MAP562 Optimal design of structures

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Exercise 1

A second heat optimization problem.
In this exercise we consider another simplied heat optimization problem. For a given smooth bounded domain $\Omega \subset\in \mathbb{R}^d$ and for any $z \in \mathbb{R}^d$, we investigate the minimization of the cost functional:

$$J(z) = \int_\Omega |T - T_0|^2 \, dx$$

where $T_0(x)$ is a given smooth temperature field and $T \equiv T(x, z)$ is the solution of the following boundary value problem for the x variable

$$\begin{aligned} -\Delta T &= f_z \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f_z(x) = f(x - z)$ and $f(x)$ a smooth non-negative function with compact support. In the numerical applications, take $d = 2$, $T_0(x) \equiv 0.1$, $\Omega = (0, 10)^2$ and

$$f(x) = \begin{cases} 1 - |x|^2 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

1. For a given direction $e \in \mathbb{R}^d$, find the problem solved by the directional derivative $v \equiv \nabla_z T \cdot e$.
2. Compute the derivative of $J(z)$ in the direction e in terms of v.
3. Introduce the Lagrangian corresponding to the minimization of $J(z)$.
4. Deduce the adjoint problem for this minimization problem.
5. Find an explicit formula for the gradient $\nabla_z J(z)$.
6. Implement a gradient algorithm for this problem in FreeFem++.

Exercise 2

Optimization of a heater (part II from TD2).
We continue our investigation of an optimal heat source. As reminder, we recall that we want to optimize the temperature T of a room Ω by the mean of a heater located in $\omega \subset \Omega$ whose heat flux $v(x) \in L^2(\omega)$ is controlled. In extension to the previous problem (TD 2), an air stream of velocity u fills the room. We assume that the air is incompressible, that is u is assumed to be divergence free. Moreover, u is also assumed to be regular. The temperature on the boundary of the room is equal to the external temperature, assumed to be zero. The temperature in the room satisfies the following convection-diffusion equation

$$-\Delta T + u \cdot \nabla T = 1_\omega v \quad \text{ in } \Omega.$$

a. Determine the variational problem satisfied by the solution $T \equiv T(v)$ of the convection-diffusion equation. Prove that it admits an unique solution depending continuously on the data.
b. We want to optimize the value of the function $v(x)$ in order to maintain the temperature T to a desired value T_0. To this end, we introduce the cost function

$$J(v) = \int_\Omega |T(v) - T_0|^2 dx$$

which we want to minimize. Compute the derivatives of $T(v)$ and $J(v)$ with respect to v. Can the expressions obtained be used to implement a gradient type algorithm applied to the minimization of J ?
c. The gradient of J can be explicitly expressed by introducing an adjoint state. To this end, we first introduce the Lagrangian

$$\mathcal{L}(v, T, p) = \int_\Omega |T - T_0|^2 dx + \int_\Omega \nabla p \cdot \nabla T + (u \cdot \nabla T) p dx - \int_\omega p v dx,$$

where $T, p \in H_0^1(\Omega)$ and $v \in L^2(\omega)$. Prove that finding the minimizer of J is equivalent of solving the following min-max problem

$$\min_{v, T \in H_0^1(\Omega)} \sup_{p \in H_0^1(\Omega)} \mathcal{L}(v, T, p).$$

Determine the derivatives of $\mathcal{L}$ with respect to T and v.
d. By noticing that $J(v) = \mathcal{L}(v, T(v), p)$ for all $p \in H_0^1(\Omega)$, find a new expression of the differential of J depending on the derivatives of $\mathcal{L}$. Prove that a particular choice for p enables us to get rid of the term that depends on $\partial T / \partial v$. Deduce a new (and workable) version of the gradient of J.

Exercise 3

Optimal control of ODEs.

