MAP562 Optimal design of structures

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Exercise 1

Eigenvalues of the Laplacian.
In this exercise we study the eigenvalues of the Laplacian. Let Ω be a regular bounded and connected open set of $\mathbb{R}^n$. A real λ is said to be an eigenvalue of the Laplacian with Dirichlet boundary conditions if there exists a non zero function $u \in H_0^1(\Omega)$ such that

$$-\Delta u = \lambda u.$$

The function u is called the eigenfunction (or eigenmode) associated to λ.

1. *Positivity of the eigenvalues.* Prove that all the eigenvalues of the Laplacian with Dirichlet BC are non negative (and even positive).
2. *First eigenvalue.* Let β defined by

$$\beta = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

   Prove that β is positive and that there exists $u \in H_0^1(\Omega)$ such that

$$\beta = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

   What is the relationship between β and the Poincaré inequality ?
3. *First eigenfunction.* Prove that the functional

$$J(u) = \int_\Omega |u|^2 dx$$

   is Gâteaux differentiable, i.e., there exists a continuous linear form L on $H_0^1(\Omega)$ such that for all $w \in H_0^1(\Omega)$,

$$\lim_{\delta \to 0} \frac{J(u + \delta w) - J(u)}{\delta} = L(w)$$

   Prove that the functional

$$F(u) = \int_\Omega |\nabla u|^2 dx$$

is also Gâteaux differentiable. Deduce that the function u maximizing J on the set

$$K = \{v \in H_0^1(\Omega) : F(u) \leq 1\},$$

is an eigenfunction of the Laplacian with Dirichlet boundary conditions.

4. Implement in FreeFem++ a projected gradient algorithm for

$$\min_{u \in H_0^1(\Omega)} \int_\Omega |\nabla u|^2 \, dx,$$

under the constraint $\int_\Omega u^2 \, dx = 1$.

Exercise 2

Quadratic optimization.

In this exercise we recapitulate basic properties of quadratic optimization. Consider

$$\min_{x \in \mathbb{R}^n, \ Bx = c} J(x) = \frac{1}{2} Ax \cdot x - b \cdot x$$

where A is a $n \times n$ symmetric, positive definite matrix, $b \in \mathbb{R}^n$, B is a $m \times n$ matrix with rank equal to $m \leq n$ and $c \in \mathbb{R}^m$.

1. State the optimality condition
2. Deduce the optimal solution

Exercise 3

Optimization of a non-linear system.

If linear PDEs do correctly describe the behavior of physical systems close to their equilibrium state, non linear phenomena can appear in more general situations. The aim of this exercise is to extend the analysis performed on a linear system to some non-linear cases.

Let Ω be a bounded open set of $\mathbb{R}^N$, $p \in \mathbb{N}$ such that $p > 2$ and $f \in L^2(\Omega)$ the source term (or control). We consider the PDE (whose unknown is u)

$$\begin{aligned} -\operatorname{div}((1 + |\nabla u|^{p-2}) \nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

For $|\nabla u| \ll 1$ (small perturbations), we recover the standard Poisson equation.

1. What is the related energy minimization problem? [**Hint:** Multiply by u and integrate the PDE.] We introduce the Banach space $W_0^{1,p}(\Omega)$ defined by

$$W_0^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega), \text{ et } u = 0 \text{ on } \partial\Omega\},$$

Prove that the energy is strictly convex. We admit that the minimization problem has a unique minimizer.

2. State the variational formulation of (1).
3. Compute (formally) the derivative of u with respect to f.
4. We look for the best control $f \in L^2(\Omega)$ so that the state of the system u is as close as possible to a target function u_0. To this end, we introduce the cost function
$$J(f) = \int_\Omega |u(f) - u_0|^2 dx.$$
Compute (formally) the derivative of J with respect to f. Can the expression obtained be used to implement a gradient type algorithm applied to the minimization of J ?
5. In order to obtain a more convenient expression of the gradient of J, we are going to introduce an adjoint state.
a. Reformulate the minimization problem of J as a min-max problem. To this end, the Lagrangian $\mathcal{L}$ associated to the minimization of $\|u - u_0\|^2_{L^2(\Omega)}$ under the constraint "u solution of (1)" will be introduced.
b. Compute the partial derivatives of the Lagrangian introduced.
c. Give a new expression of the differential of J depending on an adjoint state to define. Prove that if u is regular the adjoint system admits an unique solution in $H^1_0(\Omega)$.
Remark: Even if the state equation is non linear, the adjoint problem is always linear.
6. Write down a Newton scheme for the p-Laplace problem (1).

Exercise 4

Newton's method

Let Ω be a smooth bounded open set of $\mathbb{R}^n$. Let $f \in L^2(\Omega)$. Let $a(v)$ be a smooth function from $\mathbb{R}$ into $\mathbb{R}$ which is bounded uniformly, as well as all its derivatives on $\mathbb{R}$, and satisfies
$$0 < C^- \leq a(v) \leq C^+ < +\infty \quad \forall v \in \mathbb{R}.$$
We consider the minimization problem
$$\min_{v \in H^1_0(\Omega)} E(v) = \frac{1}{2} \int_\Omega a(v(x)) |\nabla v(x)|^2 \, dx - \int_\Omega f(x)\, v(x)\, dx \,. \tag{2}$$
In the sequel $u \in H^1_0(\Omega)$ is assumed to be a smooth function.

1. Compute the first order directional derivative of E at u in the direction of v, that we shall denote by $\langle E'(u), v\rangle$.
2. Show that the first-order optimality condition for (2) is a variational formulation for a non-linear partial differential equation, which should be explicitly exhibited.

3. Compute the second order derivative of E at u, in the directions v and w, that we shall denote by $E''(u)\big(v,w\big)$.
4. Prove that the bilinear form $(v,w) \to E''(u)\big(v,w\big)$ is symmetric and continuous on $H_0^1(\Omega)$. From now on we assume that the first and second order derivatives of $v \to a(v)$ are uniformly small on $\mathbb{R}$. Deduce that the bilinear form is coercive on $H_0^1(\Omega)$. Hint: use integration by parts and the Poincaré inequality in Ω.
5. Define explicitly the Newton algorithm for minimizing (2).
6. Prove that, at each iteration of the Newton algorithm, there exists indeed a unique descent direction. Hint: use question 4 and assume that the previous iterate is a smooth function.

Exercise 5

A first optimization problem.

In this exercise we consider a simplied heat optimization problem. Specifically, let $u \in \mathbb{R}$ and investigate the minimization of the cost functional:

$$J(u) = \int_\Omega |T - T_0|^2 \, dx$$

where the forward problem is given by

$$\begin{aligned} -\Delta T &= 1_\omega u \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and $\Omega = (0,10)^2$ and ω a ball with radius 1.

1. Show that
$$T(u) = T(1)u$$
2. Derive the variational form of the forward problem.
3. What is the physical meaning of the cost functional?
4. Compute the derivatives of $T(u)$ and $J(u)$ w.r.t. u.
5. Formulate a gradient algorithm to solve the minimization problem.

Exercise 6

Plan for practical exercises with FreeFem++:

- Eigenvalue problem (changing the domain size and boundary conditions);
- A nonlinear problem (p-Laplace) using a gradient method and Newton's method.

The programs will be available on

`http://www.cmap.polytechnique.fr/~MAP562/`