Grégoire ALLAIRE

TWO-SCALE CONVERGENCE : A NEW METHOD IN PERIODIC HOMOGENIZATION

1. Introduction.

In many fields of science and technology one has to solve boundary value problems in periodic media. Quite often the size of the period is small compared to the size of a sample of the medium, and, denoting by ε their ratio, an asymptotic analysis, as ε goes to zero, is called for. In other words, starting from a microscopic description of a problem, we seek a macroscopic, or effective, description. This process of making an asymptotic analysis and seeking an averaged formulation is called homogenization (there is a vast body of literature on that topic, see [5], [6], [11] for an introduction, and additional references). Here, we focus on the homogenization of periodic structures, but we recall that homogenization is not restricted to that particular case and can be applied to any kind of disordered media (cf. the Γ-convergence of E. DeGiorgi [7], the H-convergence of L. Tartar [13], [8], or the G-convergence of S. Spagnolo [12]).

To fix ideas, we consider the well-known model problem in homogenization : a linear second-order partial differential equation with periodically oscillating coefficients. Such an equation models, for example, the heat conduction in a periodic composite medium. We call Ω the material domain (a bounded open set in $\mathbb{R}^N$), ε the period, and Y the rescaled unit cell (i.e. $Y = [0;1]^N$). Denoting by f the source term (a function of $L^2(\Omega)$), and enforcing a Dirichlet boundary condition for the unknown u_ε, this equation reads as

$$\begin{cases} - div\left[A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right] = f & in\ \Omega \\ u_\varepsilon = 0 & on\ \partial\Omega \end{cases} \tag{1.1}$$

where $A(y)$ is a $L^\infty(Y)$-matrix (the diffusion coefficients), Y-periodic in y, such that there exists two positive constants $0 < \alpha \le \beta$ satisfying

$$\alpha|\xi|^2 \le \sum_{i,j=1}^{N} A_{ij}(x,y)\xi_i\xi_j \le \beta|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N. \tag{1.2}$$

Under assumption (1.2), it is well-known that equation (1.1) admits a unique solution u_ε in $H_0^1(\Omega)$ which satisfies the a priori estimate

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \le C\,\|f\|_{L^2(\Omega)} \tag{1.3}$$

where C is a positive constant which depends only on Ω and α, and not on ε. In view of (1.3), the sequence of solutions u_ε is uniformly bounded in $H_0^1(\Omega)$ as ε goes to zero, and thus there exists a limit u such that, up to a subsequence, u_ε converges weakly to u in $H_0^1(\Omega)$. The homogenization of (1.1) amounts to find a "homogenized" equation which admits the limit u as its unique solution.

In section 2 we are going to recall the usual process of homogenization which relies on the successive application of two different methods. In a first step, two-scale asymptotic expansions are used to formally obtain the homogenized equation. In a second step, the so-called energy method of Tartar is applied to prove that the limit u is indeed the unique solution of the homogenized equation deduced from the first step. This way of proceeding is not entirely satisfactory since it involves two different methods which barely interact and are somehow redundant. In particular, if carefully used, two-scale asymptotic expansions give the right form of the homogenized problem. Unfortunately, this method is only formal and needs to be rigorously justified by the energy method. On the other hand the latter method doesn't use much of the information gained by the asymptotic expansions and is sometimes difficult to work out (this is not surprising since it was not conceived by L. Tartar for periodic problems, but rather in the more general and more difficult context of H-convergence).

Thus, there is room for a more efficient homogenization method, dedicated to partial differential equations with periodically oscillating coefficients : the two-scale convergence method. Section 3 is devoted to the definition of two-scale convergence and related results. In section 4 this new method is applied to the homogenization of our model problem (1.1).

