TOPOLOGY OPTIMIZATION AND OPTIMAL SHAPE DESIGN USING HOMOGENIZATION

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ABSTRACT. We study the shape optimization of a two-dimensional elastic body loaded in plane stress. The design criteria are compliance and weight. A relaxed formulation obtained by homogenization is used, whereby perforated composite materials are admitted as structural components. This approach has the advantage of placing no implicit restriction on the topology of the design. We compare our results with those of Bendsoe, Kikuchi, and Suzuki who used an approach similar to ours.

1. The Optimal Shape Design Problem

We study the shape optimization for minimum compliance and weight of a two-dimensional elastic body loaded in plane stress. We consider a region $\Omega \subset \mathbb{R}^2$, occupied by a linearly elastic material with Hooke's law A_0. Its boundary $\partial\Omega$ is loaded by a given function $f : \partial\Omega \to \mathbb{R}^2$ (other boundary conditions are also possible, e.g. part of $\partial\Omega$ may have a specified displacement). We intend to remove a subset $H \subset \Omega$, consisting of one or more holes, the new boundaries created this way being traction-free. The equations of elasticity for the resulting structure are :

$$\begin{cases} \sigma = A_0\, e(u)\,, \quad e(u) = \tfrac{1}{2}(\nabla u + {}^t\nabla u) \\ div\ \sigma = 0 \quad in\ \Omega \setminus H \\ \sigma.n = f \quad on\ \partial\Omega\,, \quad \sigma.n = 0 \quad on\ \partial H. \end{cases} \tag{1.1}$$

A global measure of rigidity for the structure $\Omega \setminus H$ is the compliance defined by

$$c(H) = \int_{\partial\Omega} f.u = \int_{\Omega\setminus H} \langle A_0 e(u), e(u) \rangle\,. \tag{1.2}$$

Our goal is to minimize the compliance and the weight. Introducing a positive Lagrange multiplier λ, our optimal shape design problem is :

$$\text{Min}_{\{H \subset \Omega\}} \left[c(H) + \lambda\, |\Omega \setminus H| \right]. \tag{1.3}$$

M. P. Bendsøe and C.A. Mota Soares (eds.), Topology Design of Structures, 207–218.

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As stated, problem (1.3) has two major difficulties. First, from a mathematical point of view, (1.3) may have no "classical" solution (i.e. a regular shape) ; the point is that it may be advantageous to use infinitely many small holes rather than a few large ones. In other words, the optimal structure may be achieved only by a composite material made by perforation. Second, from a numerical point of view, the "standard approach" for computing solutions of (1.3) is that of boundary variations. Usually, one starts from an initial guess H_0 with a free boundary subject to optimization ; no new boundaries are created, and the topology (layout) of the resulting "optimal" shape is the same as that of the initial guess. Here, instead, we are interested in the topology optimization of the initial guess and, therefore, of the optimal shape.

To remedy both difficulties, we propose a relaxed formulation of problem (1.3) obtained by homogenization of perforated microstructures ; this has the net effect of enlarging the set of structural components by allowing for composite materials made of an intricate mixture of void and material. The idea of using homogenization for relaxing optimal design problems is by now well-known in the mathematical community (see e.g. [6], [7], [8]).

2. The relaxed formulation, and composite materials

We begin with a precise formulation of what we mean by composite material obtained by perforation. Such a composite is determined by two functions : its volume fraction (or density) $\theta(x)$, taking values between 0 and 1, and its effective Hooke's law $A(x)$. Not all values of $A(x)$ are attainable by perforation of the original material A_0, and the homogenization theory (see e.g. [2]) tells us that $A(x)$ is restricted to a subspace $G_{\theta(x)}$ of fourth order tensors, defined as the set of all effective Hooke's laws obtained by perforation of A_0 with volume fraction $1-\theta(x)$ of holes. We remark that G_0 is restricted to $\{0\}$ and G_1 to $\{A_0\}$, but in general the set G_θ is not explicitly known. The equations of elasticity become :

$$\begin{cases} \sigma = A(x)\, e(u) \\ div\ \sigma = 0 \quad in\ \Omega \\ \sigma.n = f \quad on\ \partial\Omega, \end{cases} \tag{2.1}$$

and the compliance is

$$c(A) = \int_{\partial\Omega} f.u = \int_{\Omega} \langle A(x)e(u), e(u)\rangle\,. \tag{2.2}$$

