Introduction

In this paper we are concerned with the homogenization of the Stokes and Navier-Stokes equations, with a Dirichlet boundary condition, in a domain containing many tiny obstacles. The main goal of this paper is to present, in an easy and self-contained form, some results on this topic, we proved in two different articles (see [1] and [2]). Here, many technical details are skipped, and the emphasis is put on the signification of the mathematical theorems, rather than on their proof. The starting point of this study is the numerical simulation of viscous fluid flows past an array of fixed solid obstacles: for example, flows in porous media, or through porous walls or mixing grids. Such flows are governed by the Stokes or Navier-Stokes equations with a no-slip (Dirichlet) boundary condition on the obstacles, and the fluid domain is mathematically represented by an open set perforated with holes (i.e. obstacles). As the number of holes increases, the flow will tend to the solution of certain effective or "homogenized" equations which are homogeneous in form (i.e. without obstacles). Homogenization is a mathematical method which provides such effective models (see, e.g. [4] and [18] for a general introduction to this topic).

In our framework, a porous medium is modeled as the periodic repetition of an elementary cell of size ε, in which lies a solid obstacle of size a_ε (see figure 1). When the holes size a_ε is of the same order of magnitude as the period ε, it has been proved that the homogenization of the Stokes equations leads to the well-known Darcy's law (see e.g. [11], [13], and [18] for two-scales methods, and [19] for the proof of convergence; see also [3] for a generalization of [19] to the case of a connected solid part). Instead, in the sequel we always assume that the holes size a_ε is smaller than the period ε. The problem is now to find what kind of homogenized equations can be obtained, according to the scaling of the holes size a_ε. This is what section 1 is devoted to. It turns out that there are three different limit flow regimes, depending on the holes size (see figure 2). For a so-called critical size (e.g., $a_\varepsilon = \varepsilon^3$ in the three-dimensional case), the homogenized problem is a Brinkman's law. For smaller sizes the homogenized problem reduces to the initial Stokes equations, and for larger sizes it is Darcy's law (but not the same one as in the case of the holes of size ε). The so-called Brinkman's law (introduced in the late forties by H. Brinkman [6]) is obtained from the Stokes equations by adding to the momentum equation a term proportional to the velocity (this new term expresses the slowing and mixing effect of the obstacles). These results hold true either for the Stokes or the Navier-Stokes equations, because in our framework the non-linear term does not play any part in the homogenization process.

In section 2, we also derive Brinkman's law in a different geometrical situation. The obstacles are no longer distributed in volume, but on a hyperplane (see figure 3). Then for a specific critical size (different from that arising in the case of a volume distribution), the homogenized problem is still a Brinkman's law, but now the new term in the momentum equation is concentrated on the hyperplane (i.e. is equal to zero elsewhere).

Furthermore in section 3 we generalize our previous results to the case of a slip boundary condition on the obstacles, instead of a no-slip (Dirichlet) one. Basically this slip condition allows the fluid to slip on the surface of the obstacles, proportionally to the tangential stress, but still not to penetrate them. Finally let us mention that the two-dimensional case is always completely different from the three-dimensional one, because the homogenized Darcy's and Brinkman's laws do not depend on the shape and the size of the obstacles (though they do in three dimensions). We conclude this brief introduction by referring to some other works dealing with the derivation of Brinkman's law through homogenization of the Stokes equations (see the introduction of [1] for a short survey of these works): the firsts to derive Brinkman's law were V.A. Marcenko and E.Ja. Hrouslov [14], followed by E. Sanchez-Palencia [17], T. Levy [12], A. Brillard [5], and J. Rubinstein [16]. Finally we mention the works of C. Conca [8], and [9], where other effective equations in fluid mechanics are derived through homogenization of the Stokes equations.

