Generalized Radon transforms and mathematical economics

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The first microfounded model of production goes back to (Houthakker, 1955), (Johansen, 1972)

It was fruitfully applied to study of macroeconomic processes (Petrov, Pospelov, Shananin); e.g. impact of technological innovation on production (Johansen, 1972)

Globalization led to qualitative changes in production processes (increase in substitutability of factors), which are out of scope of the classical model

A natural generalization was proposed in (Sato, 1975), (Shananin, 1997), which allows to overcome this problem
Technologies are parametrized by vectors $x \in \mathbb{R}^n_{\geq 0}$.

Each technology $x$ has a capacity $f(x) \geq 0$ (number of production units using this technology).

Technologies are described by the unit cost of production $q_p(x) = q(p_1x_1, \ldots, p_nx_n)$ (here $p$ are the prices of resources).

The maximal possible profit for the industry is given by the profit function $(\Pi_q f)(p_0, p)$ ($p_0$ the price of the final product):

$$(\Pi_q f)(p_0, p) = \int_{\mathbb{R}^n_{\geq 0}} \max\{0, p_0 - q_p(x)\} f(x) \, dx$$
Neoclassical \( n \)-input, 1-output technologies \( Q(n) \): smooth, 1-homogeneous \( q: \mathbb{R}^n_{>0} \to \mathbb{R}_{>0} \) with bounded level sets

Action of \( \mathbb{R}^n_{>0} \) on \( Q(n) \), \( (p, q) \mapsto q_p \):

\[
q_p(x_1, \ldots, x_n) = q(p_1x_1, \ldots, p_nx_n)
\]

Partial composition \( \circ_i: Q(m) \times Q(n) \to Q(m + n - 1) \):

\[
(f \circ_i g)(x_1, \ldots, x_{m+n-1})
= f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots)
\]
Let $q \in Q(n)$, $h: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

- The generalized Radon transform:

  $$(R_q f)(p) = \int_{q_p^{-1}(1)} f(x) \frac{dS_x}{|\nabla q_p(x)|}$$

- Radon-type integral operators:

  $$(R^h_q f)(p) = \int_{\mathbb{R}^n_{\geq 0}} h(q_p(x)) f(x) \, dx$$

- The profit function:

  $$(\Pi_q f)(p_0, p) = \int_{\mathbb{R}^n_{\geq 0}} \max\{0, p_0 - q_p(x)\} f(x) \, dx$$
Main questions

Characterization. Determine the scope of the model. When a given profit function $\Pi$ of an industry can be constructed using our framework, i.e. $\Pi = \Pi_q f$ for some $q, f$?

Uniqueness. If $\Pi = \Pi_q f$, when such a representation is unique?

Inversion. If $\Pi = \Pi_q f$ for unique $q$ and $f$, how to find $q$ and $f$?

Identification. Given trade statistics, how to identify compatible $q$ and $f$?
Main spaces

- Weighted spaces $L^r_c(\mathbb{R}^n_{\geq 0})$ with finite norms
  \[
  \|f\|_{r,c} = \left( \int_{\mathbb{R}^n_{\geq 0}} |f(x)|^r x^{rc-I} \, dx \right)^{1/r}, \quad 1 \leq r < \infty,
  \]
  \[
  \|f\|_{\infty,c} = \inf \{ K \geq 0 : |f(x)x^c| \leq K \text{ for a.e. } x \in \mathbb{R}^n_{\geq 0} \},
  \]
  where $c \in \mathbb{R}^n_{>0}$, $I = (1, \ldots, 1)$

- $R_q$, $R^h_q$, $\Pi_q$ are continuous from $L^r_{l-c}(\mathbb{R}^n_{\geq 0})$ to $L^r_c(\mathbb{R}^n_{\geq 0})$

- The Mellin transform
  \[
  (Mf)(z) = \int_{\mathbb{R}^n_{\geq 0}} x^{z-I} f(x) \, dx, \quad z \in \mathbb{C}^n
  \]
  is isometric from $L^2_c(\mathbb{R}^n_{\geq 0})$ to $L^2(\Re z = c)$
**Question U.** If $\Pi = \Pi_q f$, when such a representation is unique?

