ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Written exam, March 14th, 2012 (2 hours)

1 Parametric optimization: 10 points

We consider the optimization of the thermal insulation of a body Ω subject to heat conduction. The domain Ω is a smooth bounded open set of $\mathbb{R}^N$. Its conductivity is constant, normalized to 1, and it is subject to a heat source $f(x) \in L^2(\Omega)$. The boundary $\partial\Omega$ is coated by a thin layer of a thermal isolant which is modeled by a positive coefficient $k(x) \in L^\infty(\partial\Omega)$ which is called thermal exchange coefficient. The boundary condition is of Fourier type, i.e., the outgoing heat flux is proportional to the temperature (the exterior temperature being assumed to be 0). The temperature is denoted by $u(x)$ which is the unique solution in $H^1(\Omega)$ of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + k\, u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The exchange coefficient belongs to the following space of admissible designs

$$\mathcal{U}_{ad} = \left\{ k \in L^\infty(\partial\Omega) \ , \quad k_{max} \geq k(x) \geq k_{min} > 0 \text{ on } \partial\Omega \right\}.$$

The goal is to minimize the objective function

$$\inf_{k \in \mathcal{U}_{ad}} \left\{ J(k) = \int_\Omega j(u(x))\, dx \right\}, \tag{2}$$

where j is a smooth functions satisfying

$$|j(v)| \leq C(|v|^2 + 1) \quad \text{ and } \quad |j'(v)| \leq C(|v| + 1).$$

1. Write the variational formulation for (1) and the Lagrangian associated to the objective function $J(k)$.

2. Define the adjoint problem, the solution of which shall be denoted by p.

3. Compute (formally) the derivative of $J(k)$.

4. We now consider the special case

$$J(k) = -\int_\Omega f(x)\, u(x)\, dx,$$

namely we want to find the best insulated design. According to the formula for $J'(k)$ what should be the optimal exchange coefficient ? (At this point, a formal answer is enough.)

Prove rigorously that the minimum of (2) is indeed attained by an optimal exchange coefficient which shall be given explicitly.

2 Geometric optimization: 7 points

We consider a problem motivated by image processing. An (infinite) black-and-white image is modeled by a function f from $\mathbb{R}^2$ into $\mathbb{R}$, the values of which are the grey intensity. We want to find the largest possible domain $\Omega \subset \mathbb{R}^2$ on which the function f is almost constant. For a given threshold $\epsilon > 0$, the goal is to find a maximizer of the constrained optimization problem

$$\max_{\Omega\subset\mathbb{R}^2,\ J(\Omega)\leq\epsilon} |\Omega|, \tag{3}$$

where $|\Omega|$ is the volume of Ω and

$$J(\Omega) = \frac{1}{|\Omega|}\int_\Omega |f(x) - \mathcal{M}_\Omega(f)|^2\, dx$$

with $\mathcal{M}_\Omega(f)$ the average of f on Ω, defined by

$$\mathcal{M}_\Omega(f) = \frac{1}{|\Omega|}\int_\Omega f(x)\, dx.$$

1. For a smooth function f, compute the shape derivative of the map $\Omega \to \mathcal{M}_\Omega(f)$. Deduce that this shape derivative is zero if and only if f is equal to its average $\mathcal{M}_\Omega(f)$ on the boundary $\partial\Omega$.

2. Compute the shape derivative of $J(\Omega)$. Prove that, if Ω is a domain such that $J(\Omega) \leq \epsilon$, f is not a constant function in Ω and f is equal to its average $\mathcal{M}_\Omega(f)$ on the boundary $\partial\Omega$, then one can enlarge Ω and find a better admissible domain in (3).

3. Show that the constraint in (3) is active at the maximum if it is attained and finite. Write the optimality condition for (3).

3 Homogenization: 3 points

Consider an isotropic composite material with homogenized tensor $A^* = a^*\mathrm{Id}$ obtained by mixing two isotropic conductors $0 < \alpha < \beta$ in respective proportions $\theta > 0$ and $(1-\theta) > 0$. Restrict the space dimension to be $N = 2$ and take the limit $\alpha \to 0$. Based on the knowledge of the Hashin-Shtrikman bounds, prove that

$$a^* \leq \frac{1-\theta}{1+\theta}\beta.$$