2. The classical method : two-scale asymptotic expansions, and the energy method of Tartar.

We briefly recall the classical method for the homogenization of the model problem (1.1). In a first step we apply the well known two-scale asymptotic expansion method [5], [6], [11] in order to find the precise form of the homogenized equation. The key of that method is to postulate the following ansatz for u_ε

$$u_\varepsilon(x) = u_0(x,\frac{x}{\varepsilon}) + \varepsilon\, u_1(x,\frac{x}{\varepsilon}) + \varepsilon^2\, u_2(x,\frac{x}{\varepsilon}) + \cdots , \tag{2.1}$$

where each term $u_i(x,y)$ is Y-periodic in y. The ansatz (2.1) is inserted in equation (1.1), and a geometric series in ε is obtained by application of the formal rule of differentiation

$$\frac{\partial}{\partial x}\left[u_i(x,\frac{x}{\varepsilon})\right] = \frac{\partial u_i}{\partial x}(x,\frac{x}{\varepsilon}) + \varepsilon^{-1}\frac{\partial u_i}{\partial y}(x,\frac{x}{\varepsilon}).$$

Then, identifying the coefficients of this series to zero leads to a cascade of equations. The first one (corresponding to the ε^{-2} term) is

$$\begin{cases} -\,div_y\left[A(y)\nabla_y u_0\right] = 0 \quad in\ Y \\ y \to u_0(x,y) \quad Y\text{-}periodic. \end{cases} \tag{2.2}$$

This implies that u_0 doesn't depend on y, namely

$$u_0(x,y) \;=\; u(x). \tag{2.3}$$

The second one (the ε^{-1} term) is

$$\begin{cases} -\,div_y\left[A(y)[\nabla_y u_1(x,y) + \nabla_x u(x)]\right] \;=\; 0 \quad in\ Y \\ y \to u_1(x,y) \quad Y\text{-}periodic. \end{cases} \tag{2.4}$$

From (2.4) we compute u_1 in terms of the gradient of u :

$$u_1(x,y) \;=\; \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y), \tag{2.5}$$

where, for $1 \le i \le N$, w_i is the unique solution of the so-called local or cell problem

$$\begin{cases} -\,div_y\left[A(y)[\nabla_y w_i(y) + e_i]\right] \;=\; 0 \quad in\ Y \\ y \to w_i(y) \quad Y\text{-}periodic. \end{cases} \tag{2.6}$$

Finally the third one (the ε^{0} term) is

$$\begin{cases} -\,div_y\left[A(y)\nabla_y u_2(x,y)\right] \;=\; f(x) + div_y\left[A(y)\nabla_x u_1(x,y)\right] \\ \qquad\qquad\qquad + div_x\left[A(y)[\nabla_y u_1(x,y) + \nabla_x u(x)]\right] \quad in\ Y \\ y \to u_2(x,y) \quad Y\text{-}periodic. \end{cases} \tag{2.7}$$

Applying the Fredholm alternative to (2.7) (the average on Y of the right hand side must be zero), and replacing u_1 by its expression (2.5) leads to the homogenized equation

$$\begin{cases} -\,div\left[A^*\nabla u(x)\right] \;=\; f \quad in\ \Omega \\ u = 0 \quad on\ \partial\Omega \end{cases} \tag{2.8}$$

where the entries of the matrix A^* are given by

$$A^*_{ij} \;=\; \int_Y A(y)[\nabla_y w_i(y) + e_i].[\nabla_y w_j(y) + e_j]\, dy. \tag{2.9}$$

This method is very simple and powerful, but unfortunately is formal since there is no reason, a priori, for the ansatz (2.1) to hold true. Thus, the two-scale asymptotic expansion method is used only to guess the form of the homogenized equation (2.8), and a second step is needed to prove the convergence of the sequence u_ε to u. To this end, the more general and powerful method is the so-called energy method of L. Tartar [13], [8]. The goal of this method is to pass to the limit in the variational formulation of equation (1.1) :

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x).\nabla\phi(x)\,dx = \int_\Omega f(x)\phi(x)\,dx \quad \textit{for any } \phi \in H_0^1(\Omega). \tag{2.10}$$

For a given test function ϕ one cannot pass to the limit in (2.10), as ε goes to zero, since the left hand side involves the product of two weakly convergent sequences. The main idea is thus to replace the fixed test function ϕ by a carefully chosen sequence ϕ_ε which permits to pass to the limit thanks to some "compensated compactness" phenomenon. The right sequence of test functions is

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon\sum_{i=1}^{N}\frac{\partial\phi}{\partial x_i}(x)\tilde{w}_i\left(\frac{x}{\varepsilon}\right), \tag{2.11}$$

where ϕ is a smooth function with compact support in Ω, and $\tilde{w}_i$ is the solution of the adjoint cell problem (i.e. equation (2.6) with ${}^tA$ instead of A). Integrating by parts in (2.10) and using the cell equation (2.6) allows us to pass to the limit and to obtain the variational formulation of the homogenized problem (2.8). The convergence of the homogenization process is thus rigorously proved.

