OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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January 28th, 2015

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CHAPTER VII

TOPOLOGY OPTIMIZATION

BY THE HOMOGENIZATION METHOD

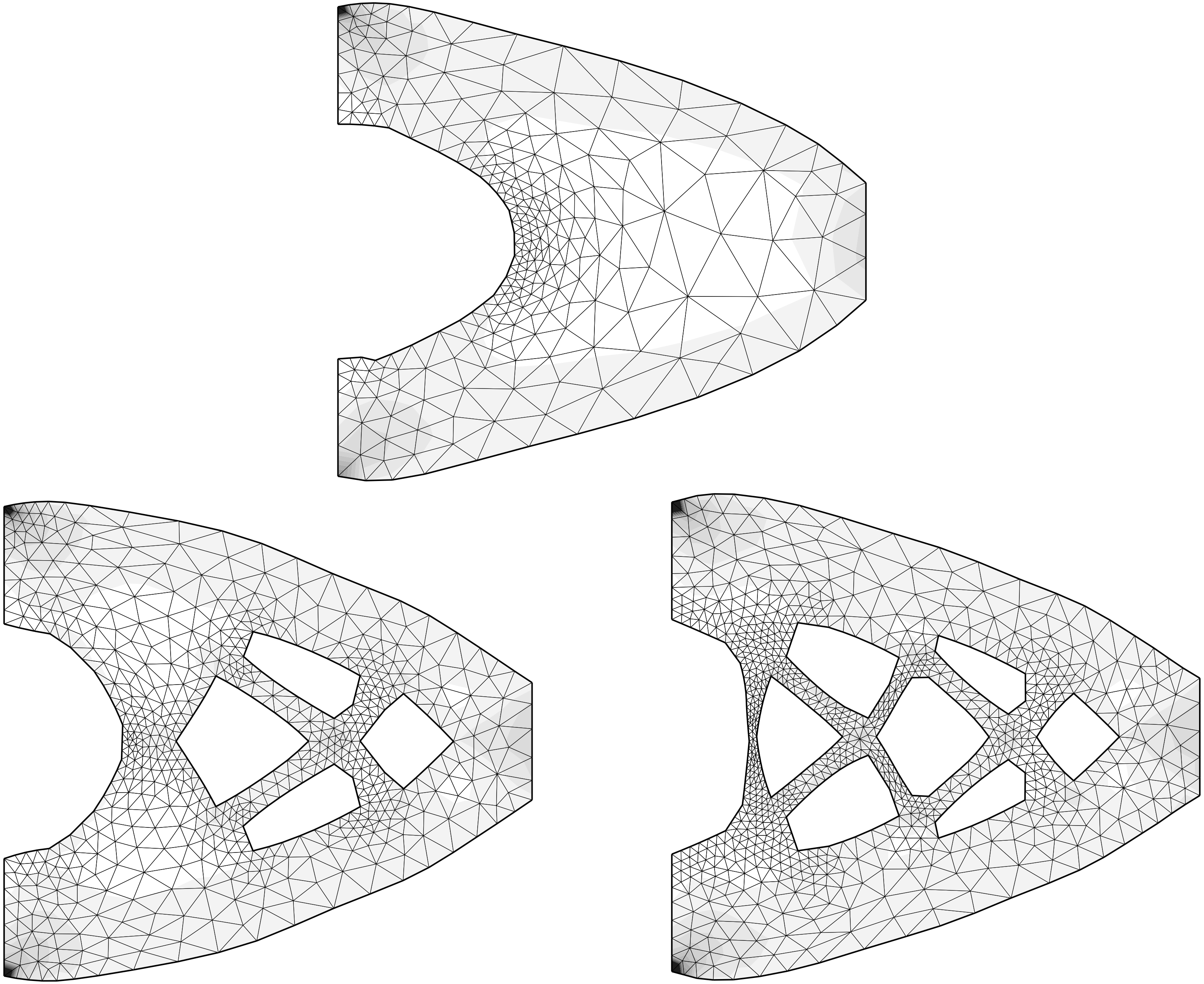
Why topology optimization ?

Drawbacks of geometric optimization:

- ☞ no variation of the topology (number of holes in 2-d),
- ☞ many local minima,
- ☞ CPU cost of remeshing (mostly in 3-d),
- ☞ ill-posed problem: non-existence of optimal solutions (in the absence of constraints). It shows up in numerics !

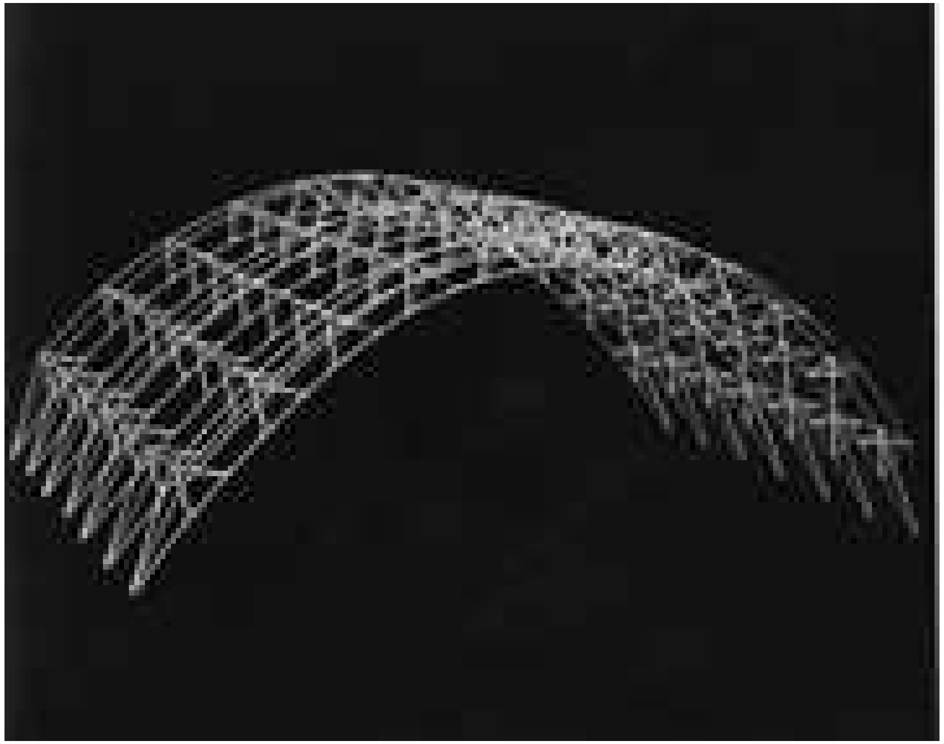
Topology optimization: we improve not only the boundary location but also its topology (i.e., its number of connected components in 2-d).

We focus on one possible method, **based on homogenization**.



The art of structure is where to put the holes.

Robert Le Ricolais, architect and engineer, 1894-1977



Principles of the homogenization method

The homogenization method is based on the concept of "relaxation": it makes ill-posed problems well-posed by enlarging the space of admissible shapes.

We introduce "generalized" shapes but not too generalized... We require the generalized shapes to be "limits" of minimizing sequences of classical shapes.

Remember the counter-example of Section 6.2.1: the minimizing sequences of shapes had a tendency to build fine mixtures of material and void.

