

OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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CHAPTER VI

GEOMETRIC OPTIMIZATION (Second Part)

”Strategy” of the course

Computing the shape derivative of the solution of a p.d.e. is not easy !

- ⇒ We explain **once** the rigorous method for computing a shape derivative.
- ⇒ It is a bit involved and quite calculus-intensive...
- ⇒ At the end we shall introduce a formal simpler method which is the one to be used **in practice**.
- ⇒ This formal method is called the Lagrangian method and you should learn how to use it !

6.3.3. Derivation of a function depending on the shape

Let $u(\Omega, x)$ be a function defined on the domain Ω .

There exist two notions of derivative:

1) Eulerian (or shape) derivative U

$$u((\text{Id} + \theta)\Omega_0, x) = u(\Omega_0, x) + U(\theta, x) + o(\theta) \quad , \quad \text{with} \quad \lim_{\theta \rightarrow 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0$$

OK if $x \in \Omega_0 \cap (\text{Id} + \theta)\Omega_0$ (local definition, makes no sense on the boundary).

2) Lagrangian (or material) derivative Y

We define the **transported** function $\bar{u}(\theta)$ on Ω_0 by

$$\bar{u}(\theta, x) = u \circ (\text{Id} + \theta) = u\left((\text{Id} + \theta)\Omega_0, x + \theta(x)\right) \quad \forall x \in \Omega_0.$$

The Lagrangian derivative Y is obtained by differentiating $\bar{u}(\theta, x)$

$$\bar{u}(\theta, x) = \bar{u}(0, x) + Y(\theta, x) + o(\theta) \quad , \quad \text{with} \quad \lim_{\theta \rightarrow 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0 \quad ,$$

Differentiating $\bar{u} = u \circ (\text{Id} + \theta)$, one can check that

$$Y(\theta, x) = U(\theta, x) + \theta(x) \cdot \nabla u(\Omega_0, x).$$

The Eulerian derivative, although being simpler, is **very delicate to use** and often not rigorous. For example, if $u \in H_0^1(\Omega)$, the space of definition varies with Ω ... Equivalently what boundary condition should the derivative satisfy ?

We recommend to use the Lagrangian derivative **to avoid mistakes**.

Remark. Computations will be made with Y but the final result is stated with U (which is simpler).

Composed shape derivative

Proposition 6.28. Let Ω_0 be a smooth bounded open set of \mathbb{R}^N , and $u(\Omega) \in L^1(\mathbb{R}^N)$. We assume that the transported function \bar{u} is differentiable at 0 from $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$, with derivative Y . Then

$$J(\Omega) = \int_{\Omega} u(\Omega) dx$$

is differentiable at Ω_0 and $\forall \theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} (u(\Omega_0) \operatorname{div} \theta + Y(\theta)) dx.$$

In other words, using the Eulerian derivative U ,

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} (U(\theta) + \operatorname{div}(u(\Omega_0)\theta)) dx.$$

Similarly, if $\bar{u}(\theta)$ is differentiable at 0 as a function from $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ into $L^1(\partial\Omega_0)$, then

$$J(\Omega) = \int_{\partial\Omega} u(\Omega) dx$$

is differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(u(\Omega_0) (\operatorname{div}\theta - \nabla\theta n \cdot n) + Y(\theta) \right) ds.$$

In other words, using the Eulerian derivative U ,

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(U(\theta) + \theta \cdot n \left(\frac{\partial u(\Omega_0)}{\partial n} + Hu(\Omega_0) \right) \right) dx.$$

6.3.4 Shape derivation of an equation

From now on, $u(\Omega)$ is the **solution of a p.d.e.** in the domain Ω .

Recall that

$$Y(\theta, x) = U(\theta, x) + \theta(x) \cdot \nabla u(\Omega_0, x).$$

The Eulerian derivative, although being simpler, is **very delicate to use** and often not rigorous. For example, if $u \in H_0^1(\Omega)$, the space of definition varies with Ω ... Equivalently what boundary condition should the derivative satisfy ?

We recommend to use the Lagrangian derivative: after getting back to the fixed reference domain Ω_0 we differentiate with respect to θ . **This is the safest and most rigorous way** for computing the shape derivative of u , but the details can be tricky.

We shall later introduce a heuristic method which is simpler.

