MAP562 Optimal design of structures

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Exercise 1

We consider the optimization of a membrane with a constant thickness in a domain $\Omega \subset \mathbb{R}^2$ (open, bounded, and sufficiently regular). The boundary is split into three parts: $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$. The parts Γ_D and Γ_N are fixed and only Γ can vary. We furthermore introduce the admissible set of shapes:

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^2 \text{ such that } (\Gamma_D \cup \Gamma_N) \subset \partial\Omega \right\}.$$

The displacement $u(x)$ of the membrane is the solution of the PDE:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{1}$$

where $f \in L^2(\mathbb{R}^2)$ is a volume force and $g \in L^2(\Gamma_N)$ is a surface traction force. Let $u_0(x) \in L^2(\mathbb{R}^2)$ be a given displacement that we want to match. The objective functional can then be formulated as:

$$\inf_{\Omega \in \mathcal{U}_{ad}} \left\{ J(\Omega) = \int_\Gamma |u - u_0|^2 ds \right\},$$

where u is the solution of (1).

1. Formulate the Lagrangian and deduce the adjoint state.
2. Calculate (formally) the shape derivative of the objective function.
3. If we find $u = u_0$ on Γ for some shape Ω, what is the value of the shape derivative?

Exercise 2

In this exercice we address another example in which the stiffness of an elastic structure is optimized. This goal is achieved by geometric optimization and finding the optimal domain. Let Ω be a bounded connected open set in $\mathbb{R}^d$. The boundary is split into three parts: $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_{free}$. We assume that the body is clamped on Γ_D and subject to surface loads $g \in [H^1(\mathbb{R}^d)]^d$ on Γ_N. On the remaining boundary parts, we deal with traction-free conditions (homogeneous Neumann). Furthermore $f \in [L^2(\mathbb{R}^d)]^d$. The goal of the optimization is the minimization of the compliance of the solid:

$$J(\Omega) = \int_\Omega f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, dx,$$

where $u \equiv u(\Omega) \in [H^1(\Omega)]^d$ denotes the vector-valued displacements of the solid in the equilibrium state.

1. Recall the variational problem verified by the displacement $u(\Omega)$ of the solid.
2. Introduce the Lagrangian $L(\Omega, u, p)$ associated to the corresponding minimization problem.
3. Compute the derivatives of L with respect to u and Ω. Assuming that the solution $u(\Omega)$ admits a shape derivative, deduce an expression for $J'(\Omega)(\theta)$.
4. In the case $f = 0$, show that one option to achieve sufficient stiffness of the solid, is to add new material.
5. Add a volume constraint such that for all admissible shapes, it holds

   $$\int_\Omega 1 dx \leq V_0.$$

   State the optimality condition of the modified system.

Exercise 3

We consider again heat optimization, which we have addressed in the optimal section some weeks ago. Let Ω be a domain standing for a room. In this room we have a heat source and its shape is given by $\omega \subset \Omega$. We assume that the temperature outside the domain is zero. In the heater we are given a constant temperature $T_1 > 0$. The state equation that describes the temperature distribution is given by:

$$\begin{aligned} -\Delta T + u \cdot \nabla T &= 0 \quad \text{in } \Omega \setminus \omega, \\ T &= 0 \quad \text{on } \partial\Omega, \\ T &= T_1 \quad \text{in } \partial\omega. \end{aligned}$$

The goal will be to achieve a constant temperature T_0 in the entire room Ω. To this end, we introduce the cost functional:

$$J(\omega) = \int_{\Omega\setminus\omega} |T(\Omega) - T_0|^2 \, dx,$$

where $T(\Omega)$ is the solution of the state equation. Note that it is ω which is the variable of optimization.