Homogenization allows as admissible shapes **composite materials** obtained by microperforation of the original material.

Notations

☞ A classical shape is parametrized by a characteristic function

$$\chi(x) = \begin{cases} 1 \text{ inside the shape,} \\ 0 \text{ inside the holes.} \end{cases}$$

☞ From now on, the holes can be microscopic as well as macroscopic $\Rightarrow$ porous composite materials !

☞ We parametrize a generalized shape by a material density $\theta(x) \in [0,1]$, and a microstructure (or holes shape).

☞ The holes shape is very important ! It induces a new optimization variable which is the effective behavior $A^*(x)$ of the composite material (defined by homogenization theory).

☞ (θ, A^*) are the two new optimization variables.

7.1.2 Model problem

Simplifying assumption: the "holes" with a free boundary condition (Neumann) are filled with a weak ("ersatz") material $\alpha << \beta$.

Membrane with two possible thicknesses $h_\chi(x) = \alpha\chi(x) + \beta(1 - \chi(x))$, with

$$\mathcal{U}_{ad} = \left\{ \chi \in L^\infty(\Omega; \{0,1\}), \ \int_\Omega \chi(x)\, dx = V_\alpha \right\}.$$

If $f \in L^2(\Omega)$ is the applied load, the displacement satisfies

$$\begin{cases} -\mathrm{div}\,(h_\chi \nabla u_\chi) = f & \text{in } \Omega \\ u_\chi = 0 & \text{on } \partial\Omega. \end{cases}$$

Optimizing the membrane's shape amounts to minimize

$$\inf_{\chi \in \mathcal{U}_{ad}} J(\chi),$$

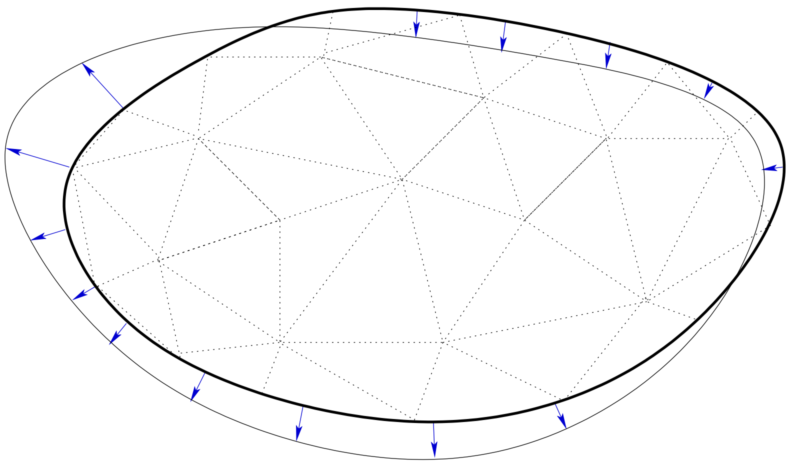
with

$$J(\chi) = \int_\Omega f u_\chi \, dx, \quad \text{or} \quad J(\chi) = \int_\Omega |u_\chi - u_0|^2 dx.$$

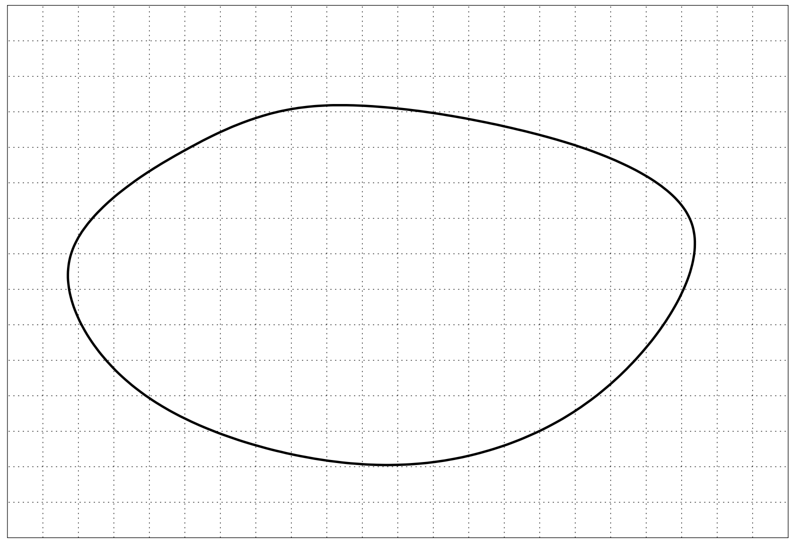
Goals of the homogenization method

- ☞ To introduce the notion of generalized shapes made of composite material.
- ☞ To show that those generalized shapes are limits of sequences of classical shapes (in a sense to be made precise).
- ☞ To compute the generalized objective function and its gradient.
- ☞ To prove an existence theorem of optimal generalized shapes (it is **not** the goal of the present course).
- ☞ To deduce new numerical algorithms for topology optimization (it is **actually** the goal of the present course).

While geometric optimization was producing **shape tracking** algorithms, topology optimization yields **shape capturing** algorithms.