Although the asymptotic expansion method leads to both the local and the homogenized problem, the energy method uses only the knowledge of the cell problem to construct the test functions. The homogenized problem is then rederived independently. Clearly the two methods don't cooperate very much, and part of the homogenization process is done twice. On the contrary, we are going to see that the two-scale convergence is efficient because it is self-contained (i.e. it works in a single step). Loosely speaking, it appears as a blend of the two above methods.

3. Two-scale convergence.

Let us begin this section by a few notations : Ω is an open set of $\mathbb{R}^N$ (not necessarily bounded), and $Y = [0;1]^N$ is the closed unit cube. We denote by $C_\#^\infty(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^N$ which are periodic of period Y, and by $C_\#(Y)$ the Banach space of continuous and Y-periodic functions.

Definition 3.1.

A sequence of functions u_ε in $L^2(\Omega)$ is said to *two-scale converge* to a limit $u_0(x,y)$ belonging to $L^2(\Omega\times Y)$ if, for any function $\psi(x,y)$ in $D[\Omega;C_\#^\infty(Y)]$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \;=\; \int\int_{\Omega Y} u_0(x,y)\psi(x,y)\, dxdy \;. \square \qquad (3.1)$$

This new notion of "two-scale convergence" makes sense because of the next compactness theorem which was first proved by G. Nguetseng [9] (here, we give a new and simpler proof).

Theorem 3.2.

From each bounded sequence u_ε in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\Omega\times Y)$ such that this subsequence two-scale converges to u_0. $\square$

Before proving Theorem 3.2, we give a few examples of two-scale convergences.

(*) Any sequence u_ε which converges strongly in $L^2(\Omega)$ to a limit $u(x)$, two-scale converges to the same limit $u(x)$.

(**) For any smooth function $a(x,y)$, being Y-periodic in y, the associated sequence $a_\varepsilon(x) = a(x,x/\varepsilon)$ two-scale converges to $a(x,y)$.

(***)For the same smooth and Y-periodic function $a(x,y)$ the other sequence defined by $b_\varepsilon(x) = a(x,\frac{x}{\varepsilon^2})$ has the same two-scale limit and weak-L^2 limit, namely $\int_Y a(x,y)\, dy$ (this is a consequence of the difference of orders in the speed of oscillations for b_ε and the test functions $\psi(x,\frac{x}{\varepsilon})$) Clearly the two-scale limit captures only the oscillations which are in resonance with those of the test functions $\psi(x,\frac{x}{\varepsilon})$.

To establish theorem 3.2, we need the following

Lemma 3.3.

Let $B(\Omega,Y)$ denote the Banach space $L^2[\Omega;C_\#(Y)]$ if Ω is unbounded, or any of the Banach spaces $L^2[\Omega;C_\#(Y)]$, $L^2_\#[Y;C(\overline{\Omega})]$, $C[\overline{\Omega};C_\#(Y)]$, if Ω is bounded. Then, this space $B(\Omega,Y)$ has the following properties :

(i) $B(\Omega,Y)$ is a separable Banach space (i.e. contains a dense countable family)

(ii) $B(\Omega,Y)$ is dense in $L^2(\Omega\times Y)$

(iii) for any $\psi(x,y) \in B(\Omega,Y)$, the function $\psi(x,\frac{x}{\varepsilon})$ is measurable and satisfies

$$\|\psi(x,\frac{x}{\varepsilon})\|_{L^2(\Omega)} \le \|\psi(x,y)\|_{B(\Omega,Y)}$$