Let $S, H \subseteq \mathbb{C}^n$. We say that

- $S$ is 1-meagre in $H$ iff $S \cap H$ is nowhere dense in $H$;
- $S$ is 2-meagre in $H$ iff $S \cap H$ has measure zero in $H$;
- $S$ is $\infty$-meagre in $H$ iff $S \cap H = \emptyset$.

**Theorem (A, Inverse Problems 2016)**

Let $q \in Q(n)$, $c \in \mathbb{R}_0^n$, $r \in \{1, 2, \infty\}$. The following statements are equivalent:

- $\Pi_q$ is injective in $L^r_{I-c} (\mathbb{R}_0^n)$.
- $R_q$ is injective in $L^r_{I-c} (\mathbb{R}_0^n)$.
- The nullset of $Me^{-q}$ is $r$-meagre in the plane $\Re z = c$. 

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*Alexey Agaltsov*  
On the generalized Radon transform
Injectivity: idea

- Recall that \((R_q f)(p) = \int_{q_p(x)=1} f(x) \frac{dS_x}{|\nabla q_p(x)|}\)
- If \(f(x) = f_1(q_{p_1}(x)) + \cdots + f_N(q_{p_N}(x))\), then \(\Pi_q\) is injective:
  \[
  \int_{\mathbb{R}^n_{\geq 0}} |f(x)|^2 dx = \sum_{k=1}^{N} \int_{0}^{\infty} f_k(t)t^{-1}(R_q f)(\frac{p_k}{t}) dt.
  \]
- Generalizes to infinite sums. When functions of the form \(\varphi(q_p(x))\) with varying \(\varphi\) and \(p\) span \(L^1_{1-c}(\mathbb{R}^n_{\geq 0})\), \(L^2_{1-c}(\mathbb{R}^n_{\geq 0})\)?
**Question.** When functions of the form $\varphi(q_p(\cdot))$ with varying $\varphi$, $p$ span $L^1_{I-c}(\mathbb{R}^n_{\geq 0})$, $L^2_{I-c}(\mathbb{R}^n_{\geq 0})$?

$\Downarrow$ Wiener Tauberian theorems: translates $\phi(\cdot - a)$ of $\phi \in L^1(\mathbb{R}^n)$ span $L^1(\mathbb{R}^n)$ iff $\mathcal{F}\varphi(\xi) \neq 0 \ \forall \xi$. Similar in $L^2(\mathbb{R}^n)$.

$\Downarrow$ Define $E_c : \mathbb{C}^{\mathbb{R}^n_{\geq 0}} \to \mathbb{C}^{\mathbb{R}^n}$ by $(E_c f)(y) = e^{cy} f(e^y)$. Then:

$E_c$ is an isometry from $L^r_c(\mathbb{R}^n_{\geq 0})$ to $L^r(\mathbb{R}^n)$, $E_c$ maps the Mellin transform to the Fourier transform $E_c$ maps the action $(p, q) \to q_p$ to the additive translation

**Conclusion.** Functions of the form $\varphi(q_p(\cdot))$ with varying $\varphi$, $p$ span $L^1_{I-c}(\mathbb{R}^n_{\geq 0})$ iff $Me^{-q}(z) \neq 0$ for $\Re z = c$. Similar in $L^2_{I-c}(\mathbb{R}^n_{\geq 0})$. 