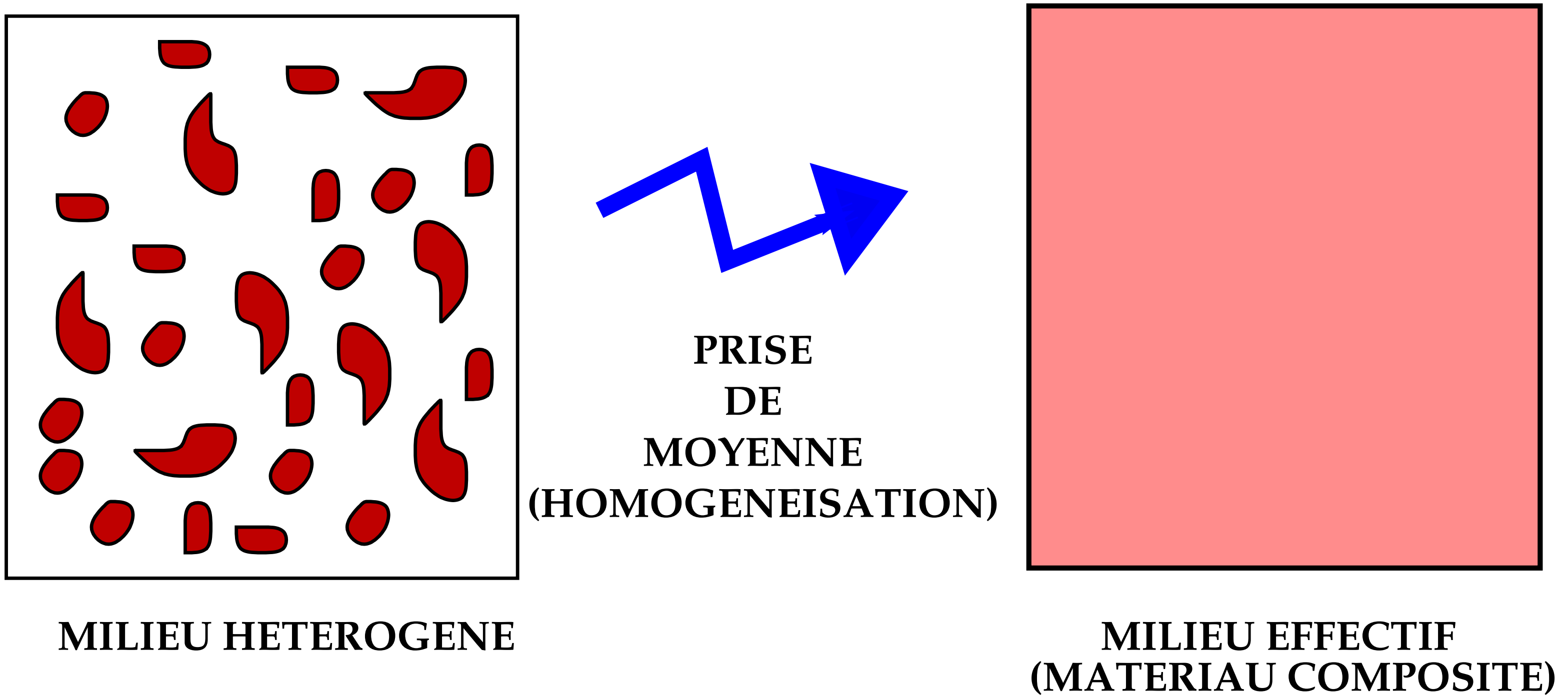


Shape tracking



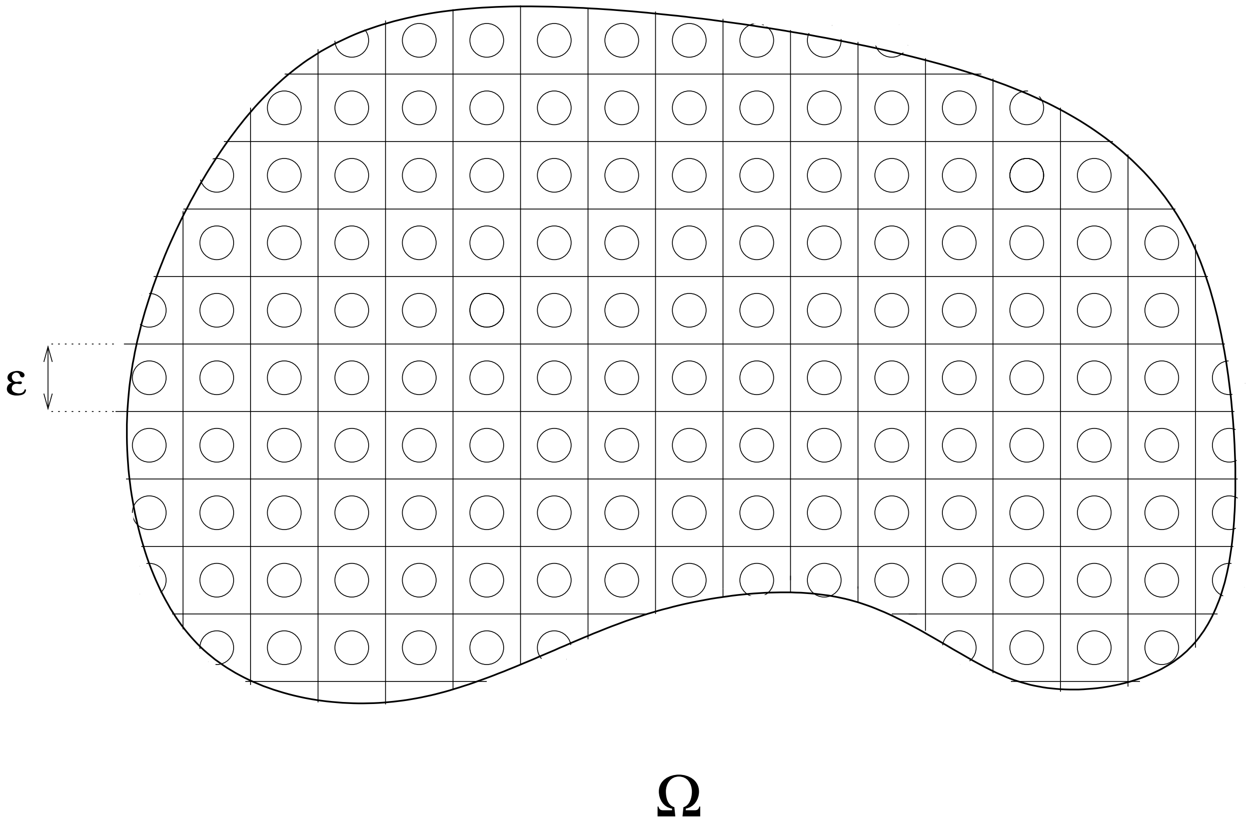
Shape capturing

7.2 Homogenization



- Averaging method for partial differential equations.
- Determination of averaged parameters (or effective, or homogenized, or equvalent, or macroscopic) for an heterogeneous medium.

Periodic homogenization



Different approaches are possible: we describe the simplest one, i.e., periodic homogenization.

Assumption: we consider periodic heterogeneous media.

Periodic homogenization (Ctd.)

- ☞ Ratio of the period with the characteristic size of the structure $= \epsilon$.
- ☞ Although, for the "true" problem under consideration, there is only one physical value ϵ_0 of the parameter ϵ, we consider a sequence of problems with smaller and smaller ϵ.
- ☞ We perform an asymptotic analysis as ϵ goes to 0.
- ☞ We shall approximate the "true" problem ($\epsilon = \epsilon_0$) by the limit problem obtained as $\epsilon \to 0$.

Model problem: elastic membrane made of composite material

For example: periodically distributed fibers in an epoxy resin.

Variable Hooke's law: $A(y)$, Y-periodic function, with $Y = (0,1)^N$.

$$A(y + e_i) = A(y) \quad \forall e_i \ i\text{-th vector of the canonical basis.}$$

We replace y by $\frac{x}{\epsilon}$:

$$x \to A\left(\frac{x}{\epsilon}\right) \text{ periodic of period } \epsilon \text{ in all axis directions.}$$

Bounded domain Ω, load $f(x)$, displacement $u_\epsilon(x)$ solution of

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

A direct computation of u_ϵ can be very expensive (since the mesh size h should satisfy $h < \epsilon$), thus we seek only the averaged values of u_ϵ.