The relaxed formulation of problem (1.3) is then

$$\mathrm{Min}_{\substack{0 \le \theta(x) \le 1 \\ A(x) \in G_{\theta(x)}}} \left[c(A) + \lambda \int_{\Omega} \theta(x)\, dx \right]. \tag{2.3}$$

Since we do not know the precise form of $G_{\theta(x)}$, problem (2.3) must be worked on a little further. One possibility is to restrict $G_{\theta(x)}$ to a special class of composites : for example, periodic arrays of rectangular holes in squared cells, as proposed by Bendsoe, Kikuchi, and Suzuki [5], [9]. Our approach is different and consists in using partial knowledge of $G_{\theta(x)}$: more precisely, we know how to minimize the complementary energy $\langle A^{-1}\sigma,\sigma\rangle$ among all tensors A in G_θ, for given σ. Thus, we are going to transform (2.3) in order to use this information. On the other hand, as a matter of theory (see [2], [4]), this process is *equivalent* to restricting $G_{\theta(x)}$ to the class of sequentially laminated composites, which thus appears as a class of *optimal* microstructures for problem (2.3).

We now turn to the derivation of a more explicit formula for the relaxed problem (2.3). Using the principle of minimum complementary energy, (2.3) is equivalent to

$$\operatorname{Min}_{\substack{0 \le \theta(x) \le 1 \\ A(x) \in G_{\theta(x)}}} \operatorname{Min}_{\substack{div\ \tau = 0\ \ in\ \Omega \\ \tau.n = f\ \ on\ \partial\Omega}} \int_{\Omega} \left[< A^{-1}(x)\tau,\tau > \ + \ \lambda\ \theta(x) \right] dx\ . \tag{2.4}$$

Next, we interchange the order of minimization in (2.4), and we put the minimization over θ and A inside the integral because they are subject only to algebraic and local constraints. We thus obtain

$$\operatorname{Min}_{\substack{div\ \tau = 0\ \ in\ \Omega \\ \tau.n = f\ \ on\ \partial\Omega}} \int_{\Omega} F_{\lambda}(\tau)\ dx\ , \tag{2.5}$$

with

$$F_{\lambda}(\tau) \ = \ \operatorname{Min}_{0 \le \theta \le 1} \operatorname{Min}_{A \in G_{\theta}} \left[< A^{-1}\tau,\tau > \ + \ \lambda\ \theta \right]. \tag{2.6}$$

We explicitly computed (2.6) in [3] : our main tool is the Hashin-Shtrikman variational principle for the minimization over G_{θ}. Finally (2.6) becomes

$$F_{\lambda}(\tau) \ = \ \begin{cases} < A_0^{-1}\tau,\tau > \ + \ \lambda & \text{if } \rho \ge 1 \\ < A_0^{-1}\tau,\tau > \ + \ \lambda\ \rho(2-\rho) & \text{if } \rho < 1 \end{cases} \tag{2.7}$$

with

$$\rho \ = \ \left(\frac{\kappa+\mu}{4\kappa\mu} \right)^{1/2} \lambda^{-1/2} \left(|\tau_1| + |\tau_2| \right) \tag{2.8}$$

where κ and μ are the bulk and shear moduli of A_0, and τ_1, τ_2 are the eigenvalues of the stress τ. The optimal value of the density θ in (2.6) is given by

$$\theta^{opt} \ = \ \operatorname{Min}\,(1,\rho)\ . \tag{2.9}$$

An optimal associated microstructure (not necessarily unique) is a sequentially laminated composite of rank 2 (in 2-D) whose parameters depend pointwise on τ (physically, this microstructure is obtained from A_0 by perforation with very thin and long holes, see [3] for details).

In the end, our relaxed problem (2.5) looks like a non-linear elasticity problem in plane stress, which is amenable to numerical computations. Its main feature are :

(i) the relaxed formulation (2.5) is equivalent to the original one (1.3). Specifically, their minimum values are equal, and every solution of (2.5) determines a minimizing sequence of "classical" designs for (1.3),

(ii) the relaxed formulation (2.5) always has a solution in 2-D.

The density $\theta(x)$ of the optimal composite (the "generalized" shape we are looking for) is recovered from equation (2.8), and may be used to deduce an optimal topology, or layout, of material.