The results depend on the type of boundary conditions.

Dirichlet boundary conditions

For $f \in L^2(\mathbb{R}^N)$ we consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits a unique solution $u(\Omega) \in H_0^1(\Omega)$.

Its **variational formulation** is: find $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

(Simplification with respect to the textbook since here $g = 0$.)

For $\Omega = (\text{Id} + \theta)(\Omega_0)$ we define the change of variables

$$x = y + \theta(y) \quad y \in \Omega_0 \quad x \in \Omega.$$

Proposition 6.30. Let $u(\Omega) \in H_0^1(\Omega)$ be the solution and $\bar{u}(\theta) \in H_0^1(\Omega_0)$ be its transported function

$$\bar{u}(\theta)(y) = u(\Omega)(x) = u\left((\text{Id} + \theta)(\Omega_0)\right) \circ (\text{Id} + \theta)(y).$$

The functional $\theta \rightarrow \bar{u}(\theta)$, from $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ into $H^1(\Omega_0)$, is differentiable at 0, and its derivative in the direction θ , called **Lagrangian derivative** is

$$Y = \langle \bar{u}'(0), \theta \rangle$$

where $Y \in H_0^1(\Omega_0)$ is the unique solution of

$$\begin{cases} -\Delta Y = -\Delta(\theta \cdot \nabla u(\Omega_0)) & \text{in } \Omega_0 \\ Y = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Proof. We perform the change of variables $x = y + \theta(y)$ with $y \in \Omega_0$ in the variational formulation

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

Take a test function $\phi = \psi \circ (\text{Id} + \theta)^{-1}$, i.e., $\psi(y) = \phi(x)$. Recall that

$$(\nabla \phi) \circ (\text{Id} + \theta) = ((I + \nabla \theta)^{-1})^t \nabla (\phi \circ (\text{Id} + \theta)).$$

We obtain: find $\bar{u} \in H_0^1(\Omega_0)$ such that, for any $\psi \in H_0^1(\Omega_0)$,

$$\int_{\Omega_0} A(\theta) \nabla \bar{u} \cdot \nabla \psi \, dy = \int_{\Omega_0} f \circ (\text{Id} + \theta) \psi \, |\det(\text{Id} + \nabla \theta)| \, dy$$

with $A(\theta) = |\det(I + \nabla \theta)| (I + \nabla \theta)^{-1} ((I + \nabla \theta)^{-1})^t$.

We differentiate with respect to θ at 0 the variational formulation

$$\int_{\Omega_0} A(\theta) \nabla \bar{u} \cdot \nabla \psi \, dy = \int_{\Omega_0} f \circ (\text{Id} + \theta) \psi |\det(\text{Id} + \nabla \theta)| \, dy$$

where ψ is a function which does not depend on θ .

We already checked in the proof of Proposition 6.22 that the right hand side is differentiable. Furthermore, the map $\theta \rightarrow A(\theta)$ is differentiable too because

$$A(\theta) = (1 + \text{div} \theta)I - \nabla \theta - (\nabla \theta)^t + o(\theta) \quad \text{with} \quad \lim_{\theta \rightarrow 0} \frac{\|o(\theta)\|_{L^\infty(\mathbf{R}^N; \mathbf{R}^{N^2})}}{\|\theta\|_{W^{1,\infty}(\mathbf{R}^N; \mathbf{R}^N)}} = 0.$$

Since $\bar{u}(\theta = 0) = u(\Omega_0)$, we get

$$\int_{\Omega_0} \nabla Y \cdot \nabla \psi \, dy + \int_{\Omega_0} \left(\operatorname{div} \theta I - \nabla \theta - (\nabla \theta)^t \right) \nabla u(\Omega_0) \cdot \nabla \psi \, dy = \int_{\Omega_0} \operatorname{div} (f \theta) \psi \, dy$$

Since $\bar{u}(\theta) \in H_0^1(\Omega_0)$, its derivative Y belongs to $H_0^1(\Omega_0)$ too. Thus Y is a solution of

$$\begin{cases} -\Delta Y = \operatorname{div} \left[\left(\operatorname{div} \theta I - \nabla \theta - (\nabla \theta)^t \right) \nabla u(\Omega_0) \right] + \operatorname{div} (f \theta) & \text{in } \Omega_0 \\ Y = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Recalling that $\Delta u(\Omega_0) = -f$ in Ω_0 , and using the identity (true for any $v \in H^1(\Omega_0)$ such that $\Delta v \in L^2(\Omega_0)$)

$$\Delta (\nabla v \cdot \theta) = \operatorname{div} \left((\Delta v) \theta - (\operatorname{div} \theta) \nabla v + \left(\nabla \theta + (\nabla \theta)^t \right) \nabla v \right),$$

leads to the final result. (gotcha !)

