ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Exercise 3

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $1 \leq d \leq 3$. Its boundary $\partial\Omega$ is divided in two disjoint sub-domains of non-zero surface measure, $\partial\Omega = \Gamma_N \cup \Gamma_D$. For given $f \in L^2(\Omega)$, consider the following boundary value problem

$$\left\{ \begin{array}{ll} -\mathrm{div}\,(h\nabla u) = f & \text{in } \Omega, \\ h\frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D, \end{array} \right. \tag{1}$$

where $h(x)$ is a coefficient which belongs to the admissible set

$$\mathcal{U}_{ad} = \left\{ h \in L^2(\Omega) \; , \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ a.e. in } \Omega \right\}.$$

Let $u_0 \in L^2(\Gamma_N)$ be a given target function. We consider the objective function

$$\inf_{h \in \mathcal{U}_{ad}} \left\{ J(h) = \int_{\Gamma_N} |u - u_0|^2 ds \right\}. \tag{2}$$

1. Write the variational formulation of (1) and prove that it admits a unique solution u.
2. Find the Lagrangian of the problem and deduce the adjoint state.
3. Compute the derivative with respect to h of the objective function.
4. For a given $R > 0$, introduce a subset of uniformly smooth admissible coefficients

   $$\mathcal{U}_{ad}^R = \left\{ h \in \mathcal{U}_{ad} \;\text{ such that } \|h\|_{H^1(\Omega)} \leq R \right\}.$$

   Prove that $J(h)$ admits a minimizer over $\mathcal{U}_{ad}^R$.