3. Numerical Procedure

The relaxed problem (2.5) is solved by finite elements on a fixed triangular mesh. To satisfy the constraint that the stress is divergence-free, we work with an Airy stress function ψ such that

$$\tau = \begin{bmatrix} \partial^2\psi/\partial y^2 & -\partial^2\psi/\partial x\,\partial y \\ -\partial^2\psi/\partial x\,\partial y & \partial^2\psi/\partial x^2 \end{bmatrix} \tag{3.1}$$

The function ψ is discretized using the Clough-Tocher finite element : each triangle being divided in three sub-triangles, it is piecewise cubic on sub-triangles, and globally C^1. To perform the minimization of the non-linear elastic energy (2.7), we used the conjugate gradient method, which requires that $F_\lambda(\tau)$ be differentiable. Unfortunately this is not the case when one of the eigenvalues τ_i of the stress is equal to zero. Therefore, $F_\lambda(\tau)$ is regularized by replacing the original ρ, defined by (2.8), with

$$\rho_\varepsilon = \left[\frac{\kappa+\mu}{4\kappa\mu}\right]^{1/2} \lambda^{-1/2} \left[\|\tau\|^2 + 2[\varepsilon^2+(\det\tau)^2]^{1/2}\right]^{1/2} . \tag{3.2}$$

Note that as ε goes to 0, ρ_ε tends to ρ ; the largest error between ρ_ε and ρ is made when $\tau = 0$ and is of order $\varepsilon^{1/2}$.

All the cases presented here have been computed with $\varepsilon = 10^{-4}$, $\kappa = 1.0$, $\mu = 0.5$, and a constant value of the loading f along selected portions of $\partial\Omega$. We recall that the effect of increasing the Lagrange multiplier λ is to decrease the weight of the final optimal design.

We present three different cases : the fillet problem (figure 1) on a square mesh of 512 triangles, the cantilever problem (figure 2) and the beam problem (figure 3), both on a rectangular mesh (its length is twice its width) of 256 triangles. Each interior triangle has three degrees of freedom.

According to the different values of the Lagrange multiplier λ and to the different loading configurations, our code gives a generalized optimal design with a varying amount of composite (the grey area on the figures ; white is void and black is pure material). Some examples of "classical" solutions (i.e. a shape with almost no composite), obtained with our code, may be found in [3]. Here, we intentionally choose three cases where large regions of composite arise. Indeed, one of the main issue of this paper is to understand the differences between these results and the ones obtained by Bendsoe, Kikuchi, and Suzuki [5], [9] which, on the contrary, exhibit microstructures on the mesh scale with very little composite.

Roughly speaking, the approach of Bendsoe, Kikuchi, and Suzuki is similar to ours, except that they use *sub-optimal* composite materials (rectangular holes in square cells). We believe this has the effect of creating an optimal microstructure on the mesh scale : it does not happen with our code since we use *optimal* microstructures on the sub-grid scale.

To support our claim, we have introduced in our code the possibility of *penalizing* composites in the minimization process for (2.5), thus forcing the final optimal design to have only small regions of composite.

4. Penalization of Composites

In order to penalize the use of composites, we add to the energy $F_\lambda(\tau)$, introduced in the relaxed formulation (2.5), a new term of the form $\theta(1-\theta)$ which is minimized for $\theta = 0$ (holes) or $\theta = 1$ (material A_0). The resulting minimization process is

$$\mathrm{Min}_{\substack{div\ \tau = 0 \ \ in\ \Omega \\ \tau.n = f \ \ on\ \partial\Omega}} \int_\Omega \left[F_\lambda(\tau) + c\ \theta^{opt}(1-\theta^{opt}) \right] dx \ , \tag{4.1}$$

where c is a positive constant (equal to 0.5 in all our computations), and θ^{opt} is given by (2.9).

We remark that, unlike the relaxed formulation (2.5), the penalized formulation (4.1) is not a well-posed minimization problem : specifically, (4.1) is not quasi-convex, and it does not necessarily have a solution. Thus, our strategy for using (4.1) is the following : first, compute an optimal design with the relaxed formulation (2.5), second, starting from that optimal design, use the penalized formulation (4.1).

The results of the above process are shown in figures 4, 5, and 6, starting from the corresponding results shown on figures 1, 2, and 3, respectively. In each case the penalization has little effect on the compliance (about 1 %). Clearly, penalizing composites (i.e. discouraging optimal micro-structures on a sub-grid scale) leads to micro-structures on the mesh scale. For the fillet problem (figure 4), the micro-structure looks like a rank-one laminate (each layer is of the size of one cell), which is the dominant sub-grid micro-structure too.

