OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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CHAPTER I

AN INTRODUCTION TO OPTIMAL DESIGN

A FEW DEFINITIONS

A problem of optimal design (or shape optimization) for structures is defined by three ingredients:

☞ a **model** (typically a partial differential equation) to evaluate (or analyse) the mechanical behavior of a structure,

☞ an **objective function** which has to be minimized or maximized, or sometimes several objectives (also called cost functions or criteria),

☞ a **set of admissible designs** which precisely defines the optimization variables, including possible constraints.

Optimal design problems can roughly be classified in three categories from the "easiest" ones to the "most difficult" ones:

- ☞ **parametric or sizing** optimization for which designs are parametrized by a few variables (for example, thickness or member sizes), implying that the set of admissible designs is considerably simplified,
- ☞ **geometric or shape** optimization for which all designs are obtained from an initial guess by moving its boundary (without changing its topology, i.e., its number of holes in 2-d),
- ☞ **topology** optimization where both the shape and the topology of the admissible designs can vary without any explicit or implicit restrictions.

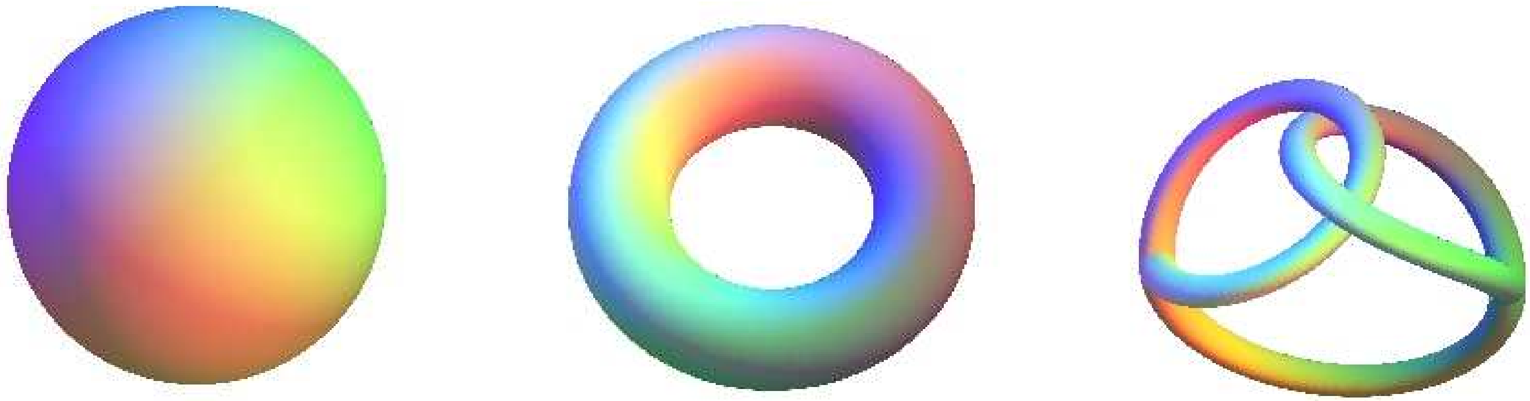
Definition of topology

Two shapes share the same topology if there exists a continuous deformation from one to the other.

In dimension 2 topology is characterized by the number of holes or of connected components of the boundary.

In dimension 3 it is quite more complicated ! Not only the hole's number matters but also the number and intricacy of "handles" or "loops".

(a ball $\neq$ a ball with a hole inside $\neq$ a torus $\neq$ a bretzel)



GOALS OF THE COURSE

1. To introduce numerical algorithms for computing optimal designs in a **"systematic" way** and not **by "trials and errors"**.

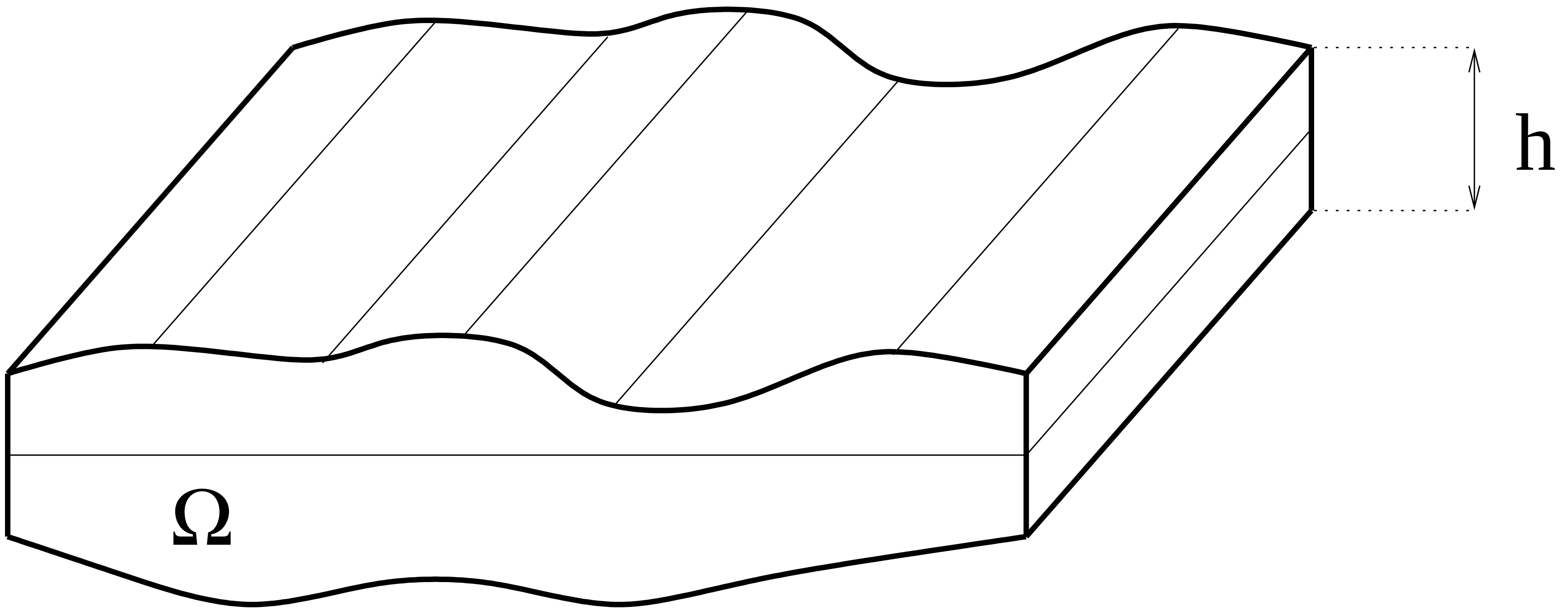
2. To obtain optimality conditions (necessary and/or sufficient) which are crucial both for the theory (characterization of optimal shapes) and for the numerics (they are the basis for gradient-type **algorithms**).

3. A (very) brief survey of theoretical results on existence, uniqueness, and qualitative properties of optimal solutions ; such issues will be discussed only when they matter for numerical purposes.

A **continuous** approach of shape optimization is prefered to a **discrete** one.

Example of sizing or parametric optimization

Thickness optimization of a membrane



- $\Omega =$ mean surface of a (plane) membrane
- $h =$ thickness in the normal direction to the mean surface

The membrane deformation is modeled by its vertical displacement $u(x) : \Omega \to \mathbb{R}$, solution of the following partial differential equation (p.d.e.), the so-called **membrane model**,

$$\begin{cases} -\operatorname{div}(h\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the thickness h, bounded by minimum and maximum values

$$0 < h_{min} \leq h(x) \leq h_{max} < +\infty.$$

The thickness h is the optimization variable.

It is a **sizing or parametric** optimal design problem because the computational domain Ω does not change.

The set of admissible thicknesses is

$$\mathcal{U}_{ad} = \left\{ h(x) : \Omega \to \mathbb{R} \text{ s. t. } 0 < h_{min} \leq h(x) \leq h_{max} \text{ and } \int_\Omega h(x)\,dx = h_0|\Omega| \right\},$$

where h_0 is an imposed average thickness.

Possible additional "feasibility" constraints: according to the production process of membranes, the thickness $h(x)$ can be discontinuous, or on the contrary continuous ; its derivative $h'(x)$ can be uniformly bounded (molding-type constraint) or even its second-order derivative $h''(x)$, linked to the curvature radius (milling-type constraint).

The **optimization criterion** is linked to some mechanical property of the membrane, evaluated through its displacement u, solution of the p.d.e.,

$$J(h) = \int_\Omega j(u)\, dx,$$

where, of course, u depends on h. For example, the global rigidity of a structure is often measured by its **compliance**, or work done by the load: the smaller the work, the larger the rigidity (be careful ! compliance = - rigidity). In such a case,

$$j(u) = fu.$$

Another example amounts to achieve (at least approximately) a **target displacement** $u_0(x)$, which means

$$j(u) = |u - u_0|^2.$$

Those two criteria are the typical examples studied in this course.