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On the generalized Radon transform
A CES function is a function of the form

\[ q(x) = C (a_1 x_1^\alpha + \cdots + a_n x_n^\alpha)^\frac{1}{\alpha}, \]

where \( C > 0, \ a_j \geq 0, \ a_1 + \cdots + a_n = 1 \) (and here \( \alpha \in (0, 1] \)) (Arrow, Solow et al, 1961)

A nested CES function is obtained from CES functions using finite compositions (Sato, 1967). It allows to take into account such effects as *capital-skill complementarity* (Griliches, 1969)


If \( q \) is a nested CES, then \( \Pi_q \) injective in \( L^r_{i-c}, \ r \in \{1, 2, \infty\} \).
Question. How the injectivity behaves with respect to composition \( \circ_i \) of technologies? Define \( \hat{f}(z) = Me^{-f(z)}/\Gamma(\Sigma(z)) \).

\[ \Pi_f \text{ is injective in } L^2_{i-c}(\mathbb{R}^n_{\geq 0}) \text{ iff } \hat{f}(z) \neq 0 \text{ for } \Re z = c \text{ a.e.} \]

\[ \text{One can show that } \hat{f} \circ_i \hat{g}(z \circ_i w) = \hat{f}(z \circ_i \Sigma(w))\hat{g}(w) \]

\[ \text{One can show that } \hat{q}(z) = \frac{a^{-z/\alpha}\Gamma(\Sigma(z))B(z/\alpha)}{\alpha^{n-1}C^{\Sigma(z)}} \text{ for CES } q \]

Conclusion. Injectivity propagates.
Question E. Determine the scope of the model. When a given profit function $\Pi$ of an industry can be constructed using our framework, i.e. $\Pi = \Pi qf$ for some $q, f$?

Question U. If $\Pi = \Pi qf$, when such a representation is unique?

$\Rightarrow f$ is uniquely determined by $q$ iff $Me^{-q}$ has no zeros


Question I. If $\Pi = \Pi qf$ for unique $q$ and $f$, how to find $q$ and $f$?
**Question.** How to decide whether $\Pi$ is in image of $\Pi_q$?

⇓ We know that $\Pi_q = R_{\Pi}^{h''}$, $h''(t) = \max\{0, 1 - t\}$.

**Question’.** How to decide whether $F$ is in image of $R_{\Pi}^{h''}$?

⇓ We know the answer for the Laplace transform $R_{\Pi}^{h'}$, where $h'(t) = e^{-t}$, $q'(x) = x_1 + \cdots + x_n$.

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**Theorem (S. Bernstein, V. Hilbert, S. Bochner)**

*Let $F \in \mathbb{R}^{\mathbb{R}^n_{\geq 0}}$. Then $F = R_{\Pi}^{h'}\mu$ for some finite Borel $\mu \geq 0$ iff $F$ is completely monotone, i.e. $F$ is smooth with non-negative even derivatives and non-positive odd derivatives.*

**Question”’.** How to relate the image of $R_{\Pi}^{h''}$ to the image of $R_{\Pi}^{h'}$?
Question”. How to relate the image of $R_{h''}^q$ to the image of $R_{h'}^q$?

- A remarkable property

$$M(R_{h''}^q F) = \Gamma^{-1} \cdot MF \cdot Me^{-q} \cdot Mh'',$$

$$M(R_{h'}^q F) = \Gamma^{-1} \cdot MF \cdot Me^{-q'} \cdot Mh'$$

- It implies

$$R_{h'}^q F = M^{-1} \frac{Me^{-q'} \cdot Mh'}{Me^{-q} \cdot Mh''} M(R_h^q F)$$

- Explicit formulas:

$$Me^{-q'}(z) = \Gamma(z_1) \cdots \Gamma(z_n),$$

$$Mh'(s) = \Gamma(s),$$

$$Mh''(s) = \frac{1}{s(s+1)}.$$
Set \( T_q = M^{-1} \rho_q M \), where \( \rho_q(z) = \frac{\Gamma(z_1 + \cdots + z_n + 2)\Gamma(z_1)\cdots\Gamma(z_n)}{Me^{-q(z)}} \).

