ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Exercise 5

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $d \geq 1$. Its boundary $\partial\Omega$ is divided in two disjoint sub-domains of non-zero surface measure, $\partial\Omega = \Gamma_N \cup \Gamma_D$. We consider the following coupled model (which is a simplification of either thermo-elasticity or the buoyancy Boussinesq model) for two unknowns u and v. For given $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, consider the first boundary value problem

$$\begin{cases} -\mathrm{div}\,(A(\rho)\nabla u) = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D. \end{cases} \tag{1}$$

Introduce a C^1 function F from $\mathbb{R}$ to $\mathbb{R}$ which, for some positive constant C, satisfies the growth condition

$$|F(s)| \leq C(1 + |s|) \quad \text{and} \quad |F'(s)| \leq C \quad \forall s \in \mathbb{R}. \tag{2}$$

For given $h \in L^2(\Omega)$, consider the second boundary value problem

$$\begin{cases} -\mathrm{div}\,(B(\rho)\nabla v) = h + F(u) & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_N, \\ v = 0 & \text{on } \Gamma_D. \end{cases} \tag{3}$$

In the above $\rho \in \mathcal{U}_{ad}$ is an optimization variable which, for $\rho_{min}, \rho_{max} \in \mathbb{R}$, belongs to the admissible set

$$\mathcal{U}_{ad} = \left\{ \rho \in H^1(\Omega)\ , \quad \rho_{max} \geq \rho(x) \geq \rho_{min} \text{ a.e. in } \Omega \right\},$$

and the coefficients $A(r), B(r)$ are C^1 functions from $\mathbb{R}$ to the set $\mathcal{M}_s^d$ of symmetric $d \times d$ matrices which, for constants $0 < c \leq C$, satisfy

$$c|\xi|^2 \leq A(r)\xi \cdot \xi, B(r)\xi \cdot \xi \leq C|\xi|^2 \quad \forall r \in \mathbb{R}, \xi \in \mathbb{R}^d.$$

Note that (3) is coupled to (1) but not the other way around. We consider the optimization problem

$$\inf_{\rho \in \mathcal{U}_{ad}} \left\{ J(\rho) = \int_\Omega j(u, v)\, dx + \frac{1}{2}\int_\Omega |\nabla\rho(x)|^2 dx \right\}, \tag{4}$$

where (u, v) are the solutions of (1)-(3) and j is a C^1 function from $\mathbb{R}^2$ into $\mathbb{R}$ which, for some positive constant C, satisfies the growth condition

$$|j(u, v)| \leq C(1+|u|^2+|v|^2) \quad \text{and} \quad |j'(u, v)| \leq C(1+|u|+|v|) \quad \forall (u, v) \in \mathbb{R}^2. \tag{5}$$

1. Prove that the coupled system (1)-(3) admits a unique solution.
2. Prove that there exists at least one minimizer for (4).
3. Check that the map

$$\begin{array}{rcl} H^1(\Omega) & \mapsto & H^1(\Omega) \times H^1(\Omega) \\ \rho & \mapsto & (u, v) \text{ solution of (1)-(3),} \end{array}$$

   is Fréchet differentiable and compute its directional derivative in a direction w.
4. Find the Lagrangian of the problem and deduce the adjoint state. Comment on the coupling in the adjoint system.
5. Compute the derivative with respect to ρ of the objective function.
6. Suggest and describe a numerical algorithm to solve (4).