Other examples of objective functions

☞ Introducing the stress vector $\sigma(x) = h(x)\nabla u(x)$, we can minimize the maximum stress norm

$$J(h) = \sup_{x\in\Omega} |\sigma(x)|$$

or more generally, for any $p \geq 1$,

$$J(h) = \left(\int_\Omega |\sigma|^p dx \right)^{1/p}.$$

☞ For a vibrating structure, introducing the first eigenfrequency ω, defined by

$$\begin{cases} -\operatorname{div}(h\nabla u) = \omega^2 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we consider $J(h) = -\omega$ to maximize it.

Other examples of criteria (ctd.)

☞ Multiple loads optimization: for n given loads $(f_i)_{1\le i\le n}$ the independent displacements u_i are solutions of

$$\begin{cases} -\operatorname{div}(h\nabla u_i) = f_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and we introduce an aggregated criteria

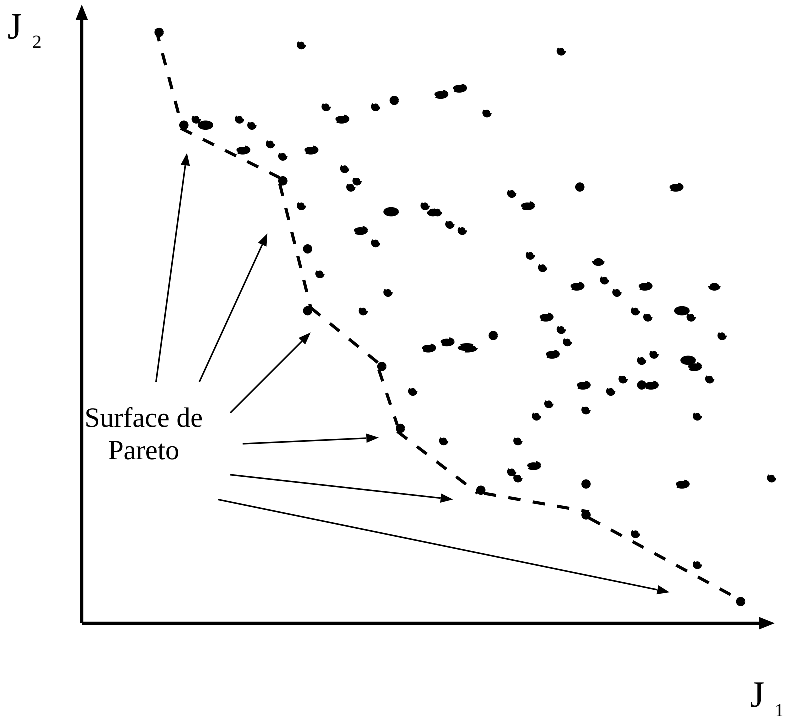
$$J(h) = \sum_{i=1}^{n} c_i \int_\Omega j(u_i)\,dx,$$

with given coefficients c_i, or

$$J(h) = \max_{1\le i\le n} \left(\int_\Omega j(u_i)\,dx \right).$$

☞ Multi-criteria optimization: notion of Pareto front (see next slide).

Multi-criteria optimization: Pareto front



Assume we have n objective functions $J_i(h)$.

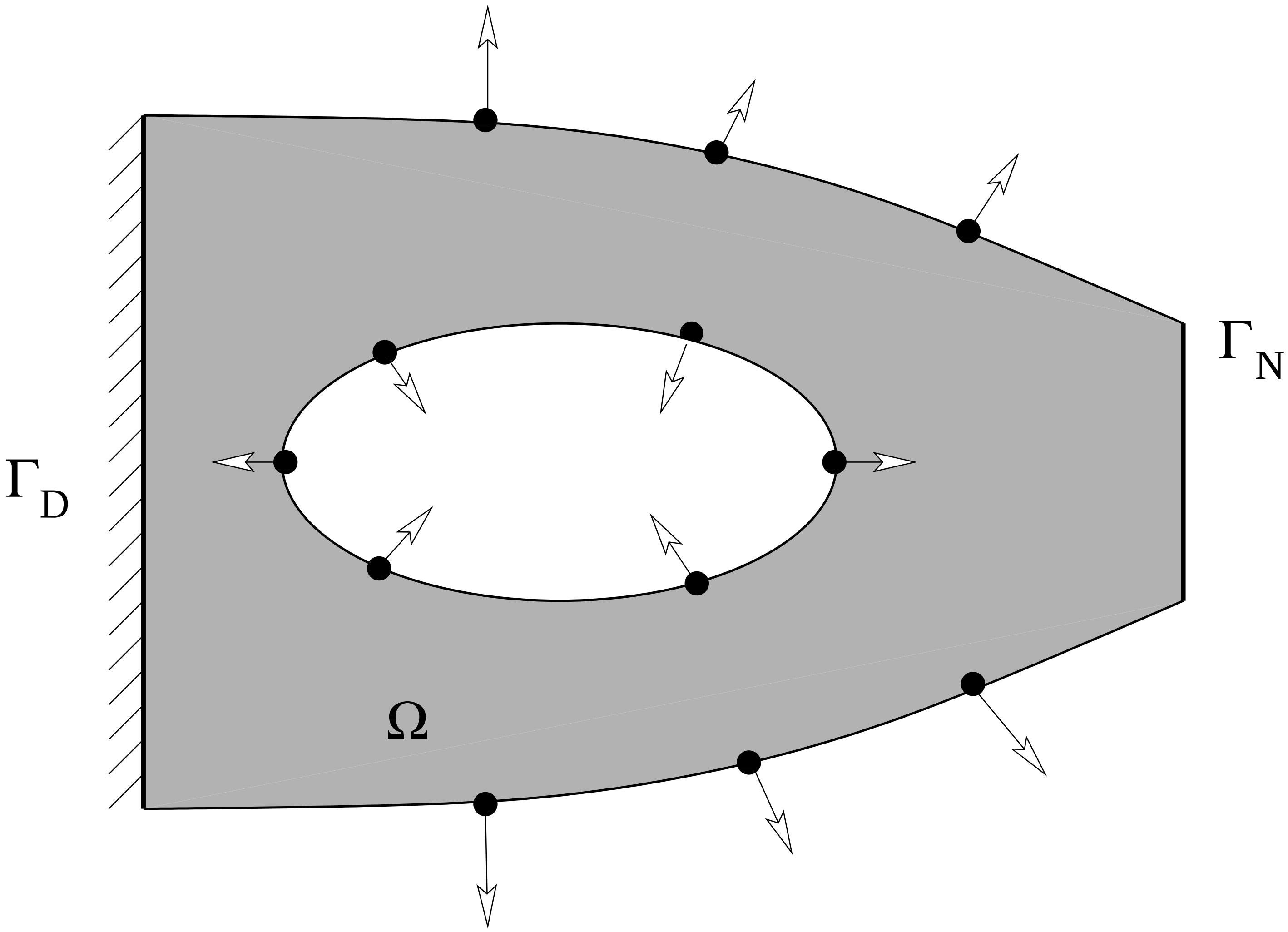
A design h is said to dominate another design $\tilde{h}$ if

$$J_i(h) \leq J_i(\tilde{h}) \quad \forall\, i \in \{1, ..., n\}$$

The Pareto front is the set of designs which are not dominated by any other.

Example of geometric optimization

Optimization of a membrane's shape



A reference domain for the membrane is denoted by Ω, with a boundary made of three disjoint parts

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

where Γ is the variable part, Γ_D is the Dirichlet (clamped) part and Γ_N is the Neumann part (loaded by g).

The vertical displacement u is the solution of the **membrane model**

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

From now on the membrane thickness is fixed, equal to 1.

The set of admissible shapes is thus

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^N \text{ such that } \Gamma_D \bigcup \Gamma_N \subset \partial\Omega \text{ and } \int_\Omega dx = V_0 \right\},$$