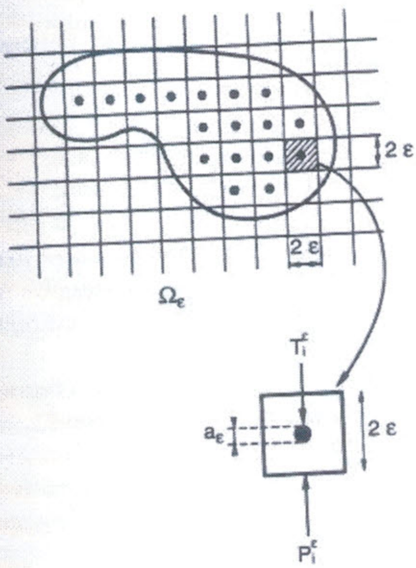

Figure 1

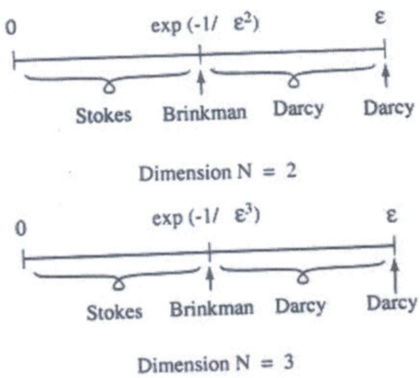

Figure 2

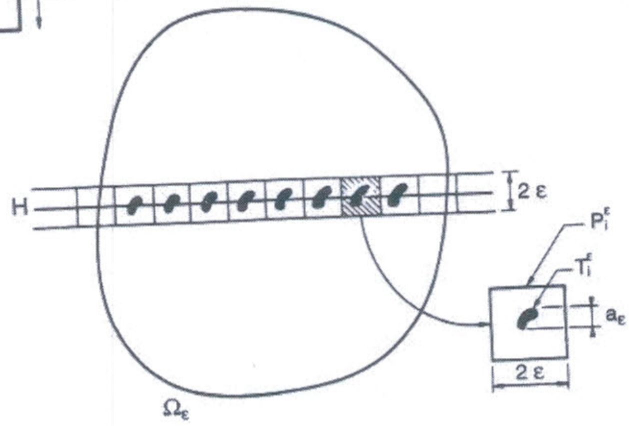

Figure 3

1.- Main Results

We consider a smooth bounded domain Ω of $\mathbb{R}^N (N \geq 2)$, wich represents a porous medium. Now we define a subdomain Ω_ε of Ω, which represents the fluid part of a porous medium. First the set Ω is covered with a regular mesh of size 2ε, each cell being a cube P_i^ε, identical to $]-\varepsilon, +\varepsilon[^N$. At the center of each cube P_i^ε, included in Ω we make a hole T_i^ε, each of them being equal to the same model obstacle T scaled to the size a_ε. Then Ω_ε is obtained by removing from Ω this collection of periodically distributed obstacles $(T_i^\varepsilon)_{1\leq i\leq N(\varepsilon)}$ (their number $N(\varepsilon)$ is of order ε^{-N}) (see figure 1).

$$\Omega_\varepsilon = \Omega - \cup_{i=1}^{N(\varepsilon)} T_i^\varepsilon$$

The holes size a_ε is always assumed to be smaller that the inter-hole distance ε, i.e. $\lim_{\varepsilon\to 0} a_\varepsilon/\varepsilon = 0$. We define the so-called critical size of the holes a_ε^{crit} :

$$a_\varepsilon^{crit.} = C_0 \varepsilon^{\frac{N}{N-2}} \text{ for } N \geq 3 \text{ and } a_\varepsilon^{crit.} = e^{-\frac{C_0}{\varepsilon^2}} \text{ for } N = 2$$

where C_0 is a strictly positive constant. We also define a ratio σ_ε between the current size of the holes and the critical one:

$$\sigma_\varepsilon = \left(\frac{\varepsilon^N}{a_\varepsilon^{N-2}}\right)^{1/2} \quad \text{for } N \geq 3$$
$$\sigma_\varepsilon = \varepsilon \left| \operatorname{Log}\left(\frac{a_\varepsilon}{\varepsilon}\right)\right|^{1/2} \quad \text{for } N = 2.$$

When the limit of σ_ε is strictly positive and finite, the current holes size is actually critical. If the limit of σ_ε is zero, then the current holes size is larger than the critical one. If the limit of σ_ε is infinite, then the current holes size is smaller than the critical one.