Further work to explain the differences in the optimal designs obtained when using optimal or sub-optimal microstructures in the relaxation process has been completed in [1].

Acknowledgements. G. A. is grateful to T. de Gramont for his assistance in producing the figures of this paper.

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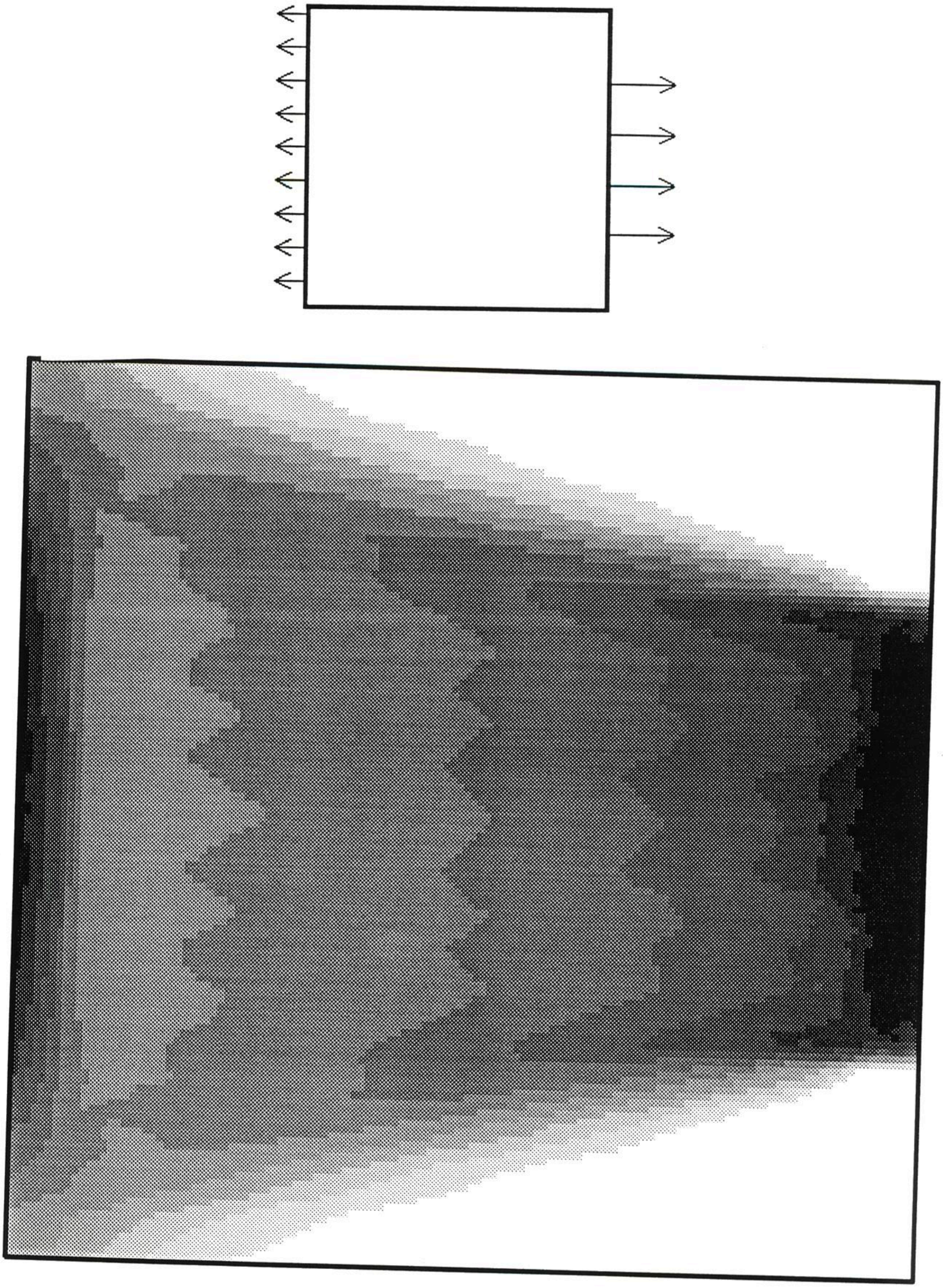