where V_0 is a given volume. The **geometric** shape optimization problem reads

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega),$$

with, as a **criteria**, the compliance

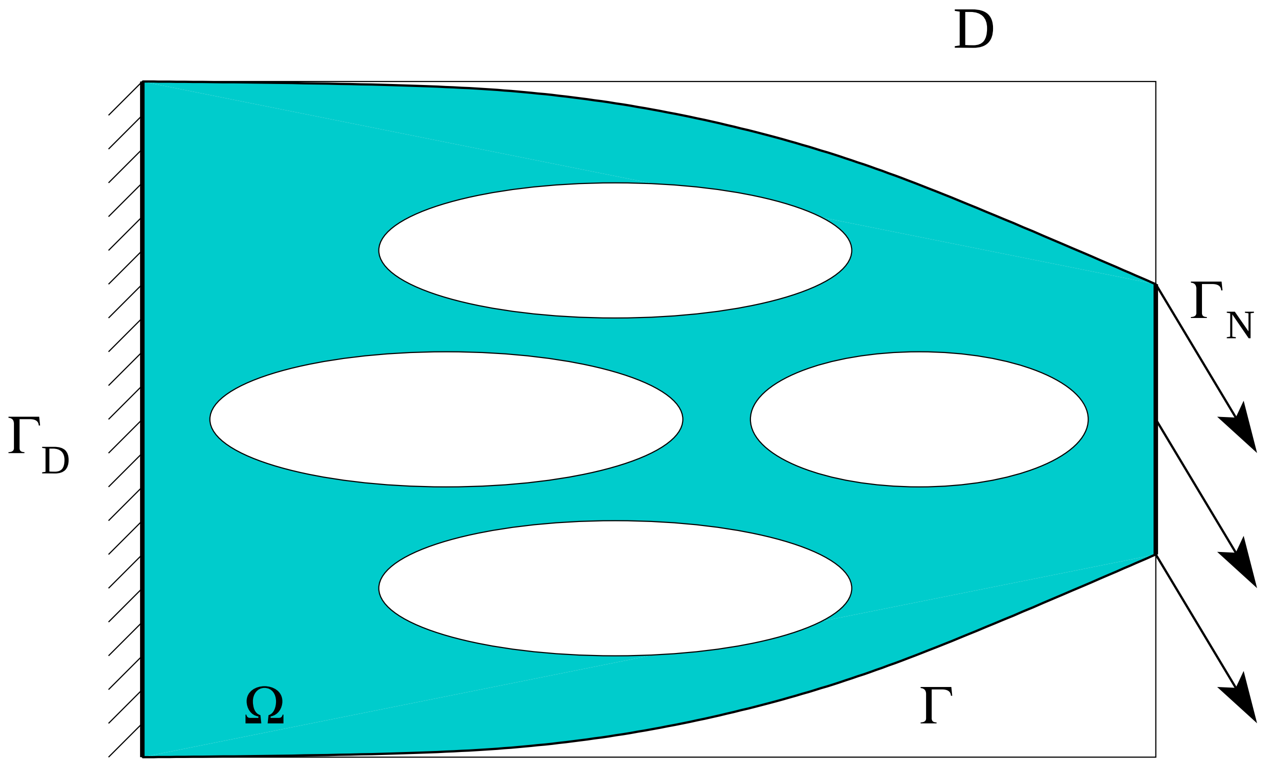
$$J(\Omega) = \int_{\Gamma_N} g u \, dx,$$

or a least square functional to achieve a target displacement $u_0(x)$

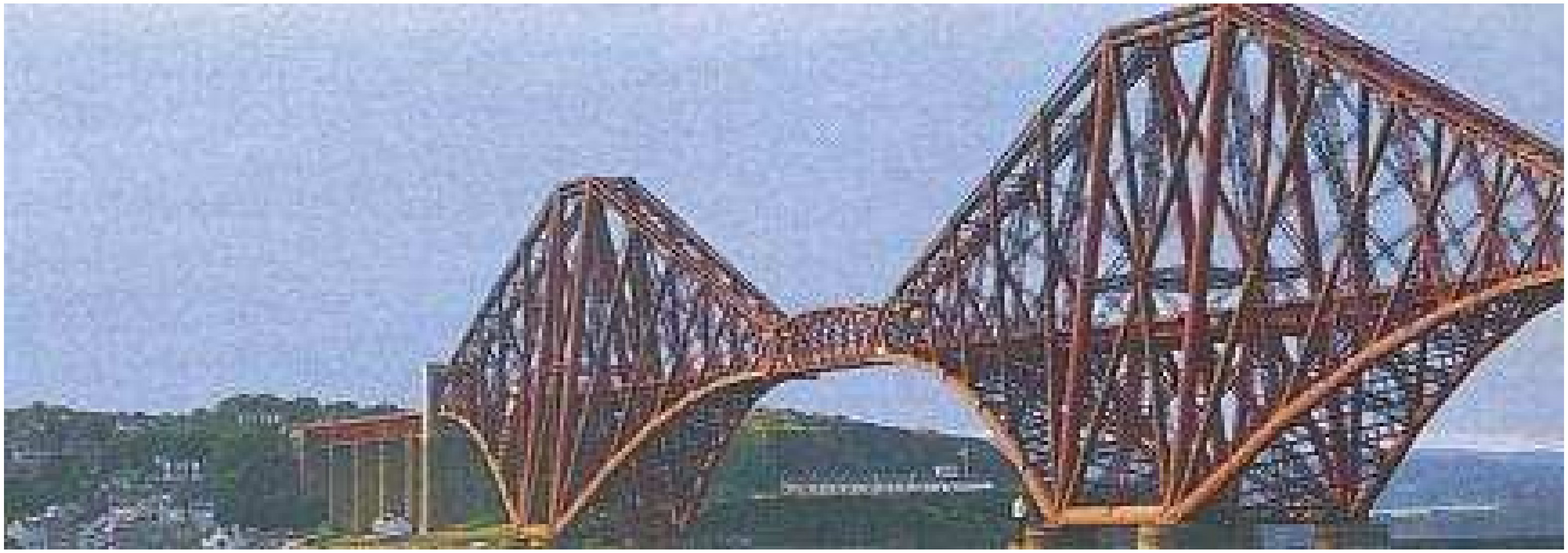
$$J(\Omega) = \int_\Omega |u - u_0|^2 dx.$$

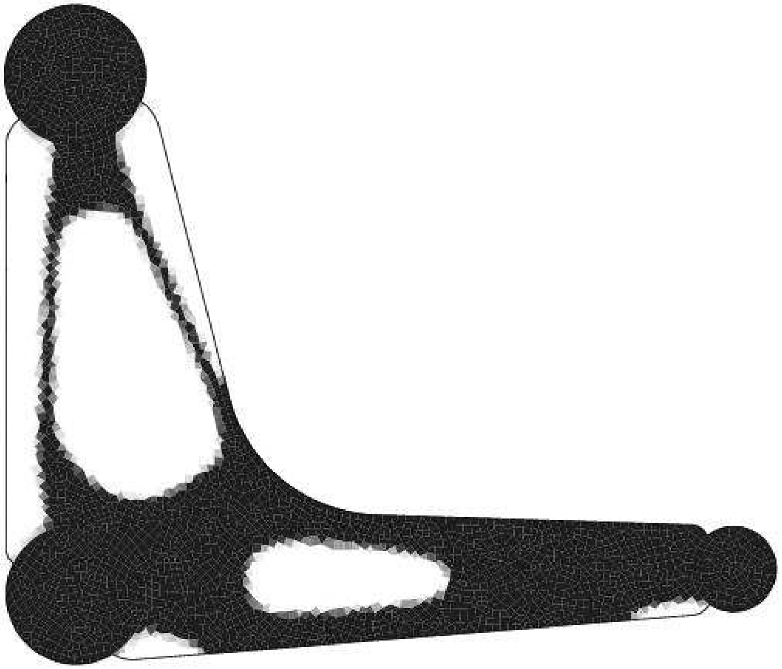
The true optimization variable is the free boundary Γ.

Example of topology optimization



Not only the shape boundaries Γ are allowed to move but new connected components (holes in 2-d) of Γ can appear or disappear. **Topology is now optimized too.**





Shape optimization in the elasticity setting

The **model of linearized elasticity** gives the displacement vector field $u(x) : \Omega \to \mathbb{R}^N$ as the solution of the system of equations

$$\begin{cases} -\operatorname{div}\left(A\, e(u)\right) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\, e(u)\big)n = g & \text{on } \Gamma_N \\ \big(A\, e(u)\big)n = 0 & \text{on } \Gamma \end{cases}$$

with $e(u) = \big(\nabla u + (\nabla u)^t\big)/2$, and $A\xi = 2\mu\xi + \lambda(tr\xi)\,\mathrm{Id}$, where μ and λ are the Lamé coefficients.

The domain boundary is again divided in three disjoint parts

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

where Γ is the free boundary, the true optimization variable.

The set of admissible shapes is again

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^N \text{ such that } \Gamma_D \bigcup \Gamma_N \subset \partial\Omega \text{ and } \int_\Omega dx = V_0 \right\},$$

where V_0 is a given imposed volume. The **criteria** is either the compliance

$$J(\Omega) = \int_{\Gamma_N} g \cdot u \, dx,$$

or a least-square criteria for the target displacement $u_0(x)$

$$J(\Omega) = \int_\Omega |u - u_0|^2 dx.$$

As before, the shape optimization problem reads

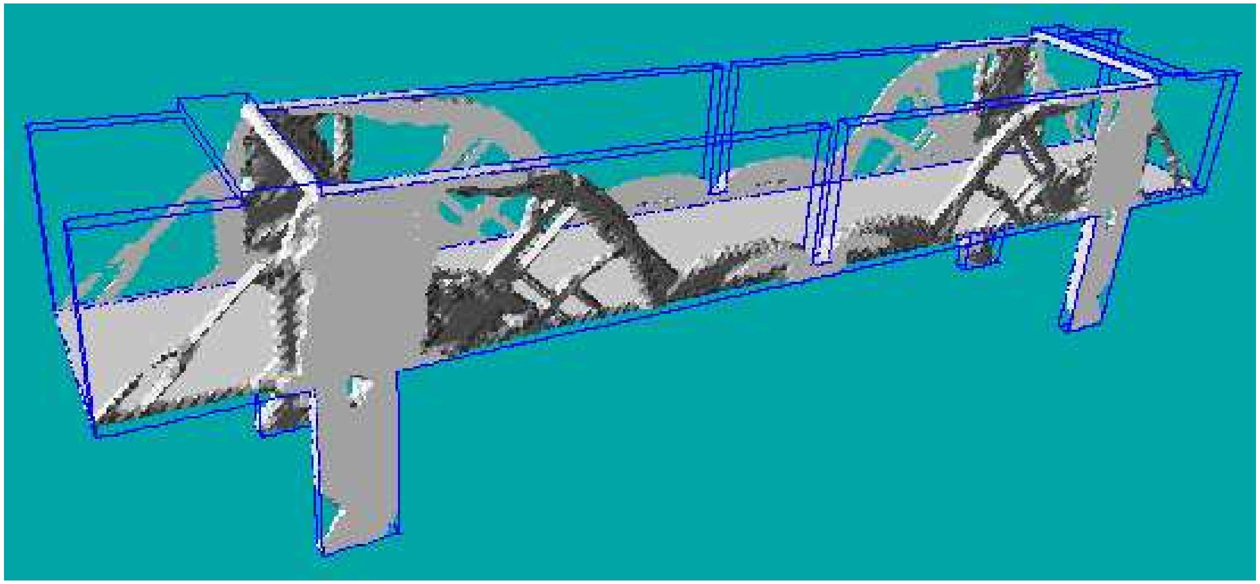
$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega).$$

Three possible approaches: **parametric**, **geometric**, **topology**.