Consider now a stationary viscous fluid flow in a porous medium (represented by Ω_ε): we model it by the Stokes equations with a no-slip condition on the surface of the obstacles (see corollary 1.4 for the case of the Navier-Stokes equations). For a given force $f \in [L^2(\Omega)]^N$ and a constant positive viscosity μ, denoting by u_ε the velocity, and by p_ε the pressure, the Stokes equations in Ω_ε, with a Dirichlet boundary condition, are:

$$(S_\varepsilon) \quad \begin{cases} \nabla p\varepsilon - \mu\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

It is well known (see e.g. [10]) that there exists a unique solution of (S_ε) such that $(u_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}]$. For different values of ε, the solutions $(u_\varepsilon, p_\varepsilon)$ do not belong to the same fixed space, because Ω_ε changes with ε. In order to overcome this inconvenient, we extend those solutions

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to the whole domain Ω. Let $\tilde{u}_\varepsilon$ and P_ε be extensions of the velocity u_ε and of the pressure p_ε respectively, defined by:

$$(i) \qquad P_\varepsilon = p_\varepsilon \text{ and } \tilde{u}_\varepsilon = u_\varepsilon \text{ in } \Omega_\varepsilon$$

$$(ii) \qquad P_\varepsilon = \frac{1}{|C_i^\varepsilon|}\int_{C_i^\varepsilon} p_\varepsilon \text{ and } \tilde{u}_\varepsilon = 0 \text{ in each hole } T_i^\varepsilon$$

where C_i^ε is a "control" volume around T_i^ε defined as the part of the ball of radius ε outside T_i^ε. Actually the most technical and difficult part of our work is the construction of an extension of the pressure, which is bounded in $[L^2(\Omega)/\mathbb{R}]$. Then, using the so-called energy method introduced by L. Tartar [20] (see also F. Murat [15]), and adapted by D. Cioranescu and F. Murat [7], we prove the following

Theorem 1.1. *According to the scaling of the holes size there are three different limit flow regimes:*

(i) *if* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$, *then* $(\tilde{u}_\varepsilon, P_\varepsilon)$ *converges strongly to* (u,p) *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u,p) *is the unique solution of the Stokes equations:*

$$\begin{aligned} \nabla p - \mu\Delta u = f & \quad \text{in } \Omega \\ \nabla\cdot u = 0 & \quad \text{in } \Omega \\ u = 0 & \quad \text{on } \partial\Omega \end{aligned}$$

(i.e. the holes are too small, and nothing happens when passing to the limit.)

(ii) *if* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma > 0$, *then* $(\tilde{u}_\varepsilon, P_\varepsilon)$ *converges weakly to* (u,p) *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u,p) *is the unique solution of the Brinkman-type law:*

$$\begin{aligned} \nabla p - \mu\Delta u + \tfrac{\mu}{\sigma^2} M u = f & \quad \text{in } \Omega \\ \nabla\cdot u = 0 & \quad \text{in } \Omega \\ u = 0 & \quad \text{on } \partial\Omega \end{aligned}$$

(i.e. for this critical size of the holes, a supplementary term appears when passing to the limit.)

(iii) *if* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, *then* $(\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}, P_\varepsilon)$ *converges strongly to* (u,p) *in* $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, *where* (u,p) *is the unique solution of the Darcy's law:*

$$\begin{aligned} u = M^{-1}(f - \nabla p) & \quad \text{in } \Omega \\ \nabla\cdot u = 0 & \quad \text{in } \Omega \\ u\cdot n = 0 & \quad \text{on } \partial\Omega \end{aligned}$$

(i.e. the holes are too large, and the Stokes flow degenerates in a Darcy flow when passing to the limit.)

