ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Exercise 4

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $1 \leq d \leq 3$. Its boundary $\partial\Omega$ is divided in two disjoint sub-domains of non-zero surface measure, $\partial\Omega = \Gamma_N \cup \Gamma_D$. For given $f \in L^2(\Omega)$, consider the following boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = v & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{1}$$

where $v \in L^2(\Gamma_N)$ is a control which belongs to the admissible set

$$\mathcal{U}_{ad} = \left\{ v \in L^2(\Gamma_N)\ , \quad v_{max} \geq v(x) \geq v_{min} \text{ a.e. in } \Gamma_N \right\}.$$

Let $u_0 \in H^1(\Omega)$ be a target field and $c > 0$. We consider the optimization problem

$$\inf_{v \in \mathcal{U}_{ad}} \left\{ J(v) = \frac{1}{2} \int_\Omega |\nabla u_v(x) - \nabla u_0(x)|^2 dx + \frac{1}{2} \int_{\Gamma_N} c|v(x)|^2 ds \right\}, \tag{2}$$

where u_v is the solution of (1) associated to the control v.

1. Write the variational formulation of (1) and prove that it admits a unique solution u_v for $v \in \mathcal{U}_{ad}$.
2. Prove that the map

$$\begin{array}{rcl} L^2(\Gamma_N) & \mapsto & H^1(\Omega) \\ v & \mapsto & u_v \text{ solution of (1)}, \end{array}$$

   is Fréchet differentiable and compute its directional derivative in a direction $w \in L^2(\Gamma_N)$.
3. Find the Lagrangian of the problem and deduce the adjoint state.
4. Compute the derivative with respect to v of the objective function.
5. Prove that (2) admits a unique minimizer.