Shape derivative U

Corollary 6.32. The **Eulerian derivative** U of the solution $u(\Omega)$, defined by formula

$$U = Y - \nabla u(\Omega_0) \cdot \theta,$$

is the solution in $H^1(\Omega_0)$ of

$$\begin{cases} -\Delta U = 0 & \text{in } \Omega_0 \\ U = -(\theta \cdot n) \frac{\partial u(\Omega_0)}{\partial n} & \text{on } \partial\Omega_0. \end{cases}$$

(Obvious proof starting from Y .)

We are going to recover **formally** this p.d.e. for U without using the knowledge of Y .

Let ϕ be a compactly supported test function in $\omega \subset \Omega$ for the variational formulation

$$\int_{\omega} \nabla u \cdot \nabla \phi \, dx = \int_{\omega} f \phi \, dx.$$

Differentiating with respect to Ω , **neither the test function, nor the domain of integration depend on Ω** . Thus it yields

$$\int_{\omega} \nabla U \cdot \nabla \phi \, dx = 0 \quad \Leftrightarrow \quad -\Delta U = 0.$$

To find the boundary condition we formally differentiate

$$\int_{\partial\Omega} u(\Omega) \psi \, ds = 0 \quad \forall \psi \in C^\infty(\mathbb{R}^N)$$

$$\Rightarrow \int_{\partial\Omega_0} U \psi \, ds + \int_{\partial\Omega_0} \left(\frac{\partial(u\psi)}{\partial n} + H u \psi \right) \theta \cdot n \, ds = 0$$

which leads to the correct result since $u = 0$ on $\partial\Omega_0$.

Remark. The direct computation of U is not always that easy !

Neumann boundary conditions

For $f \in H^1(\mathbb{R}^N)$ and $g \in H^2(\mathbb{R}^N)$ we consider the boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega \end{cases}$$

which admits a unique solution $u(\Omega) \in H^1(\Omega)$.

Its variational formulation is: find $u \in H^1(\Omega)$ such that

$$\int_{\Omega} (\nabla u \cdot \nabla \phi + u\phi) dx = \int_{\Omega} f\phi dx + \int_{\partial\Omega} g\phi ds \quad \forall \phi \in H^1(\Omega).$$

Proposition 6.34. For $\Omega = (\text{Id} + \theta)(\Omega_0)$ we define the change of variables

$$x = y + \theta(y) \quad y \in \Omega_0 \quad x \in \Omega.$$

Let $u(\Omega) \in H^1(\Omega)$ be the solution and $\bar{u}(\theta) \in H^1(\Omega_0)$ be its transported function

$$\bar{u}(\theta)(y) = u(\Omega)(x) = u\left((\text{Id} + \theta)(\Omega_0)\right) \circ (\text{Id} + \theta)(y).$$

The functional $\theta \rightarrow \bar{u}(\theta)$, from $W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ into $H^1(\Omega_0)$, is differentiable at 0, and its derivative in the direction θ , called **Lagrangian derivative** is

$$Y = \langle \bar{u}'(0), \theta \rangle$$

where $Y \in H^1(\Omega_0)$ is the unique solution of

$$\begin{cases} -\Delta Y + Y = -\Delta(\nabla u(\Omega_0) \cdot \theta) + \nabla u(\Omega_0) \cdot \theta & \text{in } \Omega_0 \\ \frac{\partial Y}{\partial n} = (\nabla \theta + (\nabla \theta)^t) \nabla u(\Omega_0) \cdot n + \nabla g \cdot \theta - g(\nabla \theta n \cdot n) & \text{on } \partial \Omega_0. \end{cases}$$

