MAP562 Optimal design of structures

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Exercise 2 (short version), Jan 11, 2017
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In this short version, most theoretical questions have been removed and the focus is on the developments that are required to implement the given problems into a software. Therefore, the goals are always the same: deriving the variational formulation, formulating the optimization problem, computing derivatives, designing solution algorithms.

Exercise 1

Quadratic optimization (already discussed in class).
In this exercise we recapitulate basic properties of quadratic optimization. Consider

$$\min_{x\in\mathbb{R}^n,\ Bx=c} J(x) = \frac{1}{2}Ax\cdot x - b\cdot x$$

where A is a $n\times n$ symmetric, positive definite matrix, $b\in\mathbb{R}^n$, B is a $m\times n$ matrix with rank equal to $m\le n$ and $c\in\mathbb{R}^m$.

1. State the optimality condition
2. Deduce the optimal solution

Exercise 2

Minimization of the p-Laplace problem
If linear PDEs do correctly describe the behavior of physical systems close to their equilibrium state, non linear phenomena can appear in more general situations. The aim of this exercise is to extend the analysis performed on a linear system to some non-linear cases.

Let Ω be a bounded open set of $\mathbb{R}^N$, $p\in\mathbb{N}$ such that $p>2$ and $f\in L^2(\Omega)$ the source term (or control). We consider the PDE (whose unknown is u)

$$\begin{aligned} -\operatorname{div}((1+|\nabla u|^{p-2})\nabla u) &= f \text{ in } \Omega\\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

For $|\nabla u|\ll 1$ (small perturbations), we recover the standard Poisson equation.

1. We introduce the Banach space $W_0^{1,p}(\Omega)$ defined by

   $$W_0^{1,p}(\Omega) = \{u\in L^p(\Omega) : \nabla u\in L^p(\Omega),\ \text{et } u=0 \text{ on } \partial\Omega\},$$

   Then, we can associate the corresponding energy functional:

   $$J(u) = \int \frac{|\nabla u|}{2} + \frac{|\nabla u|^p}{p} - fu$$

- Justify that $J(u)$ is related to the PDE; and that the PDE represents the first-order derivative, i.e., $J'(u)(\phi) = a(u, \phi)$. Thus it is equivalent to say that we minimize $J(u)$ or that we solve for a root of the PDE.
- Prove that the energy is strictly convex. Therefore, we admit that the minimization problem has a unique minimizer.

2. State the variational formulation of (1).
3. Formulate a gradient algorithm to find the solution u that minimizes $J(u)$.
4. Formulate a Newton algorithm to find the solution u that minimizes $J(u)$. Establish a justification that there exists indeed a unique descent direction. Hint: Consider the well-posedness of the second-order derivative and assume that the previous solution u^n is smooth enough.

Exercise 3

A second nonlinear problem

Let Ω be a smooth bounded open set of $\mathbb{R}^n$. Let $f \in L^2(\Omega)$. Let $a(v)$ be a smooth function from $\mathbb{R}$ into $\mathbb{R}$ which is bounded uniformly, as well as all its derivatives on $\mathbb{R}$, and satisfies

$$0 < C^- \leq a(v) \leq C^+ < +\infty \quad \forall\, v \in \mathbb{R}.$$

We consider the minimization problem

$$E(u) = \min_{v \in H_0^1(\Omega)} E(v) = \frac{1}{2} \int_\Omega a(v(x)) |\nabla v(x)|^2 \, dx - \int_\Omega f(x)\, v(x)\, dx\,. \tag{2}$$

In the sequel $u \in H_0^1(\Omega)$ is assumed to be a smooth function.

1. Compute the first order directional derivative of E at u in the direction of v, that we shall denote by $\langle E'(u), v \rangle$.
2. Show that the first-order optimality condition for (2) is a variational formulation for a non-linear partial differential equation, which should be explicitly exhibited.
3. Compute the second order derivative of E at u, in the directions δu and ϕ, that we shall denote by $E''(u)\big(\delta u, \phi\big)$.
4. **(Optional)** Prove that the bilinear form $(v, w) \to E''(u)\big(v, w\big)$ is symmetric and continuous on $H_0^1(\Omega)$. From now on we assume that the first and second order derivatives of $v \to a(v)$ are uniformly small on $\mathbb{R}$. Deduce that the bilinear form is coercive on $H_0^1(\Omega)$. Hint: use integration by parts and the Poincaré inequality in Ω.
5. Define a gradient algorithm for minimizing (2).
6. Define a Newton algorithm for minimizing (2).

7. Prove that, at each iteration of the Newton algorithm, there exists indeed a unique descent direction. Hint: use question 4 and assume that the previous iterate is a smooth function.

Exercise 4

A first optimization problem: optimal control

In this exercise we consider a simplified heat optimal control problem. In some region ω in the domain Ω we have a heat source u (the control variable). The goal is to match a certain, given, temperature T_0. This leads to the cost functional:

$$J(u) = \int_\Omega |T(u) - T_0|^2 \, dx$$

Consequently we are interested in minimizing $J(u)$. The heat distribution itself (the so-called state equation) is modeled by a diffusion equation

$$\begin{aligned} -\Delta T &= 1_\omega u \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and $\Omega = (0,1)^2$ and ω a ball with radius 0.1. The resulting problem statement is a so-called **PDE-constrained optimization problem.**

1. Show that
$$T(u) = T(1)u$$
2. Derive the variational form of the forward problem.
3. Compute the derivatives of $T(u)$ and $J(u)$ w.r.t. u. Hint: For the derivative of $J(u)$ we need to employ two times the relation $T(u) = T(1)u$. In fact in most cases such an explicit relation does not exist (and here thanks to the fact that u is constant) and then a crucial aspect in derivative-based optimization is the evaluation of the 'inner' derivatives.
4. Formulate a gradient algorithm to solve the minimization problem (including all necessary steps!).