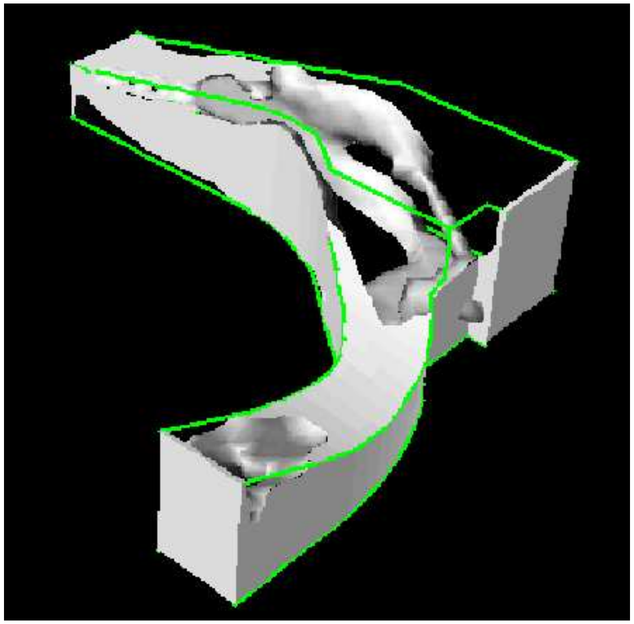
Applications

See the web site `http://www.cmap.polytechnique.fr/~optopo` (and links therein).

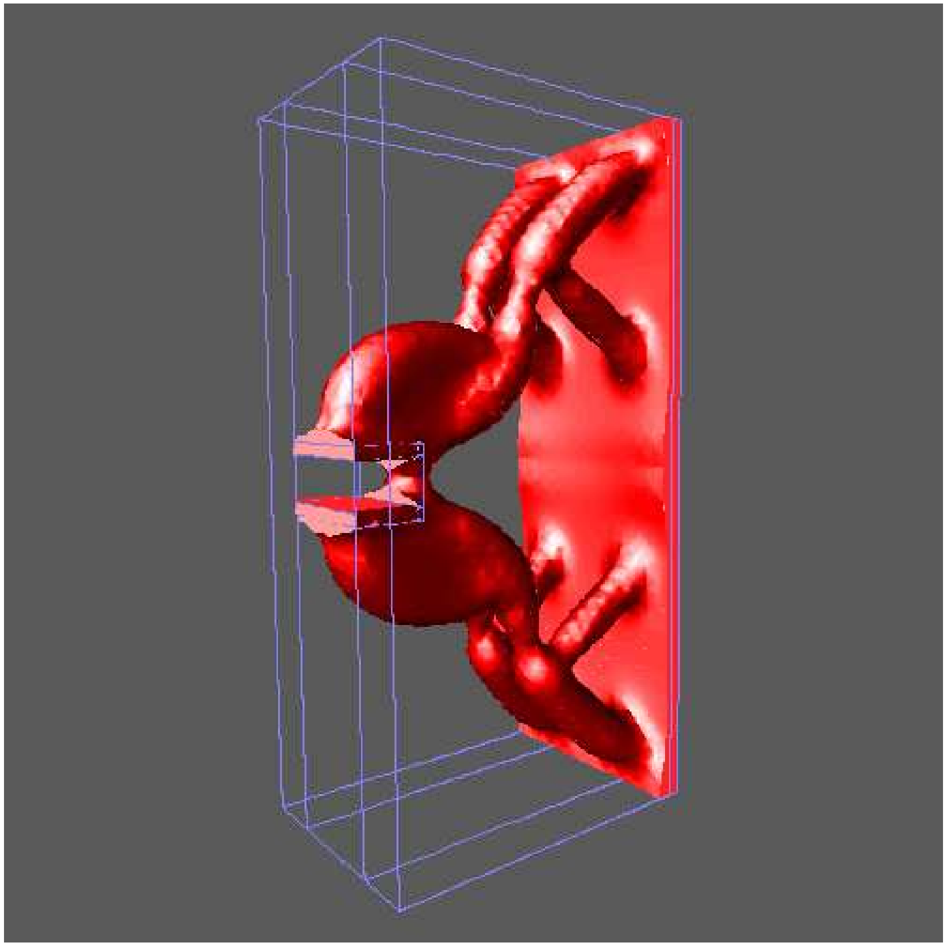
Civil engineering



Mechanical engineering



Micromechanics (MEMS)



Aeronautics

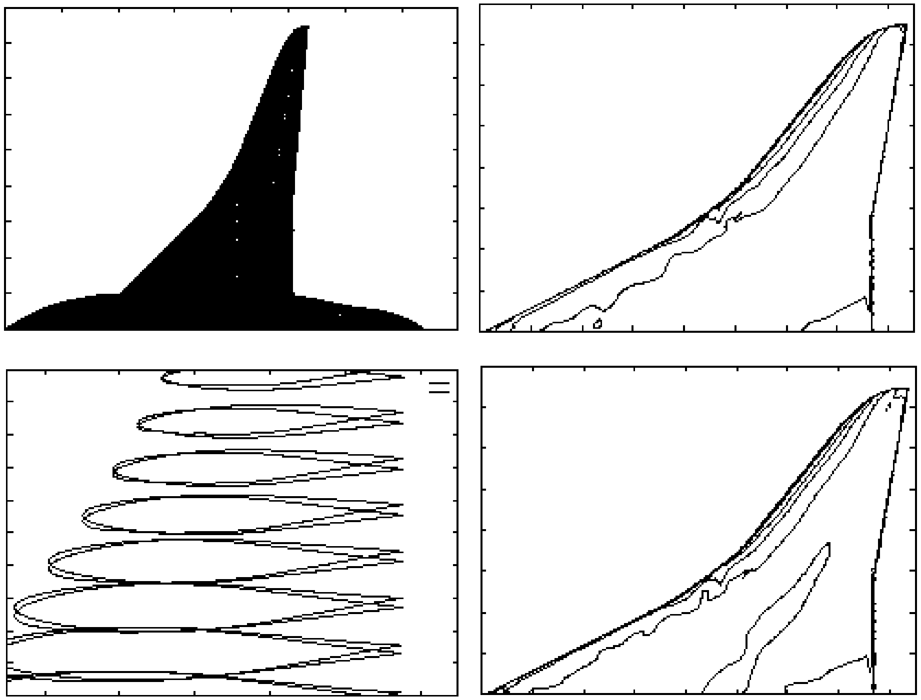
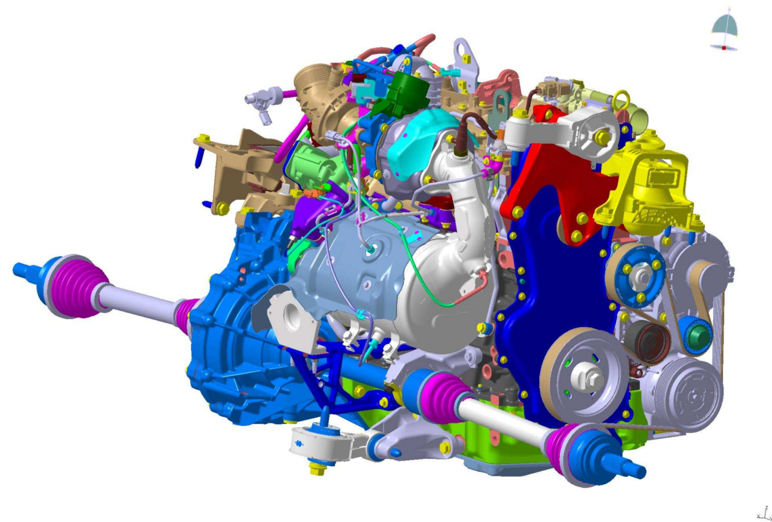


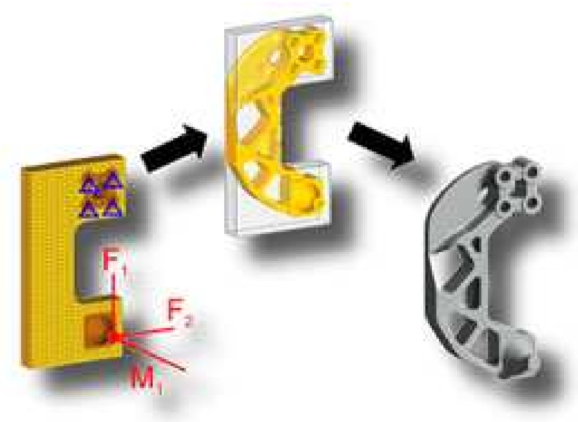
Figure 1: 3D optimization for a supersonic civil transport: top left: a view from above of the airplane with the trace of the mesh. Top right: Mach lines after optimization (minimal drag) and bottom right shows the same before optimization. Finally the bottom left figure shows cross sections of the initial (dotted lines) and optimized wings.