Two-scale asymptotic expansions

We assume that

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right),$$

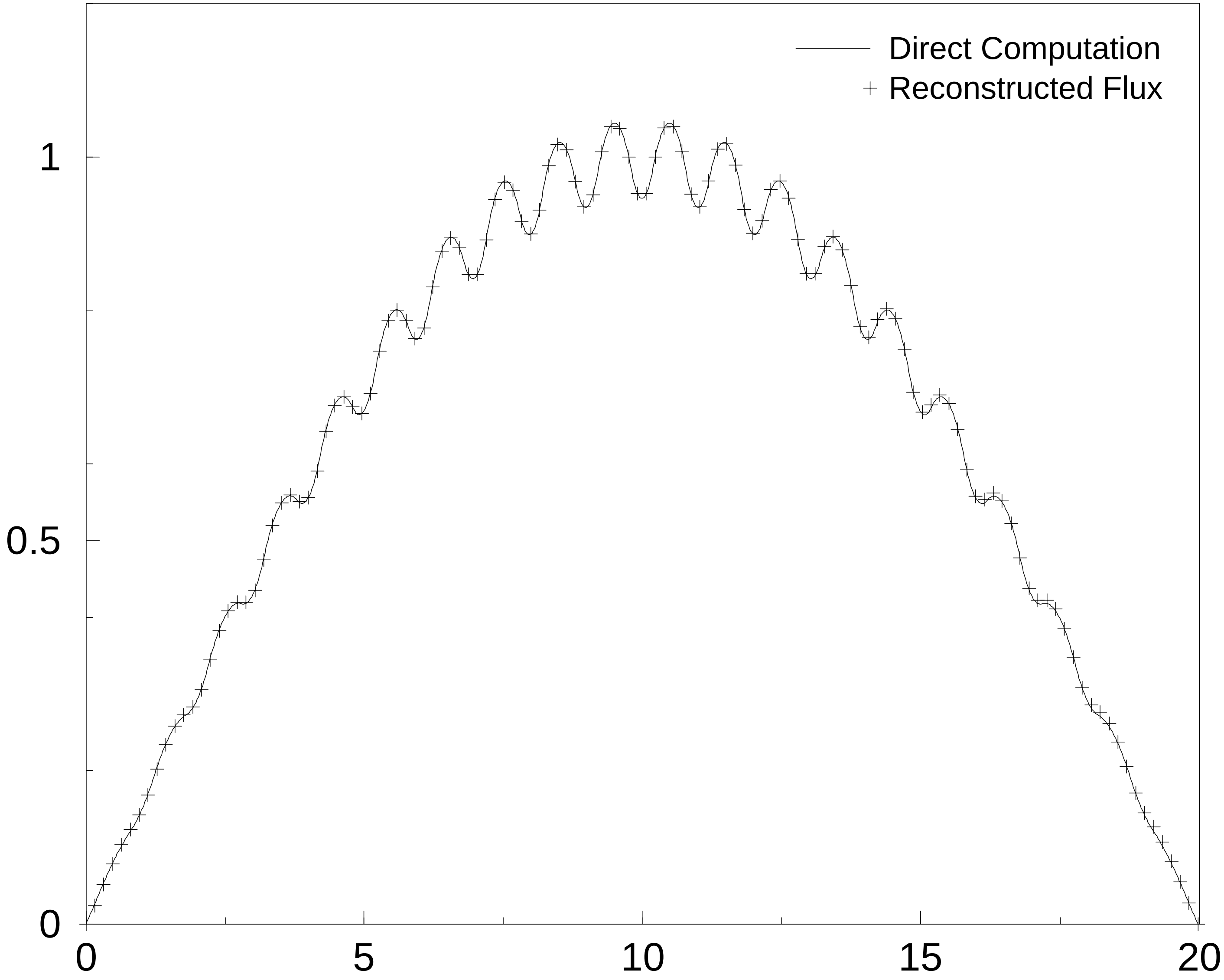
with $u_i(x, y)$ function of the two variables x and y, periodic in y of period $Y = (0,1)^N$. Plugging this series in the equation, we use the derivation rule

$$\nabla\left(u_i\left(x, \frac{x}{\epsilon}\right)\right) = \left(\epsilon^{-1}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right).$$

Thus

$$\nabla u_\epsilon(x) = \epsilon^{-1}\nabla_y u_0\left(x, \frac{x}{\epsilon}\right) + \sum_{i=0}^{+\infty} \epsilon^i \left(\nabla_y u_{i+1} + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right).$$

Typical oscillating behavior of $x \to u_i\left(x, \frac{x}{\epsilon}\right)$



The equation becomes a series in ϵ

$$-\epsilon^{-2}\left[\mathrm{div}_y\left(A\nabla_y u_0\right)\right]\left(x,\frac{x}{\epsilon}\right)$$

$$-\epsilon^{-1}\left[\mathrm{div}_y\left(A(\nabla_x u_0+\nabla_y u_1)\right)+\mathrm{div}_x\left(A\nabla_y u_0\right)\right]\left(x,\frac{x}{\epsilon}\right)$$

$$-\sum_{i=0}^{+\infty}\epsilon^i\left[\mathrm{div}_x\left(A(\nabla_x u_i+\nabla_y u_{i+1})\right)+\mathrm{div}_y\left(A(\nabla_x u_{i+1}+\nabla_y u_{i+2})\right)\right]\left(x,\frac{x}{\epsilon}\right)$$

$$=f(x).$$

☞ We identify each power of ϵ.

☞ We notice that $\phi\left(x,\frac{x}{\epsilon}\right)=0\ \forall x,\epsilon \quad \Leftrightarrow \quad \phi(x,y)\equiv 0\ \forall x,y.$

☞ Only the three first terms of the series really matter.

We start by a technical lemma.

Lemma 7.4. Take $g \in L^2(Y)$. The equation

$$\begin{cases} -\mathrm{div}_y \left(A(y)\nabla_y v(y)\right) = g(y) \text{ in } Y \\ y \to v(y) \; Y\text{-periodic} \end{cases}$$