Denote \( Q_c^{\text{reg}}(n) = \{ q \in Q(n) : \rho_q \in L^2 \cup L^\infty(\mathbb{R}z = c) \} \)


Let \( q \in Q_c^{\text{reg}}(n) \), \( c \in \mathbb{R}^n_{>0} \). Then \( \Pi \in \mathbb{R}^{\mathbb{R}^n_{\geq 0}} \) is of the form \( \Pi = \Pi_q \mu \) with Borel \( \mu \geq 0 \) such that \( \int x^{-c}d\mu < \infty \) iff

\[
\Pi \in L^2_c(\mathbb{R}^n_{\geq 0}), \quad T_q \Pi \in L^1_c(\mathbb{R}^n_{\geq 0}), \quad T_q \Pi \text{ is completely monotone}
\]
Question E. Determine the scope of the model. When a given profit function $\Pi$ of an industry can be constructed using our framework, i.e. $\Pi = \Pi_q f$ for some $q, f$?

$\Rightarrow \Pi = \Pi_q f$ for some $f$ iff $T_q \Pi$ is completely monotone, where $T_q$ is a Mellin multiplier given by an explicit formula


Question U. If $\Pi = \Pi_q f$, when such a representation is unique?

$\Rightarrow f$ is uniquely determined by $q$ iff $Me^{-q}$ has no zeros


Question I. If $\Pi = \Pi_q f$ for some $q$ and $f$, how to find $q$ and $f$?
Question. If $\Pi = \Pi_q f$ and $q$ is known, how to find $f$?

- A remarkable property:
  
  $$M(\Pi_q f) = \Gamma^{-1} \cdot Me^{-q} \cdot Mf$$

- $N$-smooth functions $C_c^{N,\sigma}(\mathbb{R}^n_0)$, $N, \sigma > n$, with finite norm

  $$\|f\|_{C_c^{N,\sigma}} = \sup_{|\alpha| \leq N, y \in \mathbb{R}^n} (1 + |y|)^\sigma \frac{\partial|\alpha|u}{\partial y^\alpha}, \quad u(y) = e^{cy}f(e^y)$$

Theorem (A, Proc. of MIPT 2014)

Let $q \in Q(n)$, $Me^{-q}(z) \neq 0$ for $\Re z = c$ a.e., $f \in C_{l-c}^{N,\sigma}(\mathbb{R}^n_0)$, $\Pi = \Pi_q f$. Set $s = z_1 + \cdots + z_n$. Then $f = f_R + f_R^{err}$, where

$$f_R(x) = (2\pi)^{-n} \int_{c+iB_R} \frac{x^{z-i} \Gamma(s+2)}{(Me^{-q})(z)} \cdot (M\Pi)(z) \, dz,$$

$$\|f_R^{err}\|_{C_{l-c}^{N,\sigma}} \leq C(n, N, \sigma) \|f\|_{C_{l-c}^{N,\sigma}} R^{n-N}.$$
Main questions

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$\Rightarrow$ explicit inversion formula

[A, Proceedings of MIPT 2014]
The class $Q_{CES}(n)$ of CES technologies

$$q(x) = C(a_1x_1^\alpha + \cdots + a_nx_n^\alpha)\frac{1}{\alpha},$$

where $C > 0$, $a_j \geq 0$, $a_1 + \cdots + a_n = 1$ (and $\alpha \in (0, 1]$).

⇒ If $\Pi_{q_1}f_1 = \Pi_{q_2}f_2$, $q_1 \neq q_2$ are $Q_{CES}(n)$, and $f_1$, $f_2 \geq 0$ decay fast, then $f_1 = f_2 = 0$. Otherwise, there are counter-examples [A, Proceedings of MIPT 2014]

⇒ There is a more simple characterization [A, Proceedings of MIPT 2013]
[1] A. Agaltsov
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Inverse problems in models of resource distribution