Proof. We perform the change of variables $x = y + \theta(y)$ with $y \in \Omega_0$ in the variational formulation. Take a test function $\phi = \psi \circ (\text{Id} + \theta)^{-1}$, i.e., $\psi(y) = \phi(x)$. We get

$$\begin{aligned} \int_{\Omega_0} A(\theta) \nabla \bar{u} \cdot \nabla \psi \, dy &+ \int_{\Omega_0} \bar{u} \psi |\det(I + \nabla \theta)| \, dy \\ &= \int_{\Omega_0} f \circ (\text{Id} + \theta) \psi |\det(I + \nabla \theta)| \, dy \\ &+ \int_{\partial \Omega_0} g \circ (\text{Id} + \theta) \psi |\det(I + \nabla \theta)| |(I + \nabla \theta)^{-t} n| \, ds \end{aligned}$$

with $A(\theta) = |\det(I + \nabla \theta)| (I + \nabla \theta)^{-1} ((I + \nabla \theta)^{-1})^t$.

We differentiate with respect to θ at 0.

The only new term is the boundary integral which can be differentiated like in Proposition 6.24.

Defining $Y = \langle \bar{u}'(0), \theta \rangle$ we deduce

$$\begin{aligned} \int_{\Omega_0} (\nabla Y \cdot \nabla \psi + Y \psi) dy + \int_{\Omega_0} (\operatorname{div} \theta I - \nabla \theta - (\nabla \theta)^t) \nabla \bar{u} \cdot \nabla \psi dy \\ + \int_{\Omega_0} \bar{u} \psi \operatorname{div} \theta dy = \int_{\Omega_0} \operatorname{div}(f \theta) \psi dy \\ + \int_{\partial \Omega_0} (\nabla g \cdot \theta + g(\operatorname{div} \theta - \nabla \theta n \cdot n)) \psi ds \end{aligned}$$

Then we recall that $\bar{u}(0) = u(\Omega_0) = u$, $\Delta u = u - f$ in Ω_0 and $\frac{\partial u}{\partial n} = g$ on $\partial \Omega_0$, and the identity

$$\Delta (\nabla v \cdot \theta) = \operatorname{div} ((\Delta v) \theta - (\operatorname{div} \theta) \nabla v + (\nabla \theta + (\nabla \theta)^t) \nabla v),$$

to get the result. **Simple in principle but computationally intensive...**

Corollary 6.36. The **Eulerian derivative** U of the solution $u(\Omega)$, defined by

$$U = Y - \nabla u(\Omega_0) \cdot \theta,$$

is a solution in $H^1(\Omega_0)$ of

$$-\Delta U + U = 0 \quad \text{in } \Omega_0.$$

and satisfies the boundary condition

$$\frac{\partial U}{\partial n} = \theta \cdot n \left(\frac{\partial g}{\partial n} - \frac{\partial^2 u(\Omega_0)}{\partial n^2} \right) + \nabla_t(\theta \cdot n) \cdot \nabla_t u(\Omega_0) \quad \text{on } \partial\Omega_0,$$

where $\nabla_t \phi = \nabla \phi - (\nabla \phi \cdot n)n$ denotes the tangential gradient on the boundary.

Proof. Easy but tedious computation.

6.4 Gradient and optimality condition

We consider the shape optimization problem

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega),$$

with $\mathcal{U}_{ad} = \{ \Omega = (\text{Id} + \theta)(\Omega_0) \text{ and } \int_{\Omega} dx = V_0 \}$. The cost function $J(\Omega)$ is either the compliance, or a least square criterion for a target displacement $u_0(x) \in L^2(\mathbb{R}^N)$

$$J(\Omega) = \int_{\Omega} f u \, dx + \int_{\partial\Omega} g u \, ds \quad \text{or} \quad J(\Omega) = \int_{\Omega} |u - u_0|^2 \, dx.$$

The function $u(\Omega)$ is the solution in $H^1(\Omega)$ of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega, \end{cases}$$

with $f \in H^1(\mathbb{R}^N)$ and $g \in H^2(\mathbb{R}^N)$.

Gradient and optimality condition

Theorem 6.38. The functional $J(\Omega) = \int_{\Omega} |u - u_0|^2 dx$ is shape differentiable

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(|u - u_0|^2 + \nabla u \cdot \nabla p + p(u - f) - \frac{\partial(gp)}{\partial n} - Hgp \right) ds,$$

where p is the adjoint state, unique solution in $H^1(\Omega_0)$ of

$$\begin{cases} -\Delta p + p = -2(u - u_0) & \text{in } \Omega_0 \\ \frac{\partial p}{\partial n} = 0 & \text{on } \partial\Omega_0, \end{cases}$$

We recover the fact that the shape derivative depends only on the normal trace of θ on the boundary.