Figure 1
fillet : $\lambda = 1$, 53.2 % of material removed, compliance = 113.7

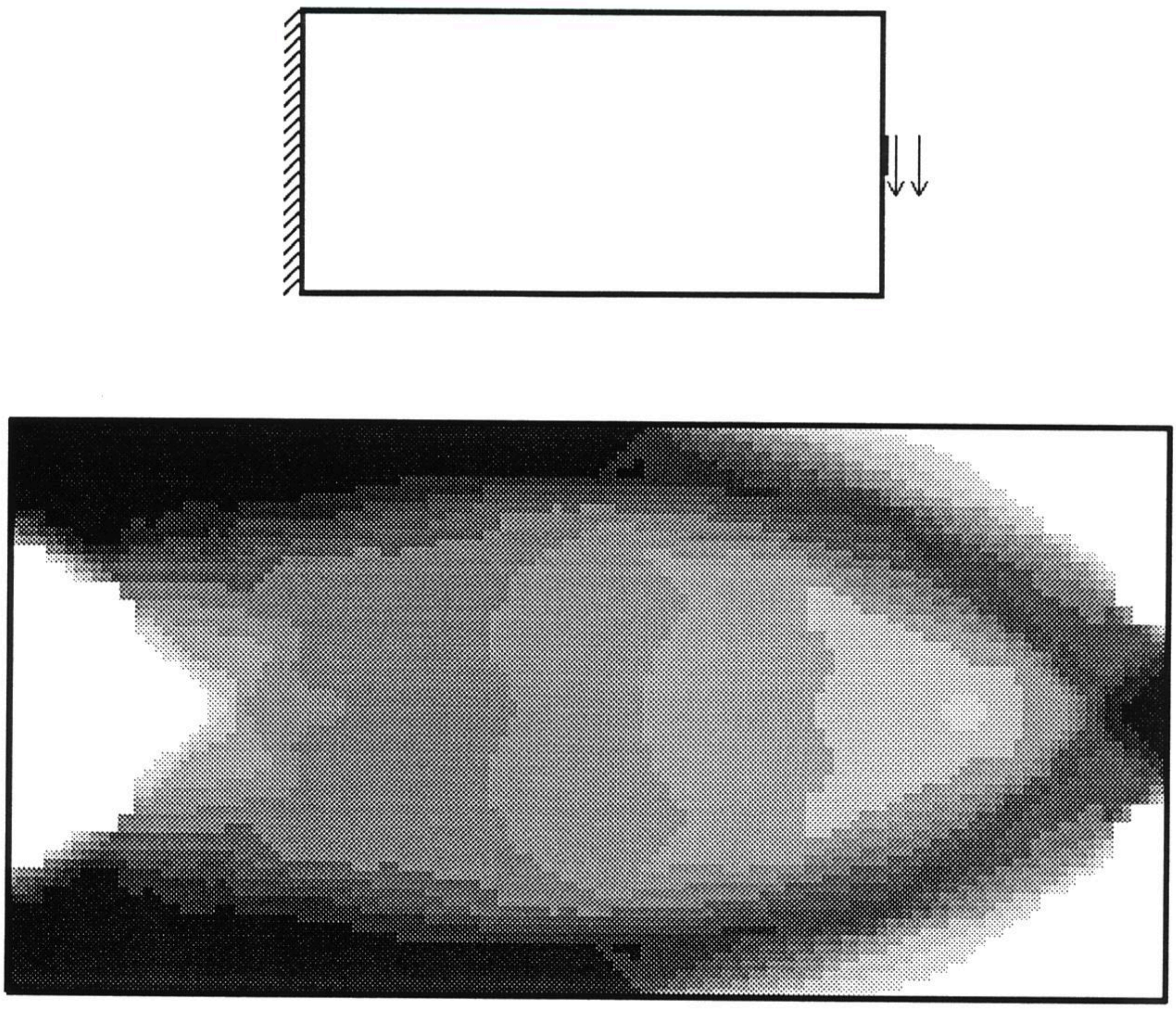

Figure 2

cantilever : $\lambda = 2$, 48.0 % of