ECOLE POLYTECHNIQUE
Master M2 "Mathematical modelling"
PDE constrained optimization (G. Allaire)

Exercise 6

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, for $d \geq 1$. Let $\alpha > 0$ be a constant and $g : \mathbb{R} \mapsto \mathbb{R}$ a C^1 function which has at most linear growth at infinity, in the sense that there exists $M > 0$ and $C > 0$ such that, if $|s| > M$, then

$$0 \leq g(s)s \leq Cs^2 \quad \text{ and } |g'(s)| \leq C. \tag{1}$$

For given $f \in L^2(\Omega)$, consider the following non-linear model

$$\begin{cases} -\Delta u + \alpha \rho g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In (2) $\rho \in \mathcal{U}_{ad}$ is an optimization variable which, for $\rho_{min}, \rho_{max} \in \mathbb{R}^+$, belongs to the admissible set

$$\mathcal{U}_{ad} = \left\{ \rho \in L^2(\Omega) \ , \quad \rho_{max} \geq \rho(x) \geq \rho_{min} \geq 0 \text{ a.e. in } \Omega \right\} .$$

For a given target field $u_0 \in H^1_0(\Omega)$, we consider the optimization problem

$$\inf_{\rho \in \mathcal{U}_{ad}} \left\{ J(\rho) = \frac{1}{2} \int_\Omega |u(x) - u_0(x)|^2 dx \right\} , \tag{3}$$

where u is the solution of (2). This is an inverse problem where we want to reconstruct the coefficient ρ in (2).

1. Prove that the boundary value problem (2) admits at least one solution in $H^1_0(\Omega)$.
2. Prove that, if $\alpha > 0$ is small enough, then there exists at most one solution of (2) in $H^1_0(\Omega)$.
3. From now on we assume that $\alpha = 1$ and that $s \mapsto g(s)$ is non-decreasing. Prove there exists at most one solution of (2) in $H^1_0(\Omega)$.
4. Prove that there exists at least one minimizer for (3).
5. Check that the map

   $$\begin{array}{ccc} L^2(\Omega) & \mapsto & H^1_0(\Omega) \\ \rho & \mapsto & u \text{ solution of (2)} \end{array}$$

   is Fréchet differentiable and compute its directional derivative in a direction w.
6. Find the Lagrangian of the problem and deduce the adjoint state.
7. Compute the derivative with respect to ρ of the objective function.
8. Suggest and describe a numerical algorithm to solve (3).