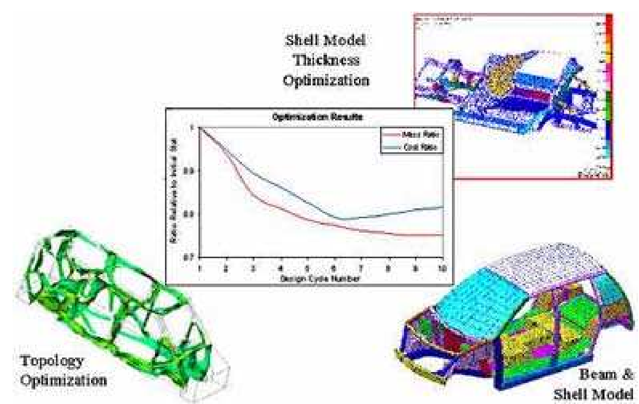
Industrial examples at EADS, Airbus, Renault...



Commercial softwares

Optistruct, Ansys DesignSpace, Genesis, MSC-Nastran, Tosca, devDept...

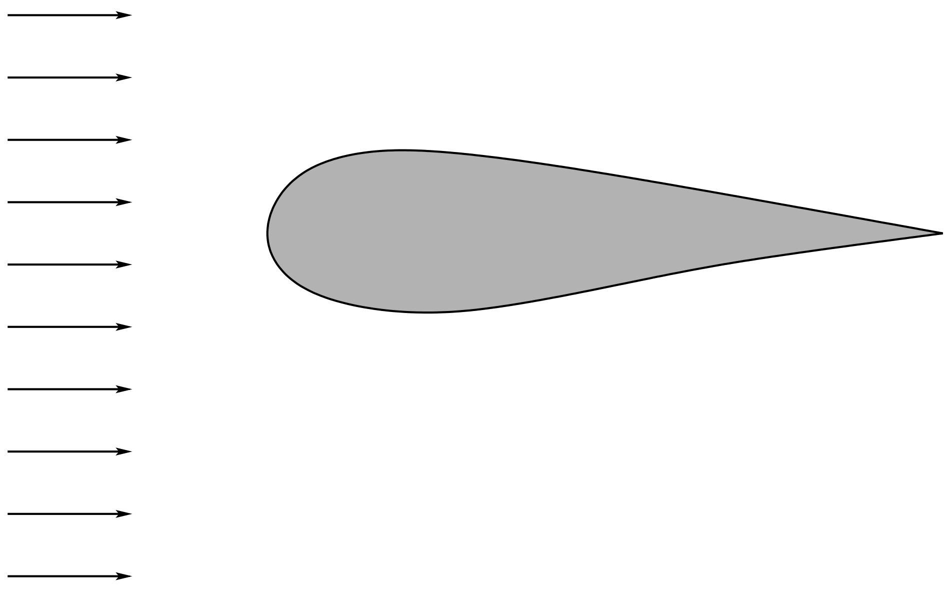




Example in fluid mechanics

Optimization of a wing profile

Drag minimization and lift maximization.



Constant velocity at infinity U_0.

Potential flow: simplification of Navier-Stokes equations for a perfect incompressible and irrotational fluid in a steady state regime. The velocity U derives from a scalar potential ϕ

$$U = \nabla\phi.$$

Bernoulli's law for the pressure

$$p = p_0 - \frac{1}{2}|\nabla\phi|^2.$$

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \\ \lim\limits_{|x|\to+\infty} (\phi(x) - U_0 \cdot x) = 0 & \text{at infinity} \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial P, \end{cases}$$

D'Alembert paradox: zero drag, zero lift !

We choose a criteria on the pressure

$$J(P) = \int_{\partial P} j(p)\, ds\,,$$

where the function j is typically a least square criteria for a target pressure

$$j(p) = |p - p_{target}|^2.$$

The **geometric shape optimization** problem reads

$$\inf_{P \in \mathcal{U}_{ad}} J(P).$$

A priori, there is no need of topology optimization for a wing profile...

Parametric optimization of a thin profile (in 2-d)

Example on how to reduce a geometric optimization problem into a parametric one.

Thin profile P with upper and lower boundaries (extrados and intrados) defined by

$$y = f^+(x) \quad \text{for} \quad 0 \leq x \leq L, \quad y = f^-(x) \quad \text{for} \quad 0 \leq x \leq L,$$

where L is the length of the profile's chord. We assume that the velocity at infinity U_0 is aligned with the x-axis. The Neumann boundary condition for the potential is

$$\frac{\partial \phi}{\partial y} - \frac{df^\pm}{dx}\frac{\partial \phi}{\partial x} = 0 \text{ on } \partial P,$$

which, at first order, becomes

$$\frac{\partial \phi}{\partial y} = U_0 \frac{df^\pm}{dx} \text{ on the chord } [0, L].$$

Parametric optimization problem with $\Sigma = [0, L]$

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \setminus \Sigma \\ \lim_{|x|\to+\infty} (\phi(x) - U_0 \cdot x) = 0 & \text{at infinity} \\ \frac{\partial \phi}{\partial y} = U_0 \frac{df^+}{dx} & \text{on } \Sigma^+ \\ \frac{\partial \phi}{\partial y} = U_0 \frac{df^-}{dx} & \text{on } \Sigma^-. \end{cases}$$

$$\inf_{f^\pm \in \mathcal{U}_{ad}} J(f^\pm),$$

with

$$\mathcal{U}_{ad} = \left\{ \begin{array}{l} f^+(x) : [0, L] \to \mathbb{R}^+ \\ f^-(x) : [0, L] \to \mathbb{R}^- \end{array} \text{ s. t. } f^+(0) = f^-(0) = f^+(L) = f^-(L) = 0 \right\}.$$

The main advantage is that the domain Ω is now **fixed**.

Modeling choices

Modeling is typically an engineering issue.

- ☞ Choice of the model: a **compromise** between accuracy and the CPU cost (optimization requires many successive analyses of the model).
- ☞ Choice of the criterion: difficulty of **measuring** a qualitative property, of **combining** several criteria.
- ☞ Choice of the admissible set: **selecting** the most appropriate constraints from the point of view of the applications but also of the numerical algorithms.

We shall not discuss this issue during the course. It is however an important aspect of the personal projects (EA).

Other fields related to shape optimization

The technical tools in this course are also useful for the following areas:

☞ Optimal control.

☞ Inverse problems.

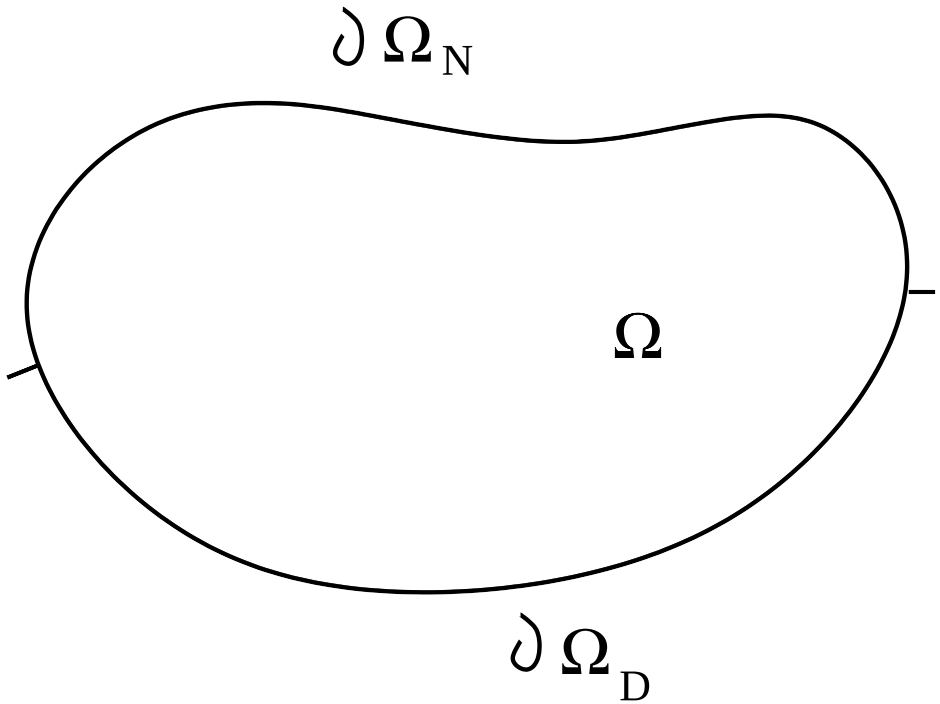
☞ Sensitivity analysis of parameters.