material removed, compliance = 167.4

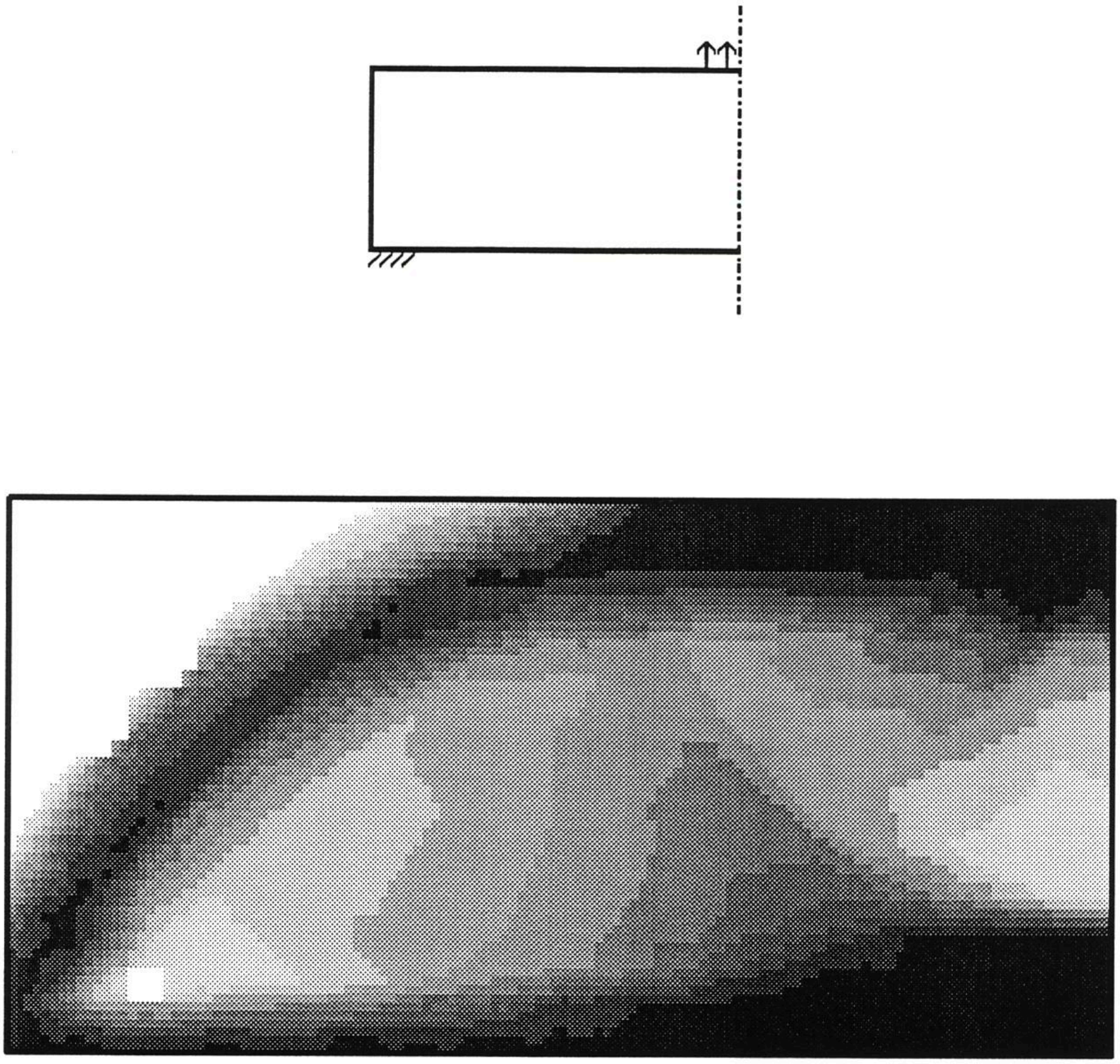

Figure 3
beam : $\lambda = 0.5$, 45.5 % of material removed, compliance = 44.6