Proof. Applying Proposition 6.28 to the cost function yields

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} (|u(\Omega_0) - u_0|^2 \operatorname{div} \theta + 2(u(\Omega_0) - u_0)(Y - \nabla u_0 \cdot \theta)) dx,$$

or equivalently, with $U = Y - \nabla u(\Omega_0) \cdot \theta$,

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} [\operatorname{div} (\theta |u(\Omega_0) - u_0|^2) + 2(u(\Omega_0) - u_0)U] dx.$$

Multiplying the adjoint equation by U

$$\int_{\Omega_0} (\nabla p \cdot \nabla U + pU) dy = -2 \int_{\Omega_0} (u(\Omega_0) - u_0) U dy,$$

then the equation for U by p

$$\int_{\Omega_0} (\nabla p \cdot \nabla U + pU) dy = \int_{\partial\Omega_0} \theta \cdot n \left(-\nabla u(\Omega_0) \cdot \nabla p - p\Delta u(\Omega_0) + \frac{\partial(gp)}{\partial n} + Hgp \right) ds,$$

we deduce the result by comparison of the two equalities.

The compliance case (self-adjoint)

Theorem 6.40. The functional $J(\Omega) = \int_{\Omega} fu \, dx + \int_{\partial\Omega} gu \, ds$ is shape-differentiable

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(-|\nabla u(\Omega_0)|^2 - |u(\Omega_0)|^2 + 2u(\Omega_0)f \right) ds \\ + \int_{\partial\Omega_0} \theta \cdot n \left(2 \frac{\partial(gu(\Omega_0))}{\partial n} + 2Hgu(\Omega_0) \right) ds,$$

Interpretation: assume $f = 0$ and $g = 0$ where $\theta \cdot n \neq 0$. The formula simplifies in

$$J'(\Omega_0)(\theta) = - \int_{\partial\Omega_0} \theta \cdot n \left(|\nabla u|^2 + u^2 \right) ds \leq 0$$

It is always advantageous to increase the domain (i.e., $\theta \cdot n > 0$) for decreasing the compliance.

Proof. Applying Proposition 6.28 to the cost function yields

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} (fu \operatorname{div} \theta + u\theta \cdot \nabla f + fY) dx \\ + \int_{\partial\Omega_0} (gu (\operatorname{div} \theta - \nabla \theta n \cdot n) + u\theta \cdot \nabla g + gY) ds,$$

or equivalently, with $U = Y - \nabla u \cdot \theta$,

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} (\operatorname{div}(fu\theta) + fU) dx + \int_{\partial\Omega_0} \left(\theta \cdot n \left(\frac{\partial(gu)}{\partial n} + Hgu \right) + gU \right) ds.$$

Multiplying the equation for u by U and that for U by u , then comparing, leads to the result.

Remark. Same type of result for a Dirichlet boundary condition (but different formulas).

6.4.3 Fast derivation: the Lagrangian method

- ⇒ The previous computations are quite tedious... but there is a simpler and faster (albeit formal) method, called the **Lagrangian method** (proposed in this context by J. C ea).
- ⇒ The Lagrangian allows us to find the correct definition of **the adjoint state** too.
- ⇒ It is easy for Neumann boundary conditions, a little more involved for Dirichlet ones.
- ⇒ That is the method to be known !

Fast derivation for Neumann boundary conditions

If the objective function is

$$J(\Omega) = \int_{\Omega} j(u(\Omega)) dx,$$

the Lagrangian is defined as the sum of J and of the variational formulation of the state equation

$$\mathcal{L}(\Omega, v, q) = \int_{\Omega} j(v) dx + \int_{\Omega} (\nabla v \cdot \nabla q + vq - fq) dx - \int_{\partial\Omega} gq ds,$$

with v and $q \in H^1(\mathbb{R}^N)$. It is important to notice that the space $H^1(\mathbb{R}^N)$ **does not depend** on Ω and thus the three variables in \mathcal{L} are clearly **independent**.