Moreover the matrix M, *which appears in the Brinkman-type law and in the Darcy's law, is the same in both cases, and depends only on the* *model hole T (not on the force f and the solution (u,p), nor on the holes size a_ε).* ■

Actually, we can compute M through a local problem around the model obstacle T :

Proposition 1.2:

• For $N \geq 3$, denoting by e_k the k^{th} unit basis vector in $I\!R^N$, for each k the local problem is:

$$\begin{cases} \nabla q_k - \Delta w_k = 0 & \text{in } I\!R^N - T \\ \nabla \cdot w_k = 0 & \text{in } I\!R^N - T \\ w_k = 0 & \text{on } \partial T \\ w_k = e_k & \text{at infinity} \end{cases}$$

Then the matrix M is given by the following formula:

$$M = \frac{1}{2^N} \left[\int_{I\!R^N - T} \nabla w_k \cdot \nabla w_i \right]_{1 \leq i,k \leq N}$$

• For $N = 2$, because of the celebrated Stokes paradox the local problem is, for $k = 1, 2$:

$$\begin{cases} \nabla q_k - \nabla w_k = 0 & \text{in } I\!R^N - T \\ \nabla \cdot w_k = 0 & \text{in } I\!R^N - T \\ w_k = 0 & \text{on } \partial T \\ w_k = (Log\ r) e_k & \text{at infinity} \end{cases}$$

(Remark the logarithmic growth at infinity). Then we have the paradoxical result:

$$M = \pi Id \quad \text{for any model hole } T. \quad ■$$

Remark that the Darcy's law which is derived at line (iii) in theorem 1.1 has nothing to do with the one obtained by the two-scales method when the holes size is exactly ε (see, e.g. [18]). Actually in both cases the permeability tensor (here the matrix M^{-1}) has a different value. Furthermore, here the local problem occurs in the entire space around the obstacle, while it takes place in a unit period of the porous medium, with a periodic boundary condition, when the obstacle and the period have the same size ε. Besides the rigorous derivation of the Brinkman's law through homogenization of the Stokes equations, theorem 1.1 gives a complete description of the different flow regimes obtained at the limit. The entire range of the holes sizes has been examined, and the only effective equations that have been found are the Stokes equations, the Brinkman's law, and the Darcy's law (see figure 2). We emphasize that, in theorem 1.1, the viscosity is always assumed to be constant, not related to the period ε of the porous medium. An interesting open problem is to find whether one can obtain other effective equations through homogenization of the Stokes equations with a viscosity, now depending on ε.

Remark 1.3:
The two-dimensional case is somehow disappointing, because all the informations on the geometry of the porous medium (i.e. the shape and the size of the obstacles) is lost when passing to the limit. Even if the porous medium was not isotropic, we find that the matrix M in the effective equations (Brinkman's or Darcy's law) is always scalar. Fortunately the three-dimensional case is more satisfactory, because the matrix M can be non scalar, and even non-diagonal. It means that the effective equations keep track not only of the slowing effect, but also of the rotating or deviating effect of the obstacles. Furthermore, the assumption that the holes are all the same is not necessary in the homogenization process (a clear hint of this is that there is no periodic boundary condition in the local problem). Thus we can obtain a matrix M which is no longer constant, but depends on the current point $x \in \Omega$. ■

For $N = 2$, or 3, the homogenization of the Navier-Stokes equations (NS_ε) proceeds exactly as in the case of the Stokes equations, because, in this framework, the non-linear term is a compact perturbation of (S_ε).

$$(NS_\varepsilon) \quad \begin{cases} \nabla p_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon - \mu \Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

Let $(u_\varepsilon, p_\varepsilon)$ be a solution of (NS_ε). Extending it as previously, we obtain

Corollary 1.4:
According to the same scaling of the holes size as for the Stokes equations, we obtain:

(i) if $\lim_{\varepsilon \to 0} \sigma_\varepsilon = +\infty$, then $(\tilde{u}_\varepsilon, P_\varepsilon)$ converges strongly to (u, p) in $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u, p) is a solution of the Navier-Stokes equations:

$$\begin{cases} \nabla p + u \cdot \nabla u - \mu \Delta u = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(ii) if $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma > 0$, then $(\tilde{u}_\varepsilon, P_\varepsilon)$ converges weakly to (u, p) in $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u, p) is a solution of the non-linear Brinkman-type law:

$$\begin{cases} \nabla p + u \cdot \nabla u - \mu \Delta u + \frac{\mu}{\sigma^2} M u = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(iii) if $\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0$, then $(\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}, P_\varepsilon)$ converges strongly to (u, p) in $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u, p) is the unique solution of the Darcy's law:

$$\begin{aligned} u &= M^{-1}(f - \nabla p) && \text{in } \Omega \\ \nabla \cdot u &= 0 && \text{in } \Omega \\ u \cdot n &= 0 && \text{on } \partial\Omega \end{aligned}$$

