ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Answers to the written exam of March 4th, 2015.

1 Parametric optimization: 14 points

1. Writing $v = \langle u'(h), k\rangle$, $\Lambda = \langle \lambda'(h), k\rangle$ and differentiating problem (1) yields

$$\begin{cases} -\mathrm{div}\,(h\nabla v) - \mathrm{div}\,(k\nabla u) = \lambda\,\rho\,h\,v + \lambda\,\rho\,k\,u + \Lambda\,\rho\,h\,u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Differentiating the normalization condition $\int_\Omega \rho h u^2 dx = 1$ we obtain

$$\int_\Omega \rho\, u(2hv + ku)\, dx = 0.$$

2. Multiplying the above problem for v by u and integrating by parts leads to

$$\int_\Omega (h\nabla v\cdot\nabla u + k\nabla u\cdot\nabla u)\; dx = \int_\Omega \left(\lambda\,\rho\,h\,v\,u + \lambda\,\rho\,k\,u^2 + \Lambda\,\rho\,h\,u^2\right) dx.$$

From the equation for u we know that

$$\int_\Omega h\nabla v\cdot\nabla u\, dx = \int_\Omega \lambda\,\rho\,h\,v\,u\, dx.$$

Therefore, from the normalization $\int_\Omega \rho h u^2 dx = 1$, we deduce

$$\Lambda = \int_\Omega k\left(|\nabla u|^2 - \lambda\,\rho\,u^2\right)\, dx.$$

In other words, we found $\lambda'(h) = |\nabla u|^2 - \lambda\,\rho\,u^2$ which is a function in $L^1(\Omega)$.

3. From the previous question and since, by assumption, $\nabla u \neq 0$ on $\partial\Omega$ and $u = 0$ on $\partial\Omega$, we find $\lambda'(h) > 0$ on $\partial\Omega$ (and thus by continuity on a neighborhood of $\partial\Omega$). Therefore, for an optimal thickness h, the lower constraint $h \geq h_{min}$ must be saturated in this region, i.e., $h = h_{min}$ near $\partial\Omega$. On the contrary, at the point where u is maximum, we have $\nabla u = 0$ and $u > 0$, thus $\lambda'(h) < 0$. It implies that, in the vicinity of the maximum of u, we must have $h = h_{max}$.

4. The objective function is

$$J(h) = \int_\Omega j\left(\frac{u(h)}{\|u(h)\|}\right)\,dx.$$

It is clear that, since j is even, i.e., $j(-w) = j(w)$, we have for any $t \neq 0$

$$j\left(\frac{tu(h)}{\|tu(h)\|}\right) = j\left(\frac{u(h)}{\|u(h)\|}\right),$$

so that the objective function is independent from the normalization of the eigenfunction.

For $h \in \mathcal{U}_{ad}$, $\hat{\lambda} \in \mathbb{R}$, $\hat{u} \in H^1_0(\Omega)$ and $\hat{p} \in H^1_0(\Omega)$ we define the Lagrangian

$$\mathcal{L}(h, \hat{\lambda}, \hat{u}, \hat{p}) = \int_\Omega j\left(\frac{\hat{u}}{\|\hat{u}\|}\right)\,dx + \int_\Omega \left(h\nabla\hat{u}\cdot\nabla\hat{p} - \hat{\lambda}\rho h\hat{u}\hat{p}\right)\,dx.$$

5. For $\hat{u} \in L^2(\Omega)$ we define

$$F(\hat{u}) = \int_\Omega j\left(\frac{\hat{u}}{\|\hat{u}\|}\right)\,dx.$$

By the chain rule lemma, since the derivative of $\|\hat{u}\|$ in the direction of ϕ is $\langle\hat{u}, \phi\rangle/\|\hat{u}\|$ with the notation $\langle\hat{u}, \phi\rangle = \int_\Omega \hat{u}\,\phi\,dx$, we obtain

$$\langle F'(\hat{u}), \phi\rangle = \int_\Omega F'(\hat{u})\,\phi\,dx = \int_\Omega j'\left(\frac{\hat{u}}{\|\hat{u}\|}\right)\left(\frac{\phi}{\|\hat{u}\|} - \frac{\langle\hat{u}, \phi\rangle}{\|\hat{u}\|^3}\hat{u}\right)dx.$$

Clearly we find $\langle F'(\hat{u}), \hat{u}\rangle = 0$. Equivalently,

$$\int_\Omega F'(\hat{u})\,\phi\,dx = \int_\Omega j'\left(\frac{\hat{u}}{\|\hat{u}\|}\right)\frac{\phi}{\|\hat{u}\|}dx - \left(\int_\Omega j'\left(\frac{\hat{u}}{\|\hat{u}\|}\right)\frac{\hat{u}}{\|\hat{u}\|^3}dx\right)\int_\Omega \hat{u}\,\phi\,dx$$

which implies

$$F'(\hat{u}) = \frac{1}{\|\hat{u}\|}j'\left(\frac{\hat{u}}{\|\hat{u}\|}\right) - \left(\int_\Omega j'\left(\frac{\hat{u}}{\|\hat{u}\|}\right)\frac{\hat{u}}{\|\hat{u}\|^3}\,dx\right)\hat{u}\,.$$

6. The variational formulation of the adjoint equation is, by definition,

$$\langle\frac{\mathcal{L}}{\partial\hat{u}}(h, \lambda, u, p), \phi\rangle = 0 \quad \forall\,\phi \in H^1_0(\Omega).$$