The partial derivative of \mathcal{L} with respect to q in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\left\langle \frac{\partial \mathcal{L}}{\partial q}(\Omega, v, q), \phi \right\rangle = \int_{\Omega} \left(\nabla v \cdot \nabla \phi + v \phi - f \phi \right) dx - \int_{\partial \Omega} g \phi ds,$$

which, upon equating to 0, gives the **variational formulation of the state**.

The partial derivative of \mathcal{L} with respect to v in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\left\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \right\rangle = \int_{\Omega} j'(v) \phi dx + \int_{\Omega} \left(\nabla \phi \cdot \nabla q + \phi q \right) dx,$$

which, upon equating to 0, gives the **variational formulation of the adjoint**.

The partial derivative of \mathcal{L} with respect to Ω in the direction θ is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, v, q)(\theta) = \int_{\partial \Omega} \theta \cdot n \left(j(v) + \nabla v \cdot \nabla q + v q - f q - \frac{\partial(gq)}{\partial n} - H g q \right) ds.$$

When evaluating this derivative with the state $u(\Omega_0)$ and the adjoint $p(\Omega_0)$, we precisely find the **derivative of the objective function**

$$\frac{\partial \mathcal{L}}{\partial \Omega} \left(\Omega_0, u(\Omega_0), p(\Omega_0) \right) (\theta) = J'(\Omega_0)(\theta)$$

Indeed, if we differentiate the equality

$$\mathcal{L}(\Omega, u(\Omega), q) = J(\Omega) \quad \forall q \in H^1(\mathbb{R}^N),$$

the chain rule lemma yields

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u(\Omega_0), q)(\theta) + \left\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega_0, u(\Omega_0), q), u'(\Omega_0)(\theta) \right\rangle$$

Taking $q = p(\Omega_0)$, the last term cancels since $p(\Omega_0)$ is the solution of the adjoint equation.

Thanks to this computation, the “correct” result can be guessed for $J'(\Omega_0)$ without using the notions of shape or material derivatives.

Nevertheless, in full rigor, this “fast” computation of the shape derivative $J'(\Omega_0)$ is valid only if we know that u is shape differentiable.

Fast derivation for Dirichlet boundary conditions

It is more involved ! Let $u \in H_0^1(\Omega)$ be the solution of

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

The “usual” Lagrangian is

$$\mathcal{L}(\Omega, v, q) = \int_{\Omega} j(v) \, dx + \int_{\Omega} (\nabla v \cdot \nabla q - fq) \, dx,$$

for $v, q \in H_0^1(\Omega)$. **The variables (Ω, v, q) are not independent !**

Indeed, the functions v and q satisfy

$$v = q = 0 \quad \text{on } \partial\Omega.$$

Another Lagrangian has to be introduced.

Lagrangian for Dirichlet boundary conditions

The Dirichlet boundary condition is **penalized**

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_{\Omega} j(v) dx - \int_{\Omega} (\Delta v + f)q dx + \int_{\partial\Omega} \lambda v ds$$

where λ is the Lagrange multiplier for the boundary condition. It is now possible to differentiate since the 4 variables $v, q, \lambda \in H^1(\mathbb{R}^N)$ are independent.

Of course, we recover

$$\sup_{q, \lambda} \mathcal{L}(\Omega, v, q, \lambda) = \begin{cases} \int_{\Omega} j(u) dx = J(\Omega) & \text{if } v \equiv u, \\ +\infty & \text{otherwise.} \end{cases}$$

By definition of the Lagrangian:

the partial derivative of \mathcal{L} with respect to q in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\left\langle \frac{\partial \mathcal{L}}{\partial q}(\Omega, v, q, \lambda), \phi \right\rangle = - \int_{\Omega} \phi (\Delta v + f) dx,$$

which, upon equating to 0, gives the **state equation**,

the partial derivative of \mathcal{L} with respect to λ in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\left\langle \frac{\partial \mathcal{L}}{\partial \lambda}(\Omega, v, q, \lambda), \phi \right\rangle = \int_{\partial\Omega} \phi v dx,$$

which, upon equating to 0, gives the **Dirichlet boundary condition** for the state equation.