CHAPTER II

A BRIEF REVIEW

OF NUMERICAL ANALYSIS

Boundary value problems



Membrane model. $f =$ bulk force, $g =$ surface load.

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega_N \end{cases}$$

$n =$ unit normal vector,

notation: $\frac{\partial u}{\partial n} = \nabla u \cdot n$.

Key idea which **must** be mastered:

The variational approach

- Boundary value problem = p.d.e. + boundary condition
- It is proved that a boundary value problem is equivalent to its variational formulation.
- From a mechanical point of view, the variational formulation is just the principle of virtual work.
- Any variational formulation can be written as

$$\text{find } u \in V \text{ such that } a(u,v) = L(v) \quad \forall\, v \in V.$$

- This approach gives an existence theory for solutions and yields numerical methods such as finite elements for computing them.
- It is also a key tool for shape optimization.

Technical ingredients

Green's formula:

$$\int_\Omega \Delta u(x) v(x)\, dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(x) v(x)\, ds$$

Sobolev spaces (functions with finite energy):

$$u \in H^1(\Omega) \Leftrightarrow \int_\Omega \left(|\nabla u(x)|^2 + |u(x)|^2 \right) dx < +\infty$$

$$u \in H^1_0(\Omega) \Leftrightarrow u \in H^1(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega$$

- ☞ The Hilbert space V is usually a Sobolev space.
- ☞ To find a and L, the p.d.e. is multiplied by a test function.
- ☞ Integrate by parts using Green's formula.
- ☞ Use the boundary conditions for simplifying the boundary integrals.

Recipe

How to remember Green's formula ? It is enough to know the simple formula

$$\int_\Omega \frac{\partial w}{\partial x_i}(x)\,dx = \int_{\partial\Omega} w(x) n_i\,ds$$

with $n_i(x)$, the i-th component of the exterior unit normal vector to $\partial\Omega$ (to remember that it is the **exterior** normal, think about the 1-d formula !). All type of Green's formulas are deduced from this one.

As an example, take $w = v\frac{\partial u}{\partial x_i}$ and sum w.r.t. i to get

$$\int_\Omega \Delta u(x) v(x)\,dx = -\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial n}(x) v(x)\,ds$$

Variational formulation

Integration by parts yields

$$\int_\Omega f\,v\,dx = -\int_\Omega \Delta u\,v\,dx = \int_\Omega \nabla u\cdot\nabla v\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds$$

☞ The Dirichlet B.C. is imposed to the test functions.

☞ The Neumann B.C. is just put into the variational formulation.

Adequate choice of the Sobolev space:

$$V = \left\{ v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \partial\Omega_D \right\}$$

After simplification we get: Find $u \in V$ such that

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega f\,v\,dx + \int_{\partial\Omega_N} g\,v\,ds \quad \forall\, v\in V.$$

variational formulation (V.F.) $\Leftrightarrow$ boundary value problem (B.V.P.)

Lax-Milgram Theorem $\Rightarrow$ existence and uniqueness of $u \in V$

Checking the equivalence V.F $\Leftrightarrow$ B.V.P.

We already saw that u *solution of B.V.P.* $\Rightarrow$ u *solution of V.F.*

Let us check that u solution of V.F. $\Rightarrow$ u solution of B.V.P.

Let $u \in V = \left\{ v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \partial\Omega_D \right\}$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx + \int_{\partial\Omega_N} g \, v \, ds \quad \forall \, v \in V.$$

Integrating by parts (backwards) yields

$$-\int_\Omega \Delta u \, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds = \int_\Omega f \, v \, dx + \int_{\partial\Omega_N} g \, v \, ds \quad \forall \, v \in V.$$

Taking first v with compact support in Ω leads to

$$-\Delta u = f \quad \text{in} \quad \Omega.$$

Taking into account this first equality, the V.F. becomes

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds = \int_{\partial\Omega_N} g \, v \, ds \quad \forall \, v \in V.$$

In a second step, v is any function with a trace on $\partial\Omega_N$. Thus

$$\frac{\partial u}{\partial n} = g \ \text{ on } \ \partial\Omega_N.$$

The Dirichlet B.C. $u = 0$ on $\partial\Omega_D$ is recovered because $u \in V$.

Eventually, u is a (weak) solution of the B.V.P.

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega_N. \end{cases}$$

Remark: if $\partial\Omega_D = \emptyset$ (no clamping), then a necessary and sufficient condition of existence is the force equilibrium:

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0$$

Furthermore, uniqueness is obtained up to an additive constant, i.e., up to a rigid displacement.

Linearized elasticity system

$$\begin{cases} -\,\mathrm{div}\sigma = f & \text{in } \Omega \\ \text{with } \ \sigma = 2\mu e(u) + \lambda\,\mathrm{tr}(e(u))\,\mathrm{Id} & \\ u = 0 & \text{on } \partial\Omega_D \\ \sigma n = g & \text{on } \partial\Omega_N, \end{cases}$$

$$e(u) = \frac{1}{2}\Big(\nabla u + (\nabla u)^t\Big) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)_{1\le i,j\le N}$$

$$V = \left\{v \in H^1(\Omega)^N \text{ such that } v = 0 \text{ on } \partial\Omega_D\right\}$$

Variational formulation: find $u \in V$ such that

$$\int_\Omega 2\mu e(u)\cdot e(v)\,dx + \int_\Omega \lambda\,\mathrm{div}u\,\mathrm{div}v\,dx = \int_\Omega f\cdot v\,dx + \int_{\partial\Omega_N} g\cdot v\,ds\ \forall\, v \in V.$$

FINITE ELEMENT METHOD (F.E.M.)

Variational approximation

Exact variational formulation:

$$\text{Find } u \in V \text{ such that } a(u,v) = L(v) \quad \forall\, v \in V.$$

Approximate variational formulation:

$$\text{Find } u_h \in V_h \text{ such that } a(u_h,v_h) = L(v_h) \quad \forall\, v_h \in V_h$$

where $V_h \subset V$ is a finite-dimensional subspace.

The finite element method amounts to properly define simple subspaces V_h, linked to the notion of mesh of the domain Ω.

Introducing a basis $(\phi_j)_{1\leq j\leq N_h}$ of V_h, we define

$$u_h = \sum_{j=1}^{N_h} u_j\phi_j \quad \text{with} \quad U_h = (u_1, ..., u_{N_h}) \in \mathbb{R}^{N_h}$$

The approximate V.F. is equivalent to

$$\text{Find } U_h \in \mathbb{R}^{N_h} \text{ such that } a\left(\sum_{j=1}^{N_h} u_j\phi_j, \phi_i\right) = L(\phi_i) \quad \forall\, 1 \leq i \leq N_h,$$

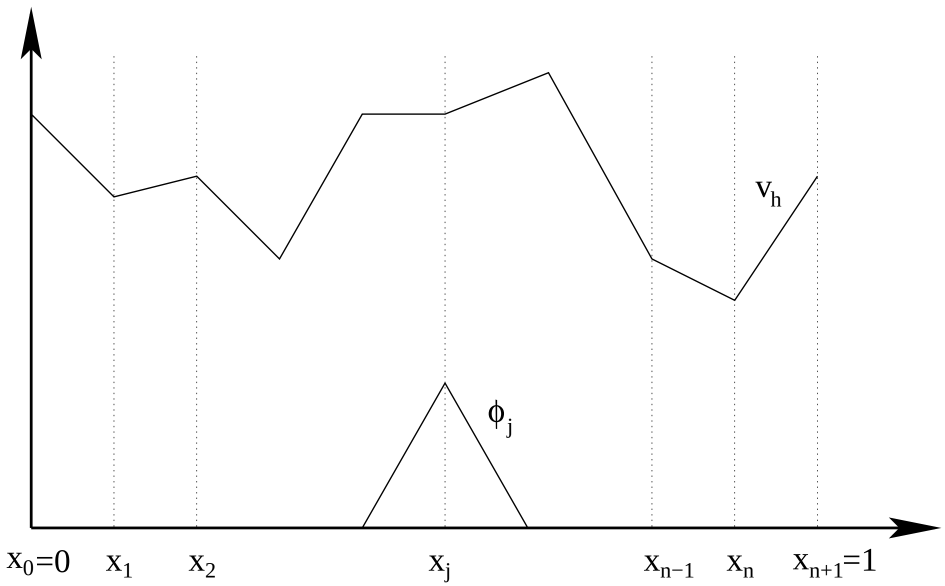
which is nothing but a linear system

$$\mathcal{K}_h U_h = b_h \quad \text{with} \quad (\mathcal{K}_h)_{ij} = a(\phi_j, \phi_i), \quad (b_h)_i = L(\phi_i).$$

Remark: the coerciveness of $a(u,v)$ implies that the rigidity matrix $\mathcal{K}_h$ is positive definite. On the same token, the symmetry of $a(u,v)$ implies that of $\mathcal{K}_h$.

Lagrange $\mathbb{P}_1$ finite elements in $N = 1$ dimension

Uniform mesh with nodes (or vertices) $(x_j = jh)_{0 \le j \le n+1}$ where $h = \frac{1}{n+1}$.