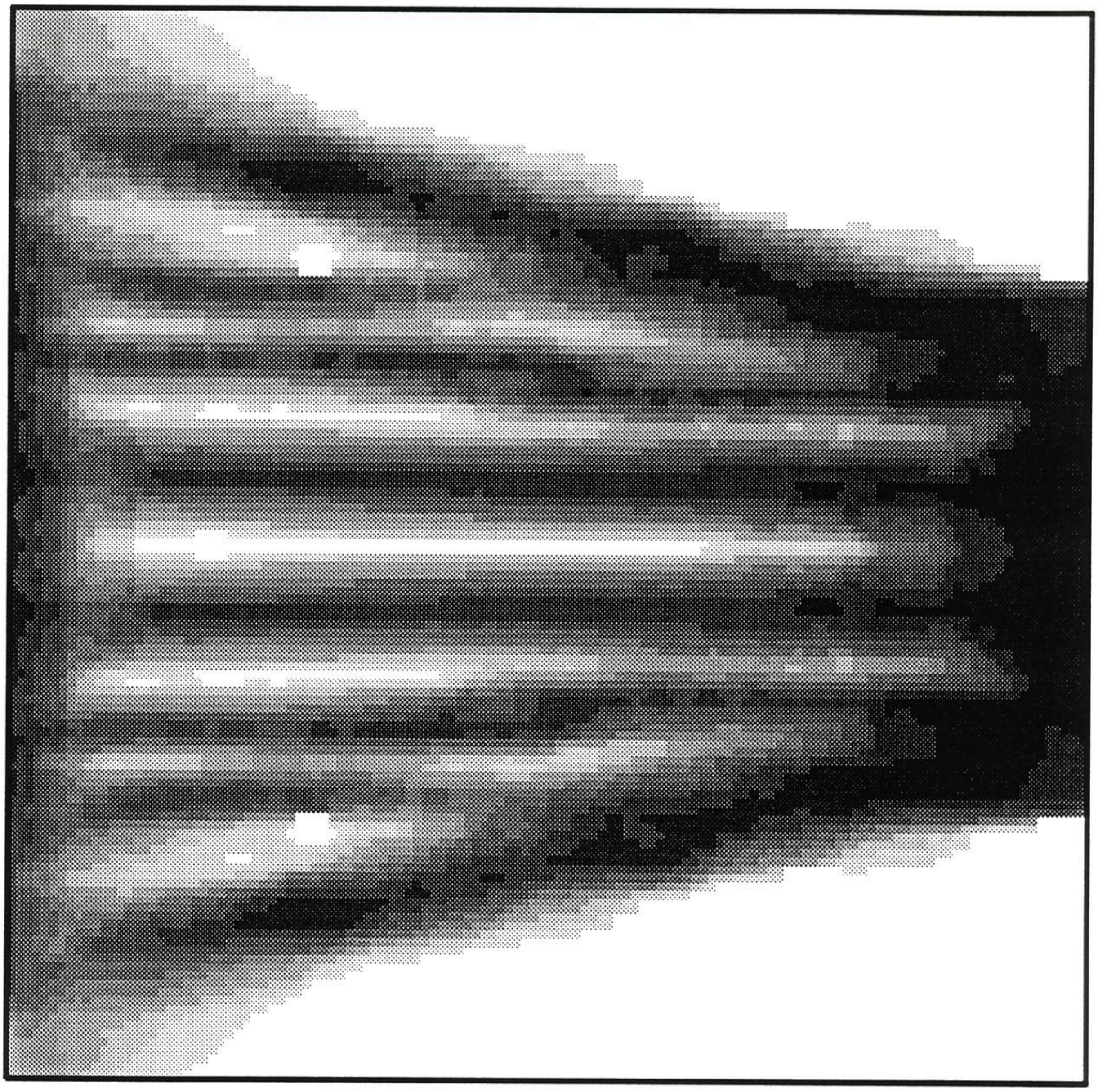

Figure 4
fillet : $\lambda = 1$, 52.6 % of material removed, compliance = 114.8

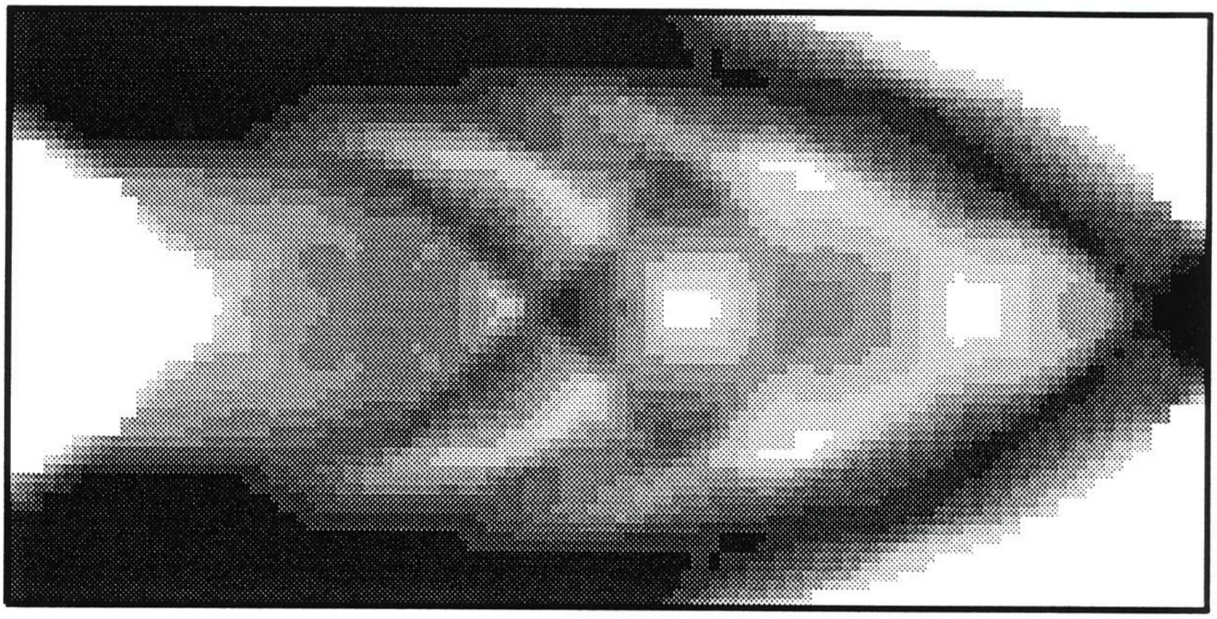

Figure 5