admits a unique solution $v \in H^1_\#(Y)/\mathbb{R}$ if and only if

$$\int_Y g(y)\, dy = 0.$$

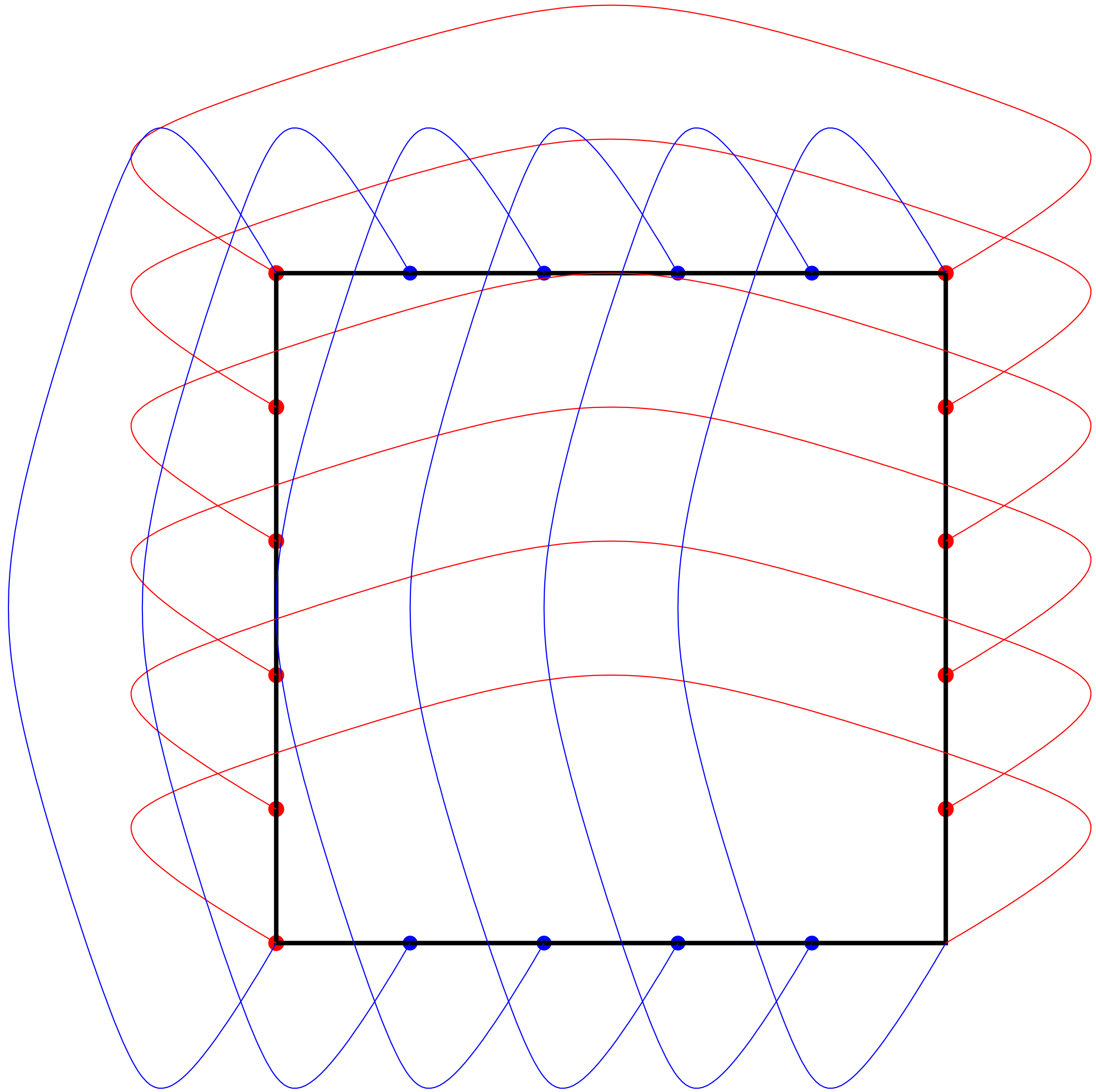
Proof. Let us check that it is a necessary condition for existence. Integrating the equation on Y

$$\int_Y \mathrm{div}_y \Big(A(y)\nabla_y v(y)\Big) dy = \int_{\partial Y} A(y)\nabla_y v(y) \cdot n\, ds = 0$$

because of the periodic boundary conditions: $A(y)\nabla_y v(y)$ is periodic but the normal n changes its sign on opposite faces of Y.

The sufficient condition is obtained by applying Lax-Milgram Theorem in $H^1_\#(Y)/\mathbb{R}$.

Periodic boundary conditions in $H^1_\#(Y)$



Equation of order ϵ^{-2}:

$$\begin{cases} -\mathrm{div}_y \left(A(y) \nabla_y u_0(x,y) \right) = 0 \text{ in } Y \\ y \to u_0(x,y) \ Y\text{-periodic} \end{cases}$$

It is a p.d.e. with respect to y (x is just a parameter).

By uniqueness of the solution (up to an additive constant), we deduce

$$\boxed{u_0(x,y) \equiv u(x)}$$

Equation of order ϵ^{-1}:

$$\begin{cases} -\mathrm{div}_y \left(A(y) \nabla_y u_1(x,y) \right) = \mathrm{div}_y \left(A(y) \nabla_x u(x) \right) \text{ in } Y \\ y \to u_1(x,y) \; Y\text{-periodic} \end{cases}$$

The necessary and sufficient condition of existence is satisfied. Thus u_1 depends linearly on $\nabla_x u(x)$.

We introduce the cell problems

$$\begin{cases} -\mathrm{div}_y \left(A(y) \left(e_i + \nabla_y w_i(y) \right) \right) = 0 & \text{in } Y \\ y \to w_i(y) & Y\text{-periodic,} \end{cases}$$

with $\left(e_i\right)_{1 \leq i \leq N}$, the canonical basis of $\mathbb{R}^N$. Then

$$\boxed{u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y)}$$

Equation of order ϵ^0:

$$\begin{cases} -\mathrm{div}_y \left(A(y)\nabla_y u_2(x,y)\right) = \mathrm{div}_y \left(A(y)\nabla_x u_1\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad +\mathrm{div}_x \left(A(y)(\nabla_y u_1 + \nabla_x u)\right) + f(x) \text{ in } Y \\ y \to u_2(x,y) \ Y\text{-periodic} \end{cases}$$

The necessary and sufficient condition of existence of the solution u_2 is:

$$\int_Y \Big(\mathrm{div}_y\big(A(y)\nabla_x u_1\big) + \mathrm{div}_x\big(A(y)(\nabla_y u_1 + \nabla_x u)\big) + f(x)\Big) dy = 0$$

We replace u_1 by its value in terms of $\nabla_x u(x)$

$$\mathrm{div}_x \int_Y A(y) \left(\sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) \nabla_y w_i(y) + \nabla_x u(x) \right) dy + f(x) = 0$$

and we find the homogenized problem

$$\begin{cases} -\mathrm{div}_x \left(A^* \nabla_x u(x)\right) = f(x) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Homogenized tensor:

$$A^*_{ji} = \int_Y A(y)(e_i + \nabla_y w_i) \cdot e_j \, dy,$$

or, integrating by parts

$$A^*_{ji} = \int_Y A(y) \left(e_i + \nabla_y w_i(y)\right) \cdot \left(e_j + \nabla_y w_j(y)\right) dy.$$

Indeed, the cell problem yields

$$\int_Y A(y) \left(e_i + \nabla_y w_i(y)\right) \cdot \nabla_y w_j(y) \, dy = 0.$$

- The formula for A^* is not fully explicit because cell problems must be solved.
- A^* does not depend on Ω, nor f, nor the boundary conditions.
- **The tensor A^* characterizes the microstructure.**
- Later, we shall compute explicitly some examples of A^*.

One can prove:

$$u_\epsilon(x) = u(x) + \epsilon\, u_1\left(x, \frac{x}{\epsilon}\right) + r_\epsilon \quad \text{with} \quad \|r_\epsilon\|_{H^1(\Omega)} \leq C\epsilon^{1/2}$$

In particular

$$\boxed{\|u_\epsilon - u\|_{L^2(\Omega)} \leq C\epsilon^{1/2}}$$

The corrector is not negligible for the strain or the stress

$$\nabla u_\epsilon(x) = \nabla_x u(x) + (\nabla_y u_1)\left(x, \frac{x}{\epsilon}\right) + t_\epsilon \quad \text{with} \quad \|t_\epsilon\|_{L^2(\Omega)} \leq C\epsilon^{1/2}$$

$$A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon(x) = A^*\nabla_x u(x) + \tau\left(x, \frac{x}{\epsilon}\right) + s_\epsilon \quad \text{with} \quad \|s_\epsilon\|_{L^2(\Omega)} \leq C\epsilon^{1/2}$$

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla u_\epsilon\, dx = \int_\Omega A^*\nabla u \cdot \nabla u\, dx + o(1)$$

Disgression: asymptotic expansions for the stress

We assume that

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right), \quad \text{and} \quad \sigma_\epsilon(x) = A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i \sigma_i\left(x, \frac{x}{\epsilon}\right),$$

with $\sigma_i(x, y)$ function of the two variables x and y, periodic in y with period $Y = (0,1)^N$. Plugging this series in the equation we find

$$-\mathrm{div}_y \sigma_0 = 0, \quad -\mathrm{div}_x \sigma_0 - \mathrm{div}_y \sigma_1 = f.$$

On the other hand,

$$\sigma_0(x, y) = A(y)\left(\nabla_x u(x) + \nabla_y u_1(x, y)\right)$$

and

$$\sigma_0(x, y) = A^* \nabla_x u(x) + \tau(x, y) \quad \text{with} \quad \int_Y \tau\, dy = 0.$$

(One can prove that τ is the solution of the dual cell problem.)