The matrix M is the same as that appearing in theorem 1.1, and is still given by proposition 1.2. ■

Remark 1.5:

One can see in theorem 1.1 that, according to the type of effective equations, there are different kinds of convergence of the solutions. If the convergence is weak, it is possible to improve it in a strong one thanks to the introduction of so-called correctors, and assuming some smoothness of the limit solution. Furthermore, it is also possible to estimate the difference (or error) between the solutions $(u_\varepsilon, p_\varepsilon)$ and the limit solution (u, p), matched with the corrector. This corrector is physically interpreted as a boundary layer of the Stokes problem in the vicinity of the obstacles. It turns out that, by rescaling the solution of the local problem around the model obstacle T, and expanding it by periodicity in the domain Ω, we actually construct boundary layers (or correctors), denoted by $(w_k^\varepsilon, q_k^\varepsilon)_{1 \leq k \leq N}$. Then, for the case of a critical size of the holes (corresponding to Brinkman's law), we obtain the following errors estimates:

$$\|\tilde{u}_\varepsilon - \sum_{k=1}^{N} u_k w_k^\varepsilon\|_{H_0^1(\Omega)} \leq C\varepsilon \|u\|_{w^{2,\infty}(\Omega)} \quad \text{and}$$

$$\|p_\varepsilon - p - \sum_{k=1}^{N} u_k q_k^\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon \|u\|_{w^{2,\infty}(\Omega)}.$$

For larger holes sizes (corresponding to Darcy's law), the convergence is already strong, thus we do not need any correctors. Nevertheless we obtain the following errors estimates:

$$\|\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} - u\|_{L^2(\Omega)} \leq C(\frac{\varepsilon}{\sigma_\varepsilon} + \sigma_\varepsilon)\|u\|_{w^{2,\infty}(\Omega)} \quad \text{and}$$

$$\|p_\varepsilon - p\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C(\frac{\varepsilon}{\sigma_\varepsilon} + \sigma_\varepsilon)\|u\|_{w^{2,\infty}(\Omega)}. \quad ■$$

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2.- Surface Distribution of the obstacles

In this section we work out the homogenization of the Stokes or Navier-Stokes equations in a domain containing obstacles distributed on a hyperplane embedded in the domain, rather than in the entire domain. Such a situation can occur in the study of fluid flows through a porous wall, a sieve, or a mixing grid (for the case of a sieve, we refer to the paper of C. Conca [8]). As in section 1, we model these devices by an array of fixed and disconnected obstacles (for example, one can think of the obstacles as the vanes of a mixing grid, which are supported by a negligible lattice), and for some critical size we derive a Brinkman-type law.

More precisely, we consider a smooth bounded domain Ω of $\mathbb{R}^N (N \geq 2)$, which intersects the hyperplane $H = \{x \in \mathbb{R}^N / x_N = 0\}$. We define a thin slice of Ω, of thickness 2ε near H :

$$H_\varepsilon = \{x \in \Omega / |x_N| < \varepsilon\}.$$

The set H_ε is covered with a regular mesh of size 2ε, each cell being a cube P_i^ε, identical to $]-\varepsilon, +\varepsilon[^N$. At the center of each cube P_i^ε included in H_ε we make a hole T_i^ε, each of them being homothetic to the same model obstacle T with a ratio a_ε. The fluid domain Ω_ε is then obtained by removing from Ω all the holes $(T_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ (their number $N(\varepsilon)$ is of order ε^{1-N}) (see figure 3):

$$\Omega_\varepsilon = \Omega - \cup_{i=1}^{N(\varepsilon)} T_i^\varepsilon$$

Remark that, if the centers of the obstacles are located on the hyperplane H, the obstacles themselves are $N-$dimensional objects, not necessarily flat (i.e. included in H).