To compute the partial derivative of \mathcal{L} with respect to v , we perform a first integration by parts

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_{\Omega} j(v) dx + \int_{\Omega} (\nabla v \cdot \nabla q - fq) dx + \int_{\partial\Omega} \left(\lambda v - \frac{\partial v}{\partial n} q \right) ds,$$

then a second integration by parts

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_{\Omega} j(v) dx - \int_{\Omega} (v\Delta q - fq) dx + \int_{\partial\Omega} \left(\lambda v - \frac{\partial v}{\partial n} q + \frac{\partial q}{\partial n} v \right) ds.$$

We now can differentiate in the direction $\phi \in H^1(\mathbb{R}^N)$

$$\left\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \right\rangle = \int_{\Omega} j'(v)\phi dx - \int_{\Omega} \phi\Delta q dx + \int_{\partial\Omega} \left(-q \frac{\partial \phi}{\partial n} + \phi \left(\lambda + \frac{\partial q}{\partial n} \right) \right) ds$$

which, upon equating to 0, gives **three relationships**, the two first ones being **the adjoint problem**.

1. If ϕ has compact support in Ω_0 , we get

$$-\Delta p = -j'(u) \quad \text{dans } \Omega_0.$$

2. If $\phi = 0$ on $\partial\Omega_0$ with any value of $\frac{\partial\phi}{\partial n}$ in $L^2(\partial\Omega_0)$, we deduce

$$p = 0 \quad \text{sur } \partial\Omega_0.$$

3. If ϕ is now varying in the full $H^1(\Omega_0)$, we find

$$\frac{\partial p}{\partial n} + \lambda = 0 \quad \text{sur } \partial\Omega_0.$$

The adjoint problem has actually been recovered but **furthermore** the optimal Lagrange multiplier λ has been characterized.

Eventually, **the shape partial derivative** is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u, p, \lambda)(\theta) = \int_{\partial \Omega_0} \theta \cdot n \left(j(u) - (\Delta u + f)p + \frac{\partial(u\lambda)}{\partial n} + Hu\lambda \right) ds$$

Knowing that $u = p = 0$ on $\partial \Omega_0$ and $\lambda = -\frac{\partial p}{\partial n}$ we deduce

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u, p, \lambda)(\theta) = \int_{\partial \Omega_0} \theta \cdot n \left(j(0) - \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \right) ds = J'(\Omega_0)(\theta)$$

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega} \left(\Omega_0, u(\Omega_0), p(\Omega_0) \right) (\theta)$$

This formula is not a surprise because differentiating

$$\mathcal{L}(\Omega, u(\Omega), q, \lambda) = J(\Omega) \quad \forall q, \lambda$$

yields

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega} (\Omega_0, u(\Omega_0), q, \lambda) (\theta) + \left\langle \frac{\partial \mathcal{L}}{\partial v} (\Omega_0, u(\Omega_0), q, \lambda), u'(\Omega_0)(\theta) \right\rangle.$$

Then, taking $q = p(\Omega_0)$ (the adjoint state) and $\lambda = -\frac{\partial p}{\partial n}(\Omega_0)$, the last term cancels and we obtain the desired formula.

Application to compliance minimization

We minimize $J(\Omega) = \int_{\Omega} f u \, dx$ with $u \in H_0^1(\Omega)$ solution of

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

The adjoint state is just $p = -u$. The shape derivative is

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(f u - \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \right) ds = \int_{\partial\Omega_0} \theta \cdot n \left(\frac{\partial u}{\partial n} \right)^2 ds \leq 0$$

It is always advantageous to shrink the domain (i.e., $\theta \cdot n < 0$) to decrease the compliance.

This is the opposite conclusion compared to Neumann b.c., but it is logical !

Another example: the drum

We optimize the shape of a **drum** (an elastic membrane) in order it produces the lowest possible tune. Let $\lambda(\Omega)$ be the eigenvalue (the square of the eigenfrequency) and $u(x)$ be the eigenmode

$$\begin{cases} -\Delta u = \lambda(\Omega)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The **fundamental mode** is the smallest eigenvalue which is also characterized by

$$\lambda(\Omega) = \min_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx}.$$

Thus we study

$$\inf_{\Omega \subset \mathbf{R}^2} \left(\lambda(\Omega) + \ell \int_{\Omega} dx \right),$$

where $\ell \geq 0$ is a given Lagrange multiplier for a constraint on the membrane area.