Two-phase mixtures

We mix two isotropic constituents $A(y) = \alpha\chi(y) + \beta(1 - \chi(y))$ with a characteristic function $\chi(y) = 0$ or 1.

Let $\theta = \displaystyle\int_Y \chi(y)\, dy$ be the volume fraction of phase α and $(1-\theta)$ that of phase β.

Definition 7.6. We define the set G_θ of all homogenized tensors A^* obtained by homogenization of the two phases α and β in proportions θ and $(1-\theta)$.

Of course, we have $G_0 = \{\beta\}$ and $G_1 = \{\alpha\}$.

But usually, G_θ is a (very) large set of tensors (corresponding to different choices of $\chi(y)$).

Non-periodic case

Homogenization works for non-periodic media too.

Let $\chi_\epsilon(x)$ be a sequence of characteristic functions ($\epsilon \neq$ period).

For $A_\epsilon(x) = \alpha\chi_\epsilon(x) + \beta\,(1 - \chi_\epsilon(x))$ and $f \in L^2(\Omega)$ we consider

$$\begin{cases} -\mathrm{div}\,(A_\epsilon(x)\nabla u_\epsilon) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 7.7. There exists a subsequence, a density $0 \leq \theta(x) \leq 1$ and an homogenized tensor $A^*(x)$ such that χ_ϵ converges "in average" (weakly) to θ, A_ϵ **converges in the sense of homogenization** to A^*, i.e., $\forall f \in L^2(\Omega)$, u_ϵ converges in $L^2(\Omega)$ to the solution u of the homogenized problem

$$\begin{cases} -\mathrm{div}\,(A^*(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, for any $x \in \Omega$, $A^*(x)$ belongs to the set $G_{\theta(x)}$, defined above.

Disgression: weak convergence or "in average"

Let $\chi_\epsilon(x)$ be a sequence of characteristic functions, $\chi_\epsilon \in L^\infty(\Omega; \{0,1\})$.

Let $\theta(x)$ be a density, $\theta \in L^\infty(\Omega; [0,1])$.

The sequence χ_ϵ is said to weakly converge to θ, and we write $\chi_\epsilon \rightharpoonup \theta$, if

$$\lim_{\epsilon \to 0} \int_\Omega \chi_\epsilon(x)\phi(x)\,dx = \int_\Omega \theta(x)\phi(x)\,dx \quad \forall \phi \in C_c^\infty(\Omega).$$

Lemma. For any sequence χ_ϵ of characteristic functions, there exists a subsequence and a limit density θ such that this subsequence weakly converges to this limit.

Remark. The space $C_c^\infty(\Omega)$ can be replaced by $L^1(\Omega)$ or any intermediate space of functions defined in Ω.

Application to shape optimization

Let χ_ϵ be a sequence (minimizing or not) of characteristic functions. We apply the preceding result

$$\chi_\epsilon(x) \rightharpoonup \theta(x) \quad A_\epsilon(x) \xrightharpoonup{H} A^*(x)$$

$$J(\chi_\epsilon) = \int_\Omega j(u_\epsilon)\,dx \to \int_\Omega j(u)\,dx = J(\theta, A^*),$$

with u, solution of the homogenized state equation

$$\begin{cases} -\mathrm{div}\,(A^*\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, the objective function is unchanged when:

$$J(\theta, A^*) = \int_\Omega f u\,dx, \quad \text{or} \quad J(\theta, A^*) = \int_\Omega |u - u_0|^2 dx.$$

Homogenized formulation of shape optimization

We define the set of admissible homogenized shapes

$$\mathcal{U}^*_{ad} = \left\{ (\theta, A^*) \in L^\infty\left(\Omega; [0,1] \times \mathbb{R}^{N^2}\right), A^*(x) \in G_{\theta(x)} \text{ in } \Omega, \int_\Omega \theta(x)\, dx = V_\alpha \right\}.$$

The relaxed or homogenized optimization problem is

$$\inf_{(\theta, A^*) \in \mathcal{U}^*_{ad}} J(\theta, A^*).$$

Remarks

- $\mathcal{U}_{ad} \subset \mathcal{U}^*_{ad}$ when $\theta(x) = \chi(x) =$ or 1.
- We have enlarged the set of admissible shapes.
- One can prove that the relaxed problem always admit an optimal solution.
- We shall exhibit very efficient numerical algorithms for computing homogenized optimal shapes.
- Homogenization does not change the problem: homogenized shapes are just the characterization of limits of sequences of classical shapes

$$\lim_{\epsilon \to 0} J\left(\chi\left(\frac{x}{\epsilon}\right)\right) = J(\theta, A^*).$$

- We need to find an explicit characterization of the set G_θ.

Strategy of the course

The goal is to find the set G_θ of all composite materials obtained by mixing α and β in proportions θ and $(1-\theta)$.

- One could do numerical optimization with respect to the geometry of the mixture $\chi(y)$ in the unit cell.
- We follow a different (and analytical) path.
- **First,** we build a class of explicit composites (so-called sequential laminates) which will "fill" the set G_θ.
- **Second,** we prove "bounds" on A^* which prove that no composite can be outside our previous guess of G_θ.

7.3 Composite materials

Theoretical study of composite materials:

- In dimension $N = 1$: explicit formula for A^*, the so-called harmonic mean.
- In dimension $N \geq 2$, for two-phase mixtures: explicit characterization of G_θ thanks to the variational principle of Hashin and Shtrikman.

Underlying assumptions:

- Linear model of conduction or membrane stiffness (it is more delicate for linearized elasticity and very few results are known in the non-linear case).
- Perfect interfaces between the phases (continuity of both displacement and normal stress): no possible effects of delamination or debonding.

$$\boxed{\text{Dimension } N = 1}$$

Cell problem: $$\begin{cases} -\Big(A(y)\,(1+w'(y))\Big)' = 0 & \text{in } [0,1] \\ y \to w(y) & \text{1-periodic} \end{cases}$$

We explicitly compute the solution

$$w(y) = -y + \int_0^y \frac{C_1}{A(t)}dt + C_2 \quad \text{with} \quad C_1 = \left(\int_0^1 \frac{1}{A(y)}dy\right)^{-1},$$

