}^\circ_{t+}(X)$ *$\mathcal{P}$-q.s. for all t.*
- *Y is the minimal $(\hat{\mathbb{F}}, \mathcal{P})$-supermartingale with $Y_T = X$.*
- $Y_t = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\hat{\mathcal{F}}_t, P)} E^{P'}[X|\hat{\mathcal{F}}_t]$ *P-a.s. for all $P \in \mathcal{P}$.*

• Y is a $\mathcal{P}$-modification of $\{\mathcal{E}^\circ_t(X),\, t \in [0,T]\}$ in regular cases but there are counterexamples.

• The process $\mathcal{E}(X) := Y$ is called the (càdlàg) $\mathcal{E}$-martingale associated with $X \in \mathcal{H}$.

Stopping Times and Optional Sampling

- Typically, the construction of $\mathcal{E}^\circ$ is not compatible with stopping times (e.g. G-expectation).
- But we can easily define $\mathcal{E}$ at a stopping time.

Theorem

Let $0 \leq \sigma \leq \tau \leq T$ be $\hat{\mathbb{F}}$-stopping times and $X \in \mathcal{H}$. Then

$$\mathcal{E}_\sigma(X) = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\hat{\mathcal{F}}_\sigma, P)} E^{P'}[X|\hat{\mathcal{F}}_\sigma] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P};$$

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Decomposition of $\mathcal{E}$-Martingales

Theorem

Let $X \in \mathcal{H}$. There exist

- *an $\hat{\mathbb{F}}$-progressive process Z^X*
- *a family $(K^P)_{P\in\mathcal{P}}$ of $\overline{\mathbb{F}}^P$-pred. increasing processes, $E^P[|K^P_T|] < \infty$,*

such that

$$\mathcal{E}_t(X) = \mathcal{E}_0(X) + {}^{(P)}\!\!\int_0^t Z^X_s\, dB_s - K^P_t, \quad P\text{-a.s. for all } P \in \mathcal{P}.$$

- Z^X does not depend on P, but the integral may do.
- Cf. **optional decomposition**: El Karoui, Quenez (95), Kramkov (96).
- Construction as in the theory of 2BSDEs: Soner, Touzi, Zhang (10)
- Here we only need Doob-Meyer decomposition + martingale represent. + pathwise integration (Bichteler 81).
- More precise results for G-expectation: Peng (07), Xu, B. Zhang (09), Soner, Touzi, Zhang (10), Song (10), Y. Hu, Peng (10).

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Superhedging

Interpretation for decomposition

$$X = \mathcal{E}_T(X) = \mathcal{E}_0(X) + {}^{(P)}\!\!\int_0^T Z_s^X \, dB_s - K_T^P :$$

- $\mathcal{E}_0(X) = \hat{\mathcal{F}}_0$-superhedging price,
- Z^X = superhedging strategy,
- K_T^P = overshoot for the scenario P
- Minimality of the overshoot:

$$\operatorname*{ess\,inf}^P_{P' \in \mathcal{P}(\hat{\mathcal{F}}_t, P)} E^{P'}\big[K_T^{P'} - K_t^{P'} \big| \hat{\mathcal{F}}_t\big] = 0 \quad P\text{-a.s. for all } P \in \mathcal{P}.$$

- replicable claims correspond to $K^P \equiv 0$ for all $P \in \mathcal{P}$.

2BSDE for $\mathcal{E}(X)$

(Y, Z) is a solution of the 2BSDE if there exists a family $(K^P)_{P\in\mathcal{P}}$ of $\overline{\mathbb{F}}^P$-adapted increasing processes satisfying $E^P[|K_T^P|] < \infty$ such that

$$Y_t = X - {}^{(P)}\!\!\int_t^T Z_s\, dB_s + K_T^P - K_t^P, \quad 0 \le t \le T, \quad P\text{-a.s. for all } P \in \mathcal{P}$$

and such that

$$\underset{P' \in \mathcal{P}(\hat{\mathcal{F}}_t, P)}{\operatorname{ess\,inf}}{}^P E^{P'}\left[K_T^{P'} - K_t^{P'} \middle| \hat{\mathcal{F}}_t\right] = 0 \quad P\text{-a.s. for all } P \in \mathcal{P}.$$

Theorem ($X \in \mathcal{H}$)

- *$(\mathcal{E}(X), Z^X)$ is the minimal solution of the 2BSDE.*
- *If (Y, Z) is a solution of the 2BSDE such that Y is of class (D,$\mathcal{P}$), then $(Y, Z) = (\mathcal{E}(X), Z^X)$.*

In particular, if $X \in \mathcal{H}^p$ for some $p \in (1, \infty)$, then $(\mathcal{E}(X), Z^X)$ is the unique solution of the 2BSDE in the class (D,$\mathcal{P}$).