cantilever : $\lambda = 2$, 47.6 % of material removed, compliance = 167.3

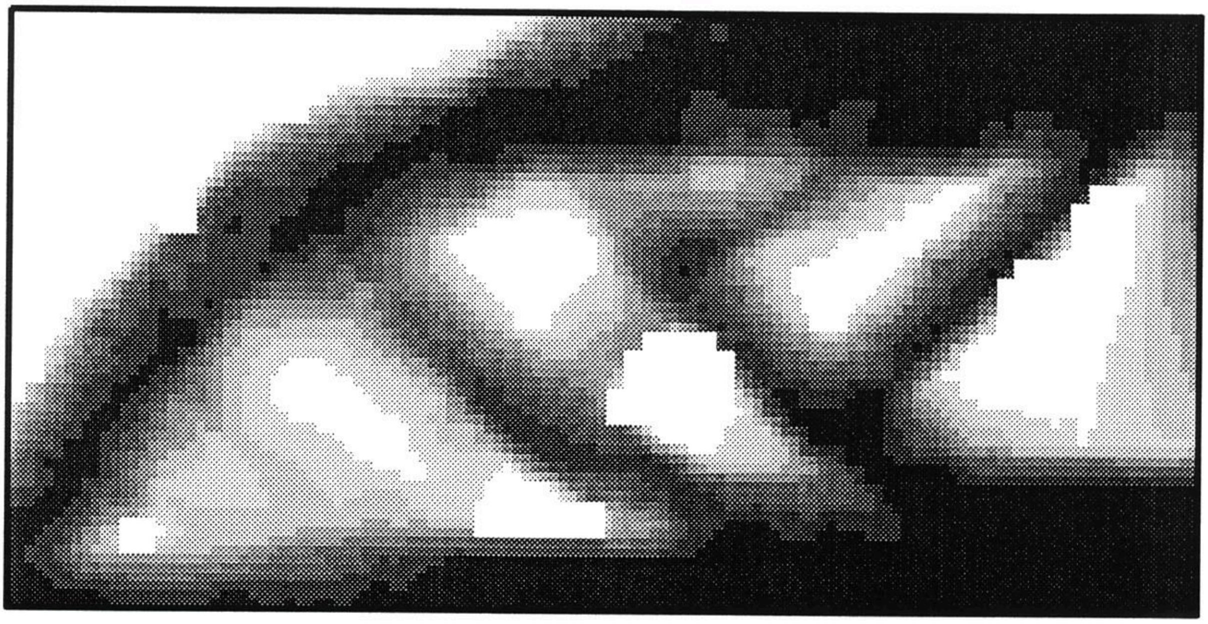

Figure 6
beam : $\lambda = 0.5$, 44.5 % of material removed, compliance = 45.2