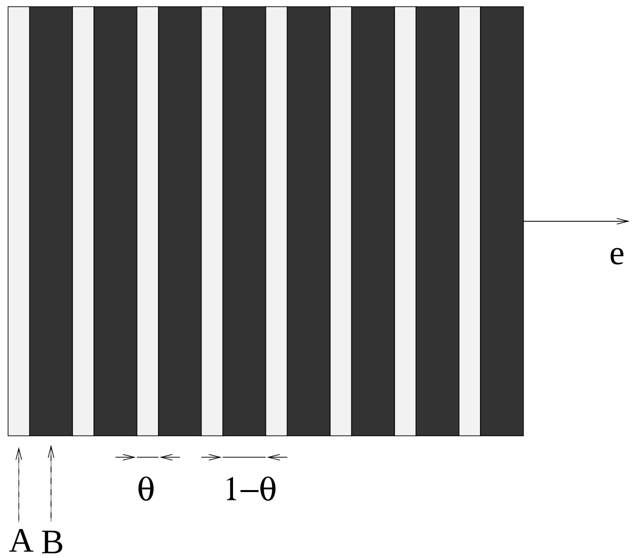
The formula for A^* is $A^* = \displaystyle\int_0^1 A(y)\,(1+w'(y))^2\,dy$, which yields the harmonic mean of $A(y)$

$$A^* = \left(\int_0^1 \frac{1}{A(y)}dy\right)^{-1}.$$

Important particular case:

$$A(y) = \alpha\chi(y) + \beta\,(1-\chi(y)) \quad \Rightarrow \quad A^* = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$$

Simple laminated composites



In dimension $N \geq 2$ we consider parallel layers of two isotropic phases α and β, orthogonal to the direction e_1

$$\chi(y_1) = \begin{cases} 1 & \text{if } 0 < y_1 < \theta \\ 0 & \text{if } \theta < y_1 < 1, \end{cases} \qquad \text{with} \quad \theta = \int_Y \chi \, dy.$$

We denote by A^* the homogenized tensor of $A(y) = \alpha\chi(y_1) + \beta\,(1 - \chi(y_1))$.

Lemma 7.9. Define $\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$ and $\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta$. We have

$$A^* = \begin{pmatrix} \lambda_\theta^- & & & 0 \\ & \lambda_\theta^+ & & \\ & & \ddots & \\ 0 & & & \lambda_\theta^+ \end{pmatrix}$$

Interpretation (resistance = inverse of conductivity). Resistances, placed in series (in the direction e_1), average arithmetically, while resistances, placed in parallel (in directions orthogonal to e_1) average harmonically.

Proof. We explictly compute the solutions $(w_i)_{1\leq i\leq N}$ of the cell problems.

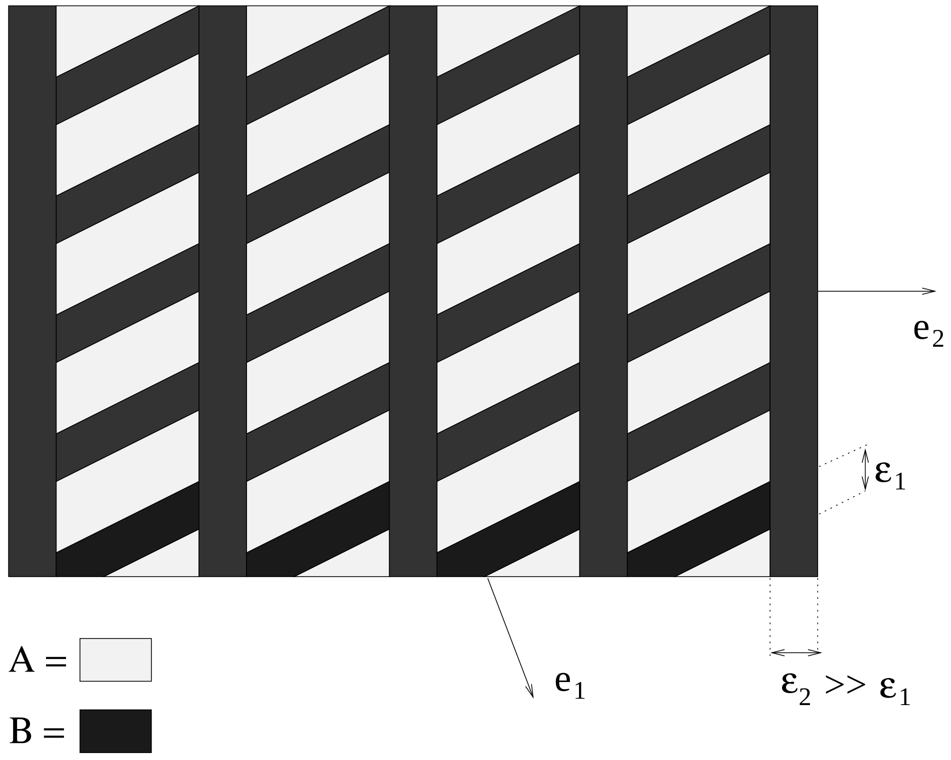
For $i=1$ we find $w_1(y)=w(y_1)$ with w the uni-dimensional solution.

For $2\leq i\leq N$ we find that $w_i(y)\equiv 0$ since, in the weak sense, we have

$$\mathrm{div}_y\Big(\alpha\chi(y_1)e_i+\beta\left(1-\chi(y_1)\right)e_i\Big)=0 \quad \text{in} \quad Y,$$

because the normal component (to the interface) of the vector $(\alpha\chi+\beta(1-\chi))e_i$ is continuous (actually zero) through the interface between the two phases.

Sequential laminated composites



We laminate again a laminated composite with one of the pure phases.

Simple laminate of two non-isotropic phases

Lemma 7.11. The homogenized tensor A^* of a simple laminate made of A and B in proportions θ and $(1-\theta)$ in the direction e_1 is

$$A^* = \theta A + (1-\theta)B - \frac{\theta(1-\theta)\ (A-B)e_1 \otimes (A-B)^t e_1}{(1-\theta)Ae_1 \cdot e_1 + \theta B e_1 \cdot e_1}.$$

If we assume that $(A-B)$ is invertible, then this formula is equivalent to

$$\boxed{\theta \left(A^* - B\right)^{-1} = \left(A-B\right)^{-1} + \frac{(1-\theta)}{Be_1 \cdot e_1} e_1 \otimes e_1}$$

Proof. By definition

$$A^*_{ji} = \int_Y A(y)(e_i + \nabla_y w_i) \cdot e_j \, dy = \int_Y A(y) \left(e_i + \nabla_y w_i(y)\right) \cdot \left(e_j + \nabla_y w_j(y)\right) dy,$$

namely

$$A^* e_i = \int_Y A(y)(e_i + \nabla_y w_i) \, dy.$$

Consequently, $\forall \xi \in \mathbb{R}^N$, we have

$$A^* \xi = \int_Y A(y) \left(\xi + \nabla_y w_\xi\right) dy,$$

with $w_\xi(y) = \sum_{i=1}^{N} \xi_i w_i(y)$ solution of

$$\begin{cases} -\mathrm{div}_y \left(A(y) \left(\xi + \nabla w_\xi(y)\right)\right) = 0 & \text{in } Y \\ y \to w_\xi(y) & Y\text{-periodic.} \end{cases}$$

Main idea: defining $u(y) = \xi \cdot y + w_\xi(y)$ we seek a solution, the gradient of which is constant in each phase

$$\nabla u(y) = a\chi(y_1) + b\Big(1 - \chi(y_1)\Big),$$

$$\Rightarrow u(y) = \chi(y_1)\,(c_a + a \cdot y) + (1 - \chi(y_1))\,(c_b + b \cdot y)\,.$$

Let Γ be the interface between the two phases.