We assume that the holes size a_ε is critical, i.e. for some strictly positive constant C_0 we have:

$$\begin{cases} a_\varepsilon = C_0 \varepsilon^{\frac{N-1}{N-2}} & \text{for } N \geq 3 \\ a_\varepsilon = e^{-\frac{C_0}{\varepsilon}} & \text{for } N = 2. \end{cases}$$

Compared with the case of a volume distribution, this definition means that, for a surface distribution, there are fewer holes, but they are bigger. Recall the Stokes equations in Ω_ε :

$$(S_\varepsilon) \quad \begin{array}{ll} \nabla p_\varepsilon - \mu \Delta u_\varepsilon = F & \text{in } \Omega_\varepsilon \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{array}$$

Then we prove

Theorem 2.1. *Let $\tilde{u}_\varepsilon$ and P_ε be extensions of the velocity u_ε and of the pressure p_ε, defined by:*

$$(i) \qquad P_\varepsilon = p_\varepsilon \text{ and } \tilde{u}_\varepsilon = u_\varepsilon \qquad \text{in } \Omega_\varepsilon$$

$$(ii) \qquad P_\varepsilon = \frac{1}{|C_i^\varepsilon|}\int_{C_i^\varepsilon} p_\varepsilon \text{ and } \tilde{u}_\varepsilon = 0 \text{ in each hole } T_i^\varepsilon$$

where C_i^ε is a "control" volume around T_i^ε defined as the part of the ball of radius ε outside T_i^ε.
Then, for any value of q' such that $1 < q' < N/(N-1)$, $(\tilde{u}_\varepsilon, P_\varepsilon)$ converges weakly to (u,p) in $[H_0^1(\Omega)]^N \times [L^{q'}(\Omega)/\mathbb{R}]$, where (u,p) is the unique solution of the following Brinkman-type law:

$$\begin{array}{ll} \nabla p - \mu\Delta u + \frac{2\mu}{\sigma^2} M\delta_H u = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}$$

where the constant σ^2 is equal to C_0^{2-N} for $N \geq 3$, and to C_0 if $N = 2$, and the matrix M is the same as in section 1 (see proposition 1.2). The symbol δ_H denotes the measure defined as the unit mass concentrated on the hyperplane H, i.e.:

$$< \delta_H, \phi >_{D', D(\mathbb{R}^N)} = \int_H \phi(s)ds \text{ for any } \phi \in D(\mathbb{R}^N). \quad \blacksquare$$

Obviously the main difference with theorem 1.1 in section 1 is that the new term appearing in the Brinkman-type law is concentrated in the hyperplane H, i.e. $\frac{2\mu}{\sigma^2} M\delta_H u = 0$ elsewhere. Remark that, for technical reasons, the convergence of the pressure is weaker than in the first section (we merely have $q' < N/(N-1) \leq 2$). As already mentioned in remark 1.3, the obstacles do not need to be all the sames, and in the three-dimensional case the matrix M may be non-diagonal. Therefore the matrix M can obviously express the mixing effect of the obstacles. That is why the Brinkman-type law obtained in theorem 2.1 seems to be an interesting model for fluid flows through mixing grids.

Remark 2.2:

As in section 1, theorem 2.1 can easely be generalized to the case of the Navier-Stokes equations. We obtain a non-linear Brinkman-type law with the same matrix M, and the convective term of the Navier-Stokes equations is kept unchanged while passing to the limit. $\blacksquare$

3.- Slip boundary condition.

This section is devoted to the generalization of all the previous results of the present paper, when the no-slip (Dirichlet) boundary condition is replaced by a slip one. More precisely we assume that on the surface of each obstacle we have:

$$\begin{cases} u_\varepsilon \cdot n = 0 & \text{on } \partial T_i^\varepsilon \\ \frac{\alpha}{a_\varepsilon} u_\varepsilon = 2(\frac{\partial u_\varepsilon}{\partial n} \cdot n)n - (\nabla u_\varepsilon + {}^t\nabla u_\varepsilon)n & \text{on } \partial T_i^\varepsilon \end{cases}$$

where the slip coefficient α is a positive constant. Remark that we have given a precise scaling of the slip coefficient. Nevertheless the following results are valid for all its scalings, because α can take the extremal values 0 and $+\infty$. The first equation expresses that the fluid does not flow through the obstacle T_i^ε. The second one is a balance relation between the tangential components of the velocity and the infinitesimal force exerted by the fluid on the obstacle. Mathematically speaking, this slip boundary condition is a mixed-type one. For another example of mixed-type boundary condition, see C. Conca [9].
Now we can state our main result which is similar to theorem 1.1, but now the matrix M depends on α. There are also some technical differences with theorem 1.1: for example, here the extension of the velocity is no longer obvious (see [2]). Anyway, we obtain