V_h = space of piecewise affine and globally continuous functions

$$\phi_j(x) = \phi\left(\frac{x - x_j}{h}\right) \quad \text{with} \quad \phi(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Resulting linear system

We have to solve the linear system $\mathcal{K}_h U_h = b_h$ where $\mathcal{K}_h$ is the rigidity matrix

$$\mathcal{K}_h = \left(\int_0^1 \phi_j'(x)\phi_i'(x)\,dx \right)_{1\le i,j\le n}, b_h = \left(\int_0^1 f(x)\phi_i(x)\,dx \right)_{1\le i\le n},$$

$$u_h(x) = \sum_{j=1}^{N_h} u_j\phi_j(x) \quad \text{with} \quad U_h = (u_1, ..., u_{N_h}) \in \mathbb{R}^{N_h}$$

A straightforward calculation shows that $\mathcal{K}_h$ is tridiagonal

$$\mathcal{K}_h = h^{-1} \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}.$$

Resulting linear system (ctd.)

To make explicit the right hand side b_h we have to compute integrals

$$(b_h)_i = \int_{x_{i-1}}^{x_{i+1}} f(x)\phi_i(x)\,dx \quad \text{for} \quad 1 \leq i \leq n.$$

For that purpose one uses quadrature formulas (or numerical integration). For example, the "trapezoidal rule"

$$\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \psi(x)\,dx \approx \frac{1}{2}\left(\psi(x_{i+1}) + \psi(x_i)\right),$$

Convergence of the F.E.M.

Theorem. Let $u \in H_0^1(0,1)$ and $u_h \in V_{0h}$ be the exact and approximate solutions, respectively. The $\mathbb{P}_1$ finite element method converges in the sense that

$$\lim_{h \to 0} \|u - u_h\|_{H^1(0,1)} = 0.$$

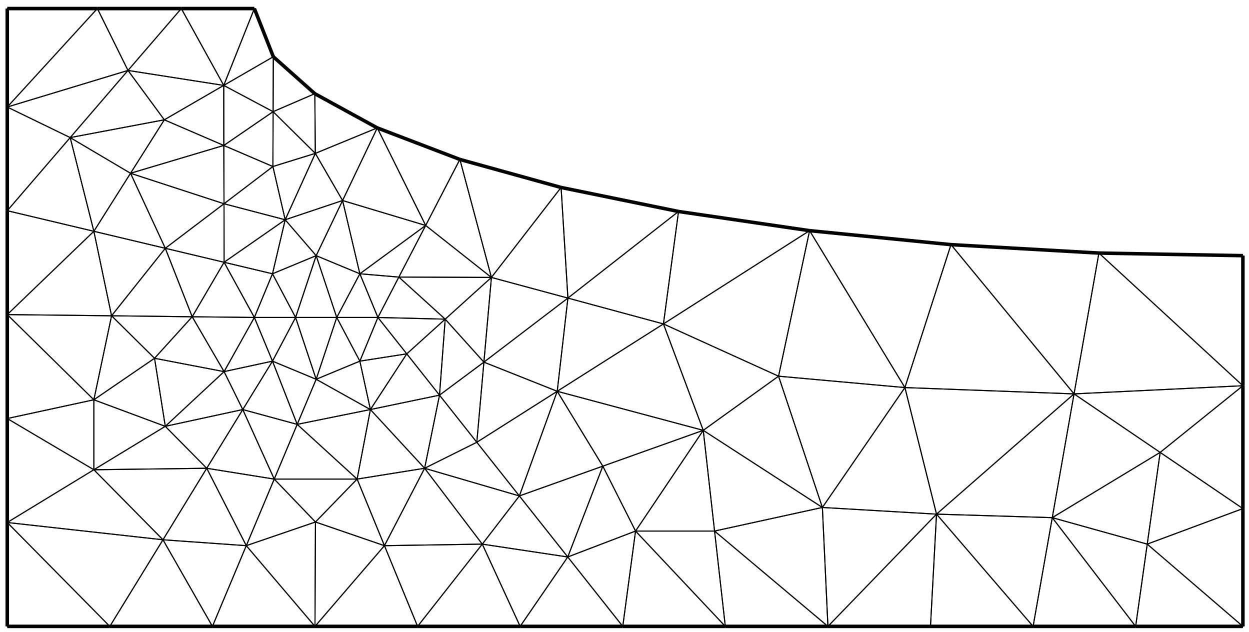
Furthermore, if $u \in H^2(0,1)$ (which is true as soon as $f \in L^2(0,1)$), then there exists a constant C, which does not depend on h, such that

$$\|u - u_h\|_{H^1(0,1)} \leq Ch\|u''\|_{L^2(0,1)} = Ch\|f\|_{L^2(0,1)}.$$

Remark. One advantage of the V.F. is that the F.E. basis functions need not to be twice differentiable but merely once.

F.E.M. IN HIGHER DIMENSIONS $N \geq 2$

Lagrange $\mathbb{P}_1$ finite elements



The domain is meshed by triangles in dimension $N = 2$ or tetrahedra in dimension $N = 3$ with vertices denoted by $(a_j)_{1 \leq j \leq N+1}$ in $\mathbb{R}^N$.

We shall use FreeFem++ `http://www.freefem.org`

Lemma Let K be a triangle or a tetrahedron with vertices $(a_j)_{1\leq j\leq N+1}$. Any affine function or polynomial $p \in \mathbb{P}_1$ can be written as

$$p(x) = \sum_{j=1}^{N+1} p(a_j)\lambda_j(x),$$

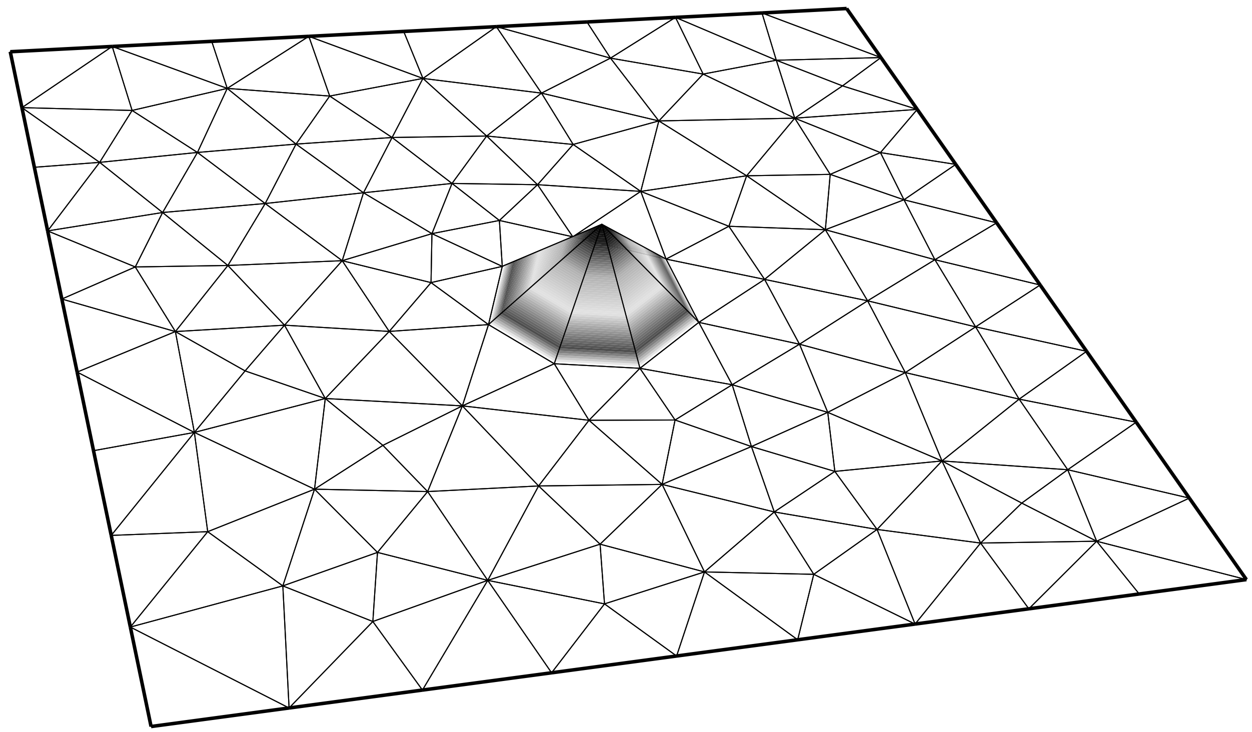
where $(\lambda_j(x))_{1\leq j\leq N+1}$ are the barycentric coordinates of $x \in \mathbb{R}^N$ defined by

$$\begin{cases} \sum_{j=1}^{N+1} a_{i,j}\lambda_j = x_i & \text{for } 1 \leq i \leq N \\ \sum_{j=1}^{N+1} \lambda_j = 1 & \end{cases}$$

In other words, any $\mathbb{P}_1$ function is uniquely characterized by its (nodal) values at the vertices or nodes of the mesh.

The Lagrange $\mathbb{P}_1$ finite element method (**triangular F.E. of order** 1) associated to a mesh $\mathcal{T}_h$ is defined by

$$V_h = \left\{ v \in \mathcal{C}(\overline{\Omega}) \text{ such that } v|_{K_i} \in \mathbb{P}_1 \text{ for any } K_i \in \mathcal{T}_h \right\}.$$



Basis function of V_h associated to one node or vertex of the mesh.

