ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Exercise 1

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $1 \leq d \leq 3$. Its boundary $\partial\Omega$ is decomposed in two disjoint subsets, with non-zero surface measure, such that $\partial\Omega = \Gamma \cup \Gamma_D$. For given $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and $\alpha > 0$, consider the following energy

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega v^4 dx + \frac{\alpha}{2}\int_\Gamma v^2 dx - \int_\Omega f v dx - \int_\Gamma g v ds$$

defined for $v \in H$ with

$$H = \left\{ v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \Gamma_D \right\}.$$

1. Prove that J admits a unique minimizer in H.
2. Compute the derivative of J and write the optimality condition.
3. Deduce the boundary value problem satisfied by the minimizer.