MAP562 Optimal design of structures

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Exercise 1

Let Ω be a regular bounded and connected open set of $\mathbb{R}^n$. Let Γ_N and Γ_D be boundary parts such that $\partial\Omega = \Gamma_N \cup \Gamma_D$ of negligible intersection , each of them being of non zero surface measure. Let $g \in L^2(\Gamma_N)$ and $f \in L^2(\Omega)$. We consider the Partial Differential Equation

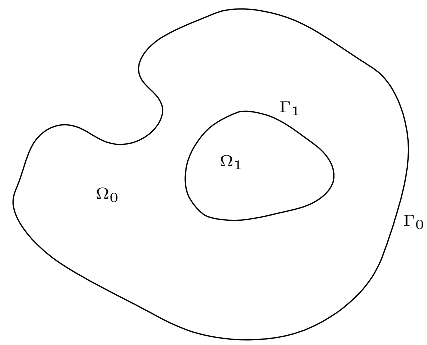
$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = g \text{ on } \Gamma_N. \end{cases} \tag{1}$$

1. Derive the variational formulation associated to (1).
2. Prove that there exists a unique solution u to the variational problem obtained. To this end, we could admit the following Lemma.

   Lemma 1. *Let Ω be a connected open set of $\mathbb{R}^N$. Let $u \in H^1(\Omega)$ such that $\nabla u = 0$ then u is constant over Ω.*
3. Assume that the solution u of the variational formulation is smooth. Then prove that it is also a solution of the initial PDE (1).
4. Suppose that $\Gamma_D = 0$ such that a pure Neumann problem is obtained. Show that a solution of the variational formulation exists only if the data satisfy a compatibility condition. Study the well-posedness of the variational formulation (in particular the coercivity) in the space $V = \{v \in H^1(\Omega) \text{ such that } \int_\Omega v(x)dx = 0\}$.

Exercise 2

In this exercise we deal with **non local boundary conditions**. Let Ω_0, Ω_1, be open bounded sets of $\mathbb{R}^2$ such that $\overline{\Omega}_1$ is included in Ω_0 assumed to be connected. We denote by Ω the open set $\Omega_0 - \overline{\Omega}_1$.



Let f be a map in $L^2(\Omega)$ and $q \in L^\infty(\Omega)$. We would like to establish the existence of a solution u of the following problem

$$\begin{cases} -\Delta u + q(x)u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = -\alpha \displaystyle\int_{\Gamma_1} u ds & \text{on } \Gamma_1 \\ u(x) = 0 & \text{on } \Gamma_0 \end{cases}$$

where α is a non negative real, $\Gamma_0 = \partial\Omega_0$ and $\Gamma_1 = \partial\Omega_1$.

Remark 1. The above situation is a simplification of a heat transfer problem. Therefore, the boundary conditions are also known as *nonlocal radiation boundary conditions.*

1. *Variational Formulation.* Prove that all regular solutions of the PDE are solutions of a variational problem to determine. Conversely, prove that all regular solutions of the variational problem are solutions of the PDE.
2. *Existence.* We assume that there exists a positive real q_0 such that for almost all $x \in \Omega$,

$$q(x) \geq q_0.$$

Prove that there exists a unique solution u to the variational problem and that this solution depends continuously on the data.
3. *Poincaré like inequality.* Prove by contradiction (reductio ad absurdum), that there exists a constant C such that for all u such that $u(x) = 0$ on Γ_0,

$$\|u\|_{H^1(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Can the conditions imposed to α and q be relaxed and still preserve the existence result ?

Exercise 3

Plan for practical exercises with FreeFem++:

- Studying the Poisson problem
- Studying an elastic beam
- Further properties of finite elements

The programs will be available on

`http://www.cmap.polytechnique.fr/~MAP562/`