Resulting linear system

We have to solve the linear system $\mathcal{K}_h U_h = b_h$ where $\mathcal{K}_h$ is the rigidity matrix

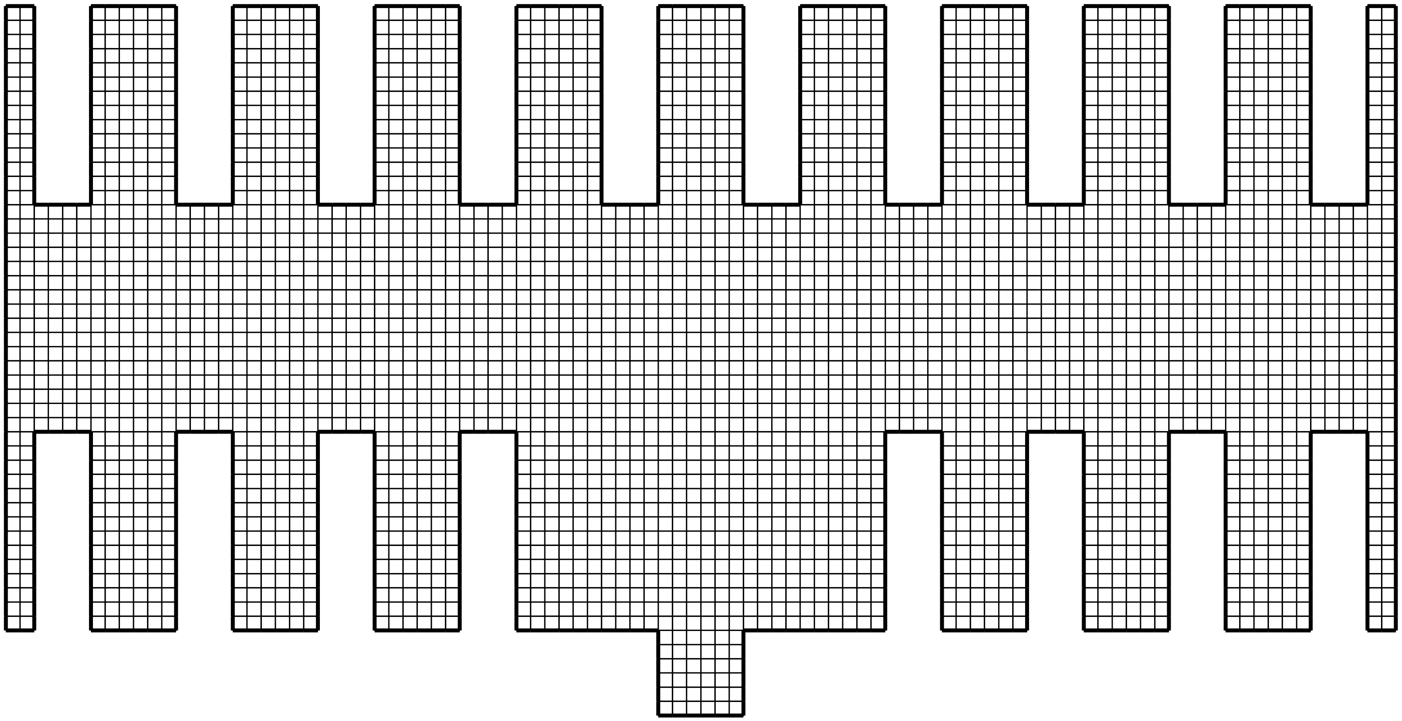
$$\mathcal{K}_h = \left(\int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx \right)_{1 \leq i,j \leq n_{dl}}, b_h = \left(\int_\Omega f \phi_i \, dx \right)_{1 \leq i \leq n_{dl}},$$

$$u_h(x) = \sum_{j=1}^{N_h} u_j \phi_j(x) \quad \text{with} \quad U_h = (u_h(\hat{a}_j))_{1 \leq j \leq n_{dl}} \in \mathbb{R}^{n_{dl}}$$

Quadrature formula for an approximate computation of integrals

$$\int_K \psi(x) \, dx \approx \frac{\text{Volume}(K)}{N+1} \sum_{i=1}^{N+1} \psi(a_i)$$

Rectangular finite elements $\mathbb{Q}_1$



A N-rectangle K in $\mathbb{R}^N$ is defined as $\prod_{i=1}^{N}[l_i, L_i]$ with $-\infty < l_i < L_i < +\infty$. Its vertices are $(a_j)_{1\leq j\leq 2^N}$.

The set $\mathbb{Q}_1$ is made of polynomials of degree less or equal to 1 **with respect to each variable** ($\neq \mathbb{P}_1$)

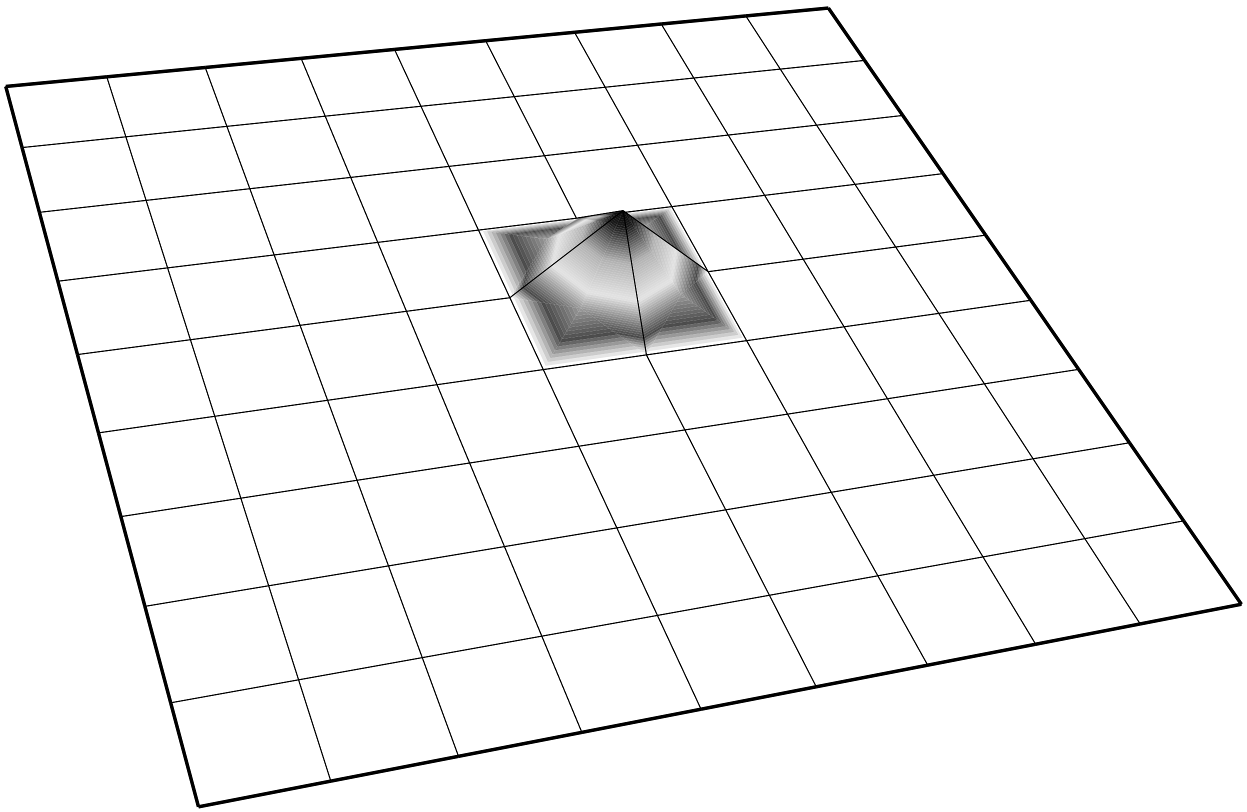
$$\mathbb{Q}_1 = \left\{ p(x) = \sum_{0 \leq i_1 \leq 1, \ldots, 0 \leq i_N \leq 1} \alpha_{i_1, \ldots, i_N} x_1^{i_1} \cdots x_N^{i_N} \text{ avec } x = (x_1, \ldots, x_N) \right\}$$

In other words, $\mathbb{Q}_1$ is defined as the tensor product of $1-d$ affine polynomials in each variable.

Any $\mathbb{Q}_1$ polynomial is uniquely characterized by its values at the vertices $(a_j)_{1 \leq j \leq 2^N}$ of a N-rectangle.

The Lagrange $\mathbb{Q}_1$ finite element method (**quadrangular F.E. of order** 1) associated to a mesh $\mathcal{T}_h$ is defined by

$$V_h = \left\{ v \in \mathcal{C}(\overline{\Omega}) \text{ such that } v|_{K_i} \in \mathbb{Q}_1 \text{ for any } K_i \in \mathcal{T}_h \right\}.$$



Basis function of V_h associated to one node or vertex of the mesh.

Section 2.2.2: Dual or complementary energy

Very important for the sequel... but we shall see that next week !