1. Determine the standard variational formulation of the state equation.
2. Recast the variational problem by expressing the constraint $T(\omega) = T_1$ on $\partial\omega$ by introducing a Lagrange multiplier $\lambda(\omega)$.
3. Determine the Lagrangian associated to the minimization problem where we do not use the standard variational formulation of the state equation. Hint: write both the strong forms of the equation and of the boundary condition on ω as contraints. We assume that both $T(\omega)$ and $\lambda(\omega)$ are differentiable with respect to ω. Compute the shape derivative $\langle \frac{\partial J}{\partial \omega}, \theta \rangle$ for all vector fields θ such that $\theta \cdot n = 0$ on $\partial\Omega$.

Exercise 4

We now turn to the practical exercise to study in more detail geometric optimization. Take the FreeFem script `cantilever.edp`, which can be found on the webpage. The goal is compliance minimization in order to obtain optimal stiffness of a cantilever. The cantilever is given in a domain $\mathbb{R}^2$ and clamped on Γ_D. On Γ_N the solid is subject to a traction force. The space for the displacements $u(\Omega)$ is defined as

$$u(\Omega) \in V := \{v \in H^1(\Omega)^2 : v = 0 \text{ on } \Gamma_D\}.$$

The state equation in variational form is given by:

$$\int_\Omega Ae(u) : e(v)\, dx = \int_{\Gamma_N} g \cdot v\, ds \quad \forall v \in V, \tag{2}$$

where A is specified (as in the other exercises in this class) by Hook's law:

$$A\xi = 2\mu\xi + \lambda\, \mathrm{tr}(e(u))I,$$

where μ and λ are the Lamé coefficients. Finally $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The objective functional is given by:

$$J_c(\Omega) = \int_{\Gamma_N} g \cdot u(\Omega)\, ds.$$

As in parametric optimization, the optimization problem is subject to a volume constraint, which we formulate in terms of penalization yielding the modified cost functional:

$$J_1(\Omega) = J_c(\Omega) + \alpha|\Omega|,$$

where α is a Lagrange multiplier. The admissible set is given by

$$U_{ad} := \{\Omega \text{ open in } \mathbb{R}^2 \text{ such that } \Gamma_N \subset \partial\Omega\}.$$

In the following we recapitule the basic steps of the optimization algorithm.

Algorithm

1. We initialize the algorithm with some initial shape $\Omega_0 \in U_{ad}$, which is contained in a large domain D.

2. We define

   $$\theta_n \in W := \{\theta : \mathbb{R}^2 \to \mathbb{R}^2 : \theta = 0 \text{ on } \Gamma_D^0 \cup \Gamma_N \text{ and } \theta \cdot n = 0 \text{ on } \partial D \cup \Gamma_D\}$$

   such that for all $\theta \in W$ it holds:

   $$(\theta_n, \theta)_W + \langle J'(\Omega_n), \theta \rangle = 0.$$

   Here $(\cdot, \cdot)_W$ denotes an appropriate scalar product on W. The shape derivative is given by:

   $$\langle J'(\Omega_n), \theta \rangle = \int_\Gamma (\delta - Ae(u) : e(u))(\theta \cdot n)\, ds.$$

   We notice that we introduce in the corners of Γ_D small 'security zones' which are not subject to optimization. The reason being the here singularities of the solution appear (which are by the way mathematically plausible!). In these zones, we have zero convection $\theta = 0$.

3. We define the new domain (obtained from Ω_n by using θ) for a small given time δt:

   $$\Omega_{n+1} = X_n(\Omega_n, \delta t),$$

   where $X_n : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is described as follows:

   $$\begin{aligned} X_n(x, 0) &= x \quad \text{for all } x \in \mathbb{R}^2, \\ \frac{\partial X_n}{\partial t}(x, t) &= \theta_n(X_n(x, t)) \quad \text{for all } (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \end{aligned}$$

4. Stopping criterion: if $\theta_n < TOL$, we stop. Otherwise we go to Step 2.

We notice that the stepsize δt could be chosen in an adaptive way in order to accelerate the speed of convergence.

1. Read the FreeFem script and become familiar with its implementation. Identify for yourself the state equation, cost functional, and optimization loop.
2. Test different initial domains Ω_0 (different exterior boundaries, different number of holes) and observe the final optimal shapes.