We consider the following linear system of ordinary differential equations, the solution of which (called the state) is a function $y(t)$ with values in $\mathbb{R}^N$

$$\left\{\begin{array}{l}\dfrac{dy}{dt} = Ay + Bv + f \text{ for } 0 \le t \le T \\ y(0) = y_0\end{array}\right. \tag{1}$$

where $y_0 \in \mathbb{R}^N$ is the initial state of the system, $f(t) \in \mathbb{R}^N$ is a source term, $v(t) \in \mathbb{R}^M$ is the control which allows us to act on the system, A and B are two constant matrices of respective dimensions $N \times N$ and $N \times M$. We shall denote by y_v the solution of (1).

We look for the optimal control v which minimizes the quadratic functional

$$\begin{aligned} J(v) = & \int_0^T Rv(t) \cdot v(t)dt + \int_0^T Q(y_v - z)(t) \cdot (y_v - z)(t)dt \\ & + D\left(y_v(T) - z_T\right) \cdot \left(y_v(T) - z_T\right), \end{aligned}$$

where $z(t)$ is a target trajectory, z_T is a target final position, and R, Q, D are three symmetric non-negative matrices, from which only R is assumed to be positive definite. Let K be a closed non-empty convex set of $\mathbb{R}^M$: we restrict the control to the admissible set $L^2(]0,T[;K)$. The minimization problem is thus

$$\inf_{v(t) \in L^2(]0,T[;K)} J(v). \tag{2}$$

We assume that, if $f(t) \in L^2(]0,T[;\mathbb{R}^N)$, there exists a unique solution of (1) $y_v(t) \in H^1(]0,T[;\mathbb{R}^N)$, which is furthermore continuous in time.

1. Prove that there exists a unique optimal control which minimizes (2).
2. Compute the derivative of the map $v \to y_v$ in the direction $w \in L^2(]0,T[;\mathbb{R}^N)$, that shall be denoted by y'_w.
3. Compute the derivative of $J(v)$ in the direction w in terms of y'_w. Explain why this formula is not useful in practice.
4. To get a simpler formula, we introduce the Lagrangian

$$\begin{aligned} \mathcal{L}(v, y, p) = & \int_0^T Rv(t) \cdot v(t)dt + \int_0^T Q(y - z)(t) \cdot (y - z)(t)dt \\ & + D\left(y(T) - z_T\right) \cdot \left(y(T) - z_T\right) + \int_0^T p \cdot \left(-\frac{dy}{dt} + Ay + Bv + f\right) dt \\ & - p(0) \cdot \left(y(0) - y_0\right), \end{aligned}$$

   defined for any $v \in L^2(]0,T[;\mathbb{R}^N)$, any $y(t) \in H^1(]0,T[;\mathbb{R}^N)$ and any $p(t) \in H^1(]0,T[;\mathbb{R}^N)$. Check that

$$\sup_{p(t) \in H^1(]0,T[;\mathbb{R}^N)} \mathcal{L}(v, y, p) = \left\{\begin{array}{ll} J(v) & \text{if (1) is satisfied,} \\ +\infty & \text{otherwise.} \end{array}\right.$$

5. Compute the partial derivative of $\mathcal{L}(v, y, p)$ with respect to y in the direction ϕ. By definition the adjoint system is obtained by writing that this partial derivative is zero for any $\phi \in H^1(]0, T[; \mathbb{R}^N)$ when $y = y_v$. We denoted by p_v the adjoint state. Check that it is a solution of the following ODE system (called the adjoint system)

$$\begin{cases} \dfrac{dp_v}{dt} = -A^* p_v - Q(y_v - z) \text{ for } 0 \leq t \leq T \\ p_v(T) = D(y_v(T) - z_T) \end{cases} \tag{3}$$

where A^* is the adjoint matrix.

6. Compute the partial derivative with respect to v, in the direction w, of $\mathcal{L}(v, y_v, p)$ where p is a function independent of v. In this partial derivative, replace p by the adjoint state p_v and deduce a formula for the derivative of $J(v)$ in the direction w. Check that

$$J'(v) = B^* p_v + Rv \ .$$