Theorem 3.1.

Let $(u_\varepsilon, p_\varepsilon)$ be the unique solution of the Stokes equations with a slip boundary condition. There exists an extension $(E_\varepsilon u_\varepsilon, P_\varepsilon)$, such that:

(i) if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$, then $(E_\varepsilon u_\varepsilon, P_\varepsilon)$ converges strongly to (u,p) in $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u,p) is the unique solution of the Stokes equations:

$$\begin{array}{ll} \nabla p - \mu\Delta u = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}$$

(ii) if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma > 0$, then $(E_\varepsilon u_\varepsilon, P_\varepsilon)$ converges weakly to (u,p) in $[H_0^1(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u,p) is the unique solution of the Brinkman-type law:

$$\begin{array}{ll} \nabla p - \mu\Delta u + \frac{\mu}{\sigma^2} M(\alpha) u = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}$$

(iii) if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, then $(\frac{E_\varepsilon u_\varepsilon}{\sigma_\varepsilon^2}, P_\varepsilon)$ converges strongly to (u,p) in $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, where (u,p) is the unique solution of the Darcy's law:

$$\begin{array}{ll} u = M(\alpha)^{-1}(f - \nabla p) & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \quad \blacksquare \end{array}$$

As in section 1, the matrix $M(\alpha)$ is the same for the Brinkman-type law and the Darcy's law, and we can compute it through a local problem around the model obstacle T :

Proposition 3.2:

Denoting by e_k the k^{th} unit basis vector in $I\!R^N$, the local problem is:

$$\begin{cases} \nabla q_k - \Delta w_k = 0 \text{ in } I\!R^n - T \\ \nabla \cdot w_k = 0 \text{ in } I\!R^N - T \\ \alpha w_k = 2(\frac{\partial w_k}{\partial n} \cdot n)n - (\nabla w_k + {}^t\nabla w_k)n \text{ on } \partial T \\ w_k \cdot n = 0 \text{ on } \partial T \\ w_k = e_k \text{ for } N \geq 3 \text{ and } w_k = (Log\ r)e_k \text{ for } N = 2 \text{ at infinity} \end{cases}$$

- For $N \geq 3$, the matrix M is given by the following formula:

$$M(\alpha) = \frac{1}{2^N}\left[\int_{I\!R^N - T} \nabla w_k \cdot \nabla w_i\right]_{1 \leq i,k \leq N}$$

- For $N = 2$, because of the logarithmic growth at infinity, we have the paradoxical result:

$$M(\alpha) = \pi Id \quad \text{for any model hole } T, \text{ and for any value of } \alpha. \quad \blacksquare$$

One can wonder if, even for $N \geq 3$, the matrix $M(\alpha)$ really depends on α. In order to show that it does actually, we give its value when the obstacle T is the unit ball:

$$M = \frac{S_N N(N-2)(2+\alpha)}{2^N(N-1)(N+\alpha)} Id \quad \text{for } N \geq 3$$

(S_N is the area of the unit sphere).

In this formula one can see that, even if the fluid slips on the boundary of the obstacles (corresponding to the limit case $\alpha = 0$), with no slowing effect due to the viscosity, the matrix M is still non-equal to zero. It means that the mere presence of the obstacles is enough to yield the same limit flow regimes, according to the same holes sizes, as in section 1. Let us conclude by claiming that, as in section 1, theorem 3.1 hold true also for the Navier-Stokes equations, and that we can easily generalize the correctors results and the errors estimates obtained for the case of a Dirichlet boundary condition.

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