By continuity of u through Γ

$$c_a + a \cdot y = c_b + b \cdot y$$

$$\Rightarrow (a - b) \cdot x = (a - b) \cdot y \quad \forall\, x, y \in \Gamma.$$

Since $(x - y) \perp e_1$, there exists $t \in \mathbb{R}$ such that $b - a = te_1$.

By continuity of $A\nabla u \cdot n$ through Γ

$$Aa \cdot e_1 = Bb \cdot e_1.$$

(In particular, it implies $-\mathrm{div}(A(y)\nabla u) = 0$ in the weak sense.)

We deduce the value of $t = \dfrac{(A-B)a \cdot e_1}{Be_1 \cdot e_1}$.

Since w_ξ is periodic, it satisfies $\int_Y \nabla w_\xi \, dy = 0$, thus

$$\int_Y \nabla u \, dy = \theta a + (1-\theta) b = \xi.$$

With these two equations we can evaluate a and b in terms of ξ.

On the other hand, by definition of A^* we have

$$A^* \xi = \int_Y A(y) \left(\xi + \nabla w_\xi \right) dy = \int_Y A(y) \nabla u \, dy = \theta A a + (1-\theta) B b.$$

An easy computation yields the desired formula

$$A^* \xi = \theta A \xi + (1-\theta) B \xi - \frac{\theta(1-\theta)(A-B)\xi \cdot e_1}{(1-\theta) A e_1 \cdot e_1 + \theta B e_1 \cdot e_1} (A-B) e_1$$

The other formula is a consequence of: M invertible implies

$$\big(M + c(Me) \otimes (M^t e) \big)^{-1} = M^{-1} - \frac{c}{1 + c(Me \cdot e)} e \otimes e.$$

Sequential lamination

We laminate again the preceding composite with always the same phase B.

Recall that the homogenized tensor A_1^* of a simple laminate is

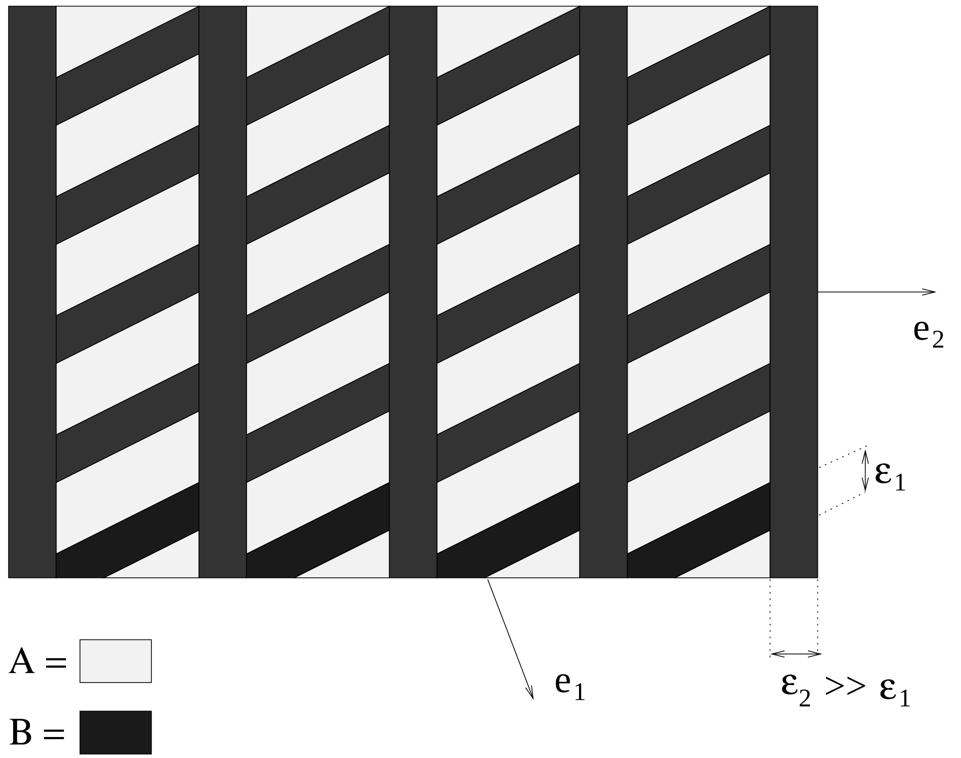
$$\theta\left(A_1^* - B\right)^{-1} = (A-B)^{-1} + (1-\theta)\frac{e_1 \otimes e_1}{Be_1 \cdot e_1}.$$

Lemma 7.14. If we laminate p times with B, we obtain a rank-p sequential laminate with matrix B and inclusion A, in proportions $(1-\theta)$ and θ

$$\theta\left(A_p^* - B\right)^{-1} = (A-B)^{-1} + (1-\theta)\sum_{i=1}^{p} m_i \frac{e_i \otimes e_i}{Be_i \cdot e_i}.$$

with

$$\sum_{i=1}^{p} m_i = 1 \text{ and } m_i \geq 0,\ 1 \leq i \leq p.$$



- A appears only at the first lamination: it is thus surrounded by B. In other words, A =inclusion and B = matrix.
- The thickness scales of the layers are very different between two lamination steps.
- Lamination parameters (m_i, e_i).

Proof. By recursion we obtain A_p^* by laminating A_{p-1}^* and B in the direction e_p and in proportions θ_p, $(1-\theta_p)$, respectively

$$\theta_p \left(A_p^* - B\right)^{-1} = \left(A_{p-1}^* - B\right)^{-1} + (1-\theta_p)\frac{e_p \otimes e_p}{Be_p \cdot e_p}.$$

Replacing $(A_{p-1}^* - B)^{-1}$ in this formula by the similar formula defining $(A_{p-2}^* - B)^{-1}$, and so on, we obtain

$$\left(\prod_{j=1}^{p} \theta_j\right) \left(A_p^* - B\right)^{-1} = (A - B)^{-1} + \sum_{i=1}^{p} \left((1-\theta_i)\prod_{j=1}^{i-1} \theta_j\right) \frac{e_i \otimes e_i}{Be_i \cdot e_i}.$$

We make the change of variables

$$(1-\theta)m_i = (1-\theta_i)\prod_{j=1}^{i-1} \theta_j \quad 1 \le i \le p$$

which is indeed one-to-one with the constraints on the m_i's and the θ_i's $(\theta = \prod_{i=1}^{p} \theta_i)$.

The same can be done when exchanging the roles of A and B.

Lemma 7.15. A rank-p sequential laminate with matrix A and inclusion B, in proportions θ and $(1-\theta)$, is defined by

$$(1-\theta)\left(A_p^* - A\right)^{-1} = (B-A)^{-1} + \theta \sum_{i=1}^{p} m_i \frac{e_i \otimes e_i}{Ae_i \cdot e_i}.$$

with

$$\sum_{i=1}^{p} m_i = 1 \text{ and } m_i \geq 0,\ 1 \leq i \leq p.$$

Remark. Sequential laminates form a very rich and explicit class of composite materials which, as we shall see, describe completely the boundaries of the set G_θ.