We compute

$$\langle\frac{\mathcal{L}}{\partial\hat{u}}(h, \lambda, u, p), \phi\rangle = \int_\Omega F'(u)\,\phi\,dx + \int_\Omega \left(h\nabla\phi\cdot\nabla p - \lambda\rho h\phi p\right)\,dx.$$

7. By integration by parts we find that the adjoint p is a solution of

$$\begin{cases} -\mathrm{div}\,(h\nabla p) - \lambda\,\rho\,h\,p = -\frac{1}{\|u\|} j'\left(\frac{u}{\|u\|}\right) + \alpha\,u & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$\alpha = \left(\int_\Omega j'\left(\frac{u}{\|u\|}\right)\frac{u}{\|u\|^3}\,dx\right)$$

We check that the right hand side is orthogonal to u since

$$\int_\Omega \frac{u}{\|u\|} j'\left(\frac{u}{\|u\|}\right) dx = \alpha \int_\Omega u^2\,dx = \alpha\|u\|^2.$$

Clearly, if p is a solution, then $p + Cu$ is another possible solution. To determine the value of the constant C we use

$$\frac{\mathcal{L}}{\partial\hat{\lambda}}(h, \lambda, u, p) = -\int_\Omega \rho h u p\,dx = 0$$

which implies that $C = -\int_\Omega \rho h u p\,dx$.

8. The derivative satisfies

$$\langle J'(h), k\rangle = \langle\frac{\partial\mathcal{L}}{\partial h}(h, \lambda, u, p), k\rangle.$$

We compute

$$\langle\frac{\partial\mathcal{L}}{\partial h}(h, \lambda, u, p), k\rangle = \int_\Omega \left(k\nabla u\cdot\nabla p - \lambda\rho k u p\right)\,dx,$$

which implies

$$J'(h) = \nabla u\cdot\nabla p - \lambda\rho u p.$$

2 Geometric optimization: 6 points

1. To define the Lagrangian we introduce two Lagrange multipliers $q \in H^1(\mathbb{R}^N)$ and $\mu \in H^1(\mathbb{R}^N)$, which, together with $v \in H^1(\mathbb{R}^N)$, are the arguments of $\mathcal{L}$

$$\mathcal{L}(\Omega, v, q, \mu) = \int_\Omega j(v)\,dx + \int_\Omega (\Delta v + f)q\,dx + \int_{\partial\Omega}(v - g)\mu\,ds.$$

2. To get the adjoint problem we differentiate the Lagrangian with respect to v and set this partial derivative equal to 0. Before that we perform two successive integration by parts

$$\mathcal{L}(\Omega, v, q, \mu) = \int_\Omega j(v)\,dx + \int_\Omega (fq - \nabla q\cdot\nabla v)\,dx + \int_{\partial\Omega}\left(\frac{\partial v}{\partial n}q + (v-g)\mu\right)\,ds$$

$$\mathcal{L}(\Omega, v, q, \mu) = \int_\Omega j(v)\,dx + \int_\Omega (fq + \Delta q v)\,dx + \int_{\partial\Omega} \left(\frac{\partial v}{\partial n} q - \frac{\partial q}{\partial n} v + (v-g)\mu \right) ds$$

For any $\phi \in H^1(\mathbb{R}^N)$ we have

$$\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q, \mu), \phi \rangle = \int_\Omega j'(v)\phi\,dx + \int_\Omega \Delta q \phi\,dx + \int_{\partial\Omega} \left(\frac{\partial \phi}{\partial n} q - \frac{\partial q}{\partial n}\phi + \phi\mu \right) ds \quad (1)$$

We first take a test function ϕ with compact support in Ω, so we deduce that the optimal value of q, the adjoint p, satisfies

$$-\Delta p = j'(u) \quad \text{in } \Omega.$$

Then we take $\phi = 0$ on $\partial\Omega$ but with no restriction on the value of $\frac{\partial \phi}{\partial n}$ on $\partial\Omega$, so that

$$p = 0 \quad \text{on } \partial\Omega.$$

This yields the adjoint problem

$$\begin{cases} -\Delta p = j'(u) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

Eventually, varying the trace of ϕ on $\partial\Omega$ gives the optimal value of the Lagrange multiplier

$$\lambda = \frac{\partial p}{\partial n} \text{ on } \partial\Omega.$$

3. Formally we know that the shape derivative is given by

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u, p)(\theta).$$

We compute the partial derivative

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, v, q, \mu)(\theta) = \int_{\partial\Omega} \Big(j(v) + fq - \nabla q \cdot \nabla v \Big) \theta \cdot n\,ds + \int_{\partial\Omega} \Big(\frac{\partial h}{\partial n} + Hh \Big) \theta \cdot n\,ds,$$

with $h = \frac{\partial v}{\partial n} q + (v-g)\mu$. Taking into account $p = 0$ and $u = g$ on $\partial\Omega$, we deduce

$$J'(\Omega)(\theta) = \int_{\partial\Omega} \Big(j(u) - \nabla p \cdot \nabla u \Big) \theta \cdot n\,ds + \int_{\partial\Omega} \Big(\frac{\partial u}{\partial n}\frac{\partial p}{\partial n} + \mu \frac{\partial (u-g)}{\partial n} \Big) \theta \cdot n\,ds.$$

Since $\lambda = \frac{\partial p}{\partial n}$ on $\partial\Omega$ and $\nabla_t p = 0$ on $\partial\Omega$, it leads to

$$J'(\Omega)(\theta) = \int_{\partial\Omega} \Big(j(u) + \frac{\partial p}{\partial n}\frac{\partial (u-g)}{\partial n} \Big) \theta \cdot n\,ds.$$