Pasting and Time Consistency

- $\mathcal{P}$ is *maximally chosen for* $\mathcal{H}$ if $\mathcal{P}$ contains all $P \in \overline{\mathcal{P}}_S$ such that $E^P[X] \leq \sup_{P' \in \mathcal{P}} E^{P'}[X]$ for all $X \in \mathcal{H}$.
- $\mathcal{P}$ is *time-consistent on* $\mathcal{H}$ if

$$\operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\mathcal{F}^\circ_s, P)} E^{P'}\Big[\operatorname*{ess\,sup}^{P'}_{P'' \in \mathcal{P}(\mathcal{F}^\circ_t, P')} E^{P''}[X|\mathcal{F}^\circ_t]\Big|\mathcal{F}^\circ_s\Big] = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\mathcal{F}^\circ_s, P)} E^{P'}[X|\mathcal{F}^\circ_s]$$

P-a.s. for all $P \in \mathcal{P}$, $0 \leq s \leq t \leq T$ and $X \in \mathcal{H}$.

Theorem

- *stability under pasting* $\Rightarrow$ *time consistency.*
- *If* $\mathcal{P}$ *is maximally chosen: time consistency* $\Rightarrow$ *stability under pasting*

- Similar results by Delbaen (06) for classical risk measures.

Time Consistency of Mappings

- Consider a family $(\mathcal{E}_t)_{0\leq t\leq T}$ of mappings $\mathcal{E}_t : \mathcal{H} \to L^1_{\mathcal{P}}(\mathcal{F}^\circ_t)$.
- $\mathcal{H}_t := \mathcal{H} \cap L^1_{\mathcal{P}}(\mathcal{F}^\circ_t)$.

Definition

$(\mathcal{E}_t)_{0\leq t\leq T}$ is called *time-consistent* if

$$\mathcal{E}_s(X) \leq (\geq)\, \mathcal{E}_s(\varphi) \quad \text{for all } \varphi \in \mathcal{H}_t \text{ such that } \mathbb{E}_t(X) \leq (\geq)\, \varphi$$

and *($\mathcal{H}_t$-) positively homogeneous* if

$$\mathcal{E}_t(X\varphi) = \mathcal{E}_t(X)\varphi \quad \text{for all bounded nonnegative } \varphi \in \mathcal{H}_t$$

for all $0 \leq s \leq t \leq T$ and $X \in \mathcal{H}$.

More on $\mathbb{L}^1_{\mathcal{P}}$

- By arguments of Denis, Hu, Peng (10):

$$\mathbb{L}^1_{\mathcal{P}} = \left\{ X \in L^1_{\mathcal{P}} \;\middle|\; \begin{array}{l} X \text{ is } \mathcal{P}\text{-quasi uniformly continuous,} \\ \lim_n \|X \mathbf{1}_{\{|X| \geq n\}}\|_{L^1_{\mathcal{P}}} = 0 \end{array} \right\}$$

- If $\mathbf{D}$ is uniformly bounded, we retrieve the space of Denis, Hu, Peng:
  - $\mathbb{L}^1_{\mathcal{P}}$ is also the closure of $C_b \subset L^1_{\mathcal{P}}$,
  - 'quasi uniformly continuous' = 'quasi continuous'.

- If $\mathbf{D}$ is uniformly bounded, $\mathcal{E}_t$ maps $\mathbb{L}^1_{\mathcal{P}}$ into $\mathbb{L}^1_{\mathcal{P}}(\mathcal{F}^\circ_t)$.
  Hence time consistency can be expressed as $\mathcal{E}_s \circ \mathcal{E}_t = \mathcal{E}_s$.