Eulerian derivation

For a test function ϕ with compact support $\omega \subset \Omega$ we derive

$$\int_{\omega} \nabla u \cdot \nabla \phi \, dx = \lambda(\Omega) \int_{\omega} u \phi \, dx$$

$$\Rightarrow \int_{\omega} \nabla U \cdot \nabla \phi \, dx = \lambda(\Omega) \int_{\omega} U \phi \, dx + \Lambda \int_{\omega} u \phi \, dx,$$

where $\Lambda = \lambda'(\Omega)(\theta)$ is the derivative of the eigenvalue (assumed to be simple).

$$\Rightarrow -\Delta U - \lambda(\Omega)U = \Lambda u \quad \text{in } \Omega.$$

To deduce the boundary condition for U we derive

$$\int_{\partial\Omega} u \psi \, ds = 0 \quad \forall \psi \in C^{\infty}(\mathbb{R}^2).$$

$$\Rightarrow \int_{\partial\Omega} \left(U \psi + \theta \cdot n \left(\frac{\partial(u\psi)}{\partial n} + H u \psi \right) \right) ds = 0,$$

which yields $U = -\frac{\partial u}{\partial n} \theta \cdot n$ since $u = 0$ on $\partial\Omega$.

Multiplying the equation for U by u and integrating by parts leads to

$$\int_{\Omega} \nabla U \cdot \nabla u \, dx = \lambda \int_{\Omega} U u \, dx + \Lambda \int_{\Omega} u^2 \, dx.$$

Multiplying the equation for u by U and integrating by parts leads to

$$\int_{\Omega} \nabla U \cdot \nabla u \, dx = \lambda \int_{\Omega} U u \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} U \, ds.$$

Thus, we deduce

$$\Lambda \int_{\Omega} u^2 \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} U \, ds = - \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 \theta \cdot n \, ds.$$

The derivative of the objective function is (self-adjoint problem)

$$J'(\Omega)(\theta) = \Lambda + \ell \int_{\partial\Omega} \theta \cdot n \, ds = \int_{\partial\Omega} \left(\ell - \frac{\left(\frac{\partial u}{\partial n} \right)^2}{\int_{\Omega} u^2 \, dx} \right) \theta \cdot n \, ds.$$

If $\ell = 0$ we have $J'(\Omega)(\theta) \leq 0$ as soon as $\theta \cdot n \geq 0$, i.e., we minimize $J(\Omega)$ if the domain Ω is enlarged.

Lagrangian method

For $\mu \in \mathbb{R}$, $v, q, z \in H^1(\mathbb{R}^N)$, we introduce the Lagrangian

$$\mathcal{L}(\Omega, \mu, v, q, z) = \mu - \int_{\Omega} (\Delta v + \mu v) q \, dx + \int_{\partial\Omega} z v \, ds$$

where z is the Lagrange multiplier for the boundary condition. Since the 5 variables are independent it is possible to differentiate.

The partial derivative $\frac{\partial \mathcal{L}}{\partial q} = 0$ gives [the state equation](#).

The partial derivative $\frac{\partial \mathcal{L}}{\partial z} = 0$ gives [the Dirichlet boundary condition](#) for the state.

The partial derivative $\frac{\partial \mathcal{L}}{\partial v} = 0$ gives [three relationships](#) including the [adjoint](#):

$$-\Delta p = \lambda p \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \quad \frac{\partial p}{\partial n} + z = 0 \quad \text{on } \partial\Omega.$$

The partial derivative $\frac{\partial \mathcal{L}}{\partial \mu} = 0$ yields

$$\int_{\Omega} up \, dx = 1$$

Since the eigenvalue λ is simple, p is a multiple of u . Thus

$$p = \frac{u}{\int_{\Omega} u^2 \, dx}.$$

Eventually, **the shape partial derivative** is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, \lambda, u, p, z)(\theta) = \int_{\partial \Omega} \theta \cdot n \left(p \Delta u + \lambda p u + \frac{\partial (uz)}{\partial n} + H u z \right) ds$$

Knowing that $u = p = 0$ on $\partial \Omega$ and $z = -\frac{\partial p}{\partial n}$ we deduce

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, \lambda, u, p, z)(\theta) = \int_{\partial \Omega} \theta \cdot n \left(-\frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \right) ds = J'(\Omega)(\theta)$$