(iv) for any $\psi(x,y) \in B(\Omega,Y)$, one has

$$\lim_{\varepsilon\to 0} \int_\Omega \psi(x,\frac{x}{\varepsilon})^2\, dx = \iint_{\Omega Y} \psi(x,y)^2\, dxdy \ . \ \Box$$

In the case where Ω is bounded and $B(\Omega,Y)$ is defined as $C[\overline{\Omega};C_\#(Y)]$, lemma 3.3 is easily proved since any function $\psi(x,y)$ in this space is continuous in both variables x and y. In the other cases the delicate point is (iv) which holds true as soon as $\psi(x,y)$ is continuous in one of its arguments (as it is the case when ψ belongs to $L^2[\Omega;C_\#(Y)]$ or $L^2_\#[Y;C(\overline{\Omega})]$). A complete proof of lemma 3.3 may be found in [2].

Proof of theorem 3.2.

Let u_ε be a bounded sequence in $L^2(\Omega)$: there exists a positive constant C such that

$$\|u_\varepsilon\|_{L^2(\Omega)} \le C.$$

For any function $\psi(x,y) \in B(\Omega,Y)$, we deduce from (iii) in lemma 3.3 that

$$\left| \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \right| \le C\, \|\psi(x,\frac{x}{\varepsilon})\|_{L^2(\Omega)} \le C\, \|\psi(x,y)\|_{B(\Omega,Y)} . \qquad (3.2)$$

Thus, for fixed ε, the left hand side of (3.2) turns out to be a bounded linear form on $B(\Omega,Y)$. Let us denote by $B'(\Omega,Y)$ the dual space of $B(\Omega,Y)$. By virtue of the Riesz representation theorem, there exists a unique function $\mu_\varepsilon \in B'(\Omega,Y)$ such that

$$< \mu_\varepsilon,\psi > = \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \qquad (3.3)$$

where the brackets in the left hand side of (3.3) denotes the duality product between $B(\Omega,Y)$ and its dual. Furthermore, in view of (3.2), the sequence μ_ε is bounded in $B'(\Omega,Y)$. Since the space $B(\Omega,Y)$ is separable (see (i) in lemma 3.3), from any bounded sequence of its dual one can extract a subsequence which converges for the weak * topology. Thus, there exists $\mu_0 \in B'(\Omega,Y)$ such that, up to a subsequence, and for any $\psi \in B(\Omega,Y)$

$$< \mu_\varepsilon,\psi > \ \to \ < \mu_0,\psi > . \qquad (3.4)$$

By combining (3.3) and (3.4) we obtain, up to a subsequence, and for any $\psi \in B(\Omega,Y)$

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx = < \mu_0,\psi > . \qquad (3.5)$$

By virtue of (iv) in lemma 3.3 we have

$$\lim_{\varepsilon\to 0} \|\psi(x,\frac{x}{\varepsilon})\|_{L^2(\Omega)} = \|\psi(x,y)\|_{L^2(\Omega\times Y)} . \qquad (3.6)$$

Now, passing to the limit in the first two terms of (3.2) with the help of (3.5) and (3.6), we deduce

$$| < \mu_0, \psi > | \leq C \, \| \psi \|_{L^2(\Omega \times Y)} .$$

By density of $B(\Omega,Y)$ in $L^2(\Omega\times Y)$ (see (ii) in lemma 3.3), μ_0 is identified with a function $u_0 \in L^2(\Omega\times Y)$, i.e.

$$< \mu_0, \psi > = \iint_{\Omega Y} u_0(x,y) \psi(x,y) \, dxdy . \tag{3.7}$$

Equalities (3.5) and (3.7) give the desired result. □

Remark that the choice of the space $B(\Omega,Y)$ is purely technical and does not affect the final result of theorem 3.2. Remark also that the test function $\psi(x,y)$ in definition 3.1 of the two-scale convergence doesn't need to be very smooth since theorem 3.2 is proved, for example, with $\psi(x,y) \in L^2[\Omega;C_\#(Y)]$.

The next theorem shows that more information is contained in a two-scale limit than in a weak-L^2 limit ; some of the oscillations of a sequence are contained in its two-scale limit. When all of them are captured by the two-scale limit (condition (3.9) below), one can even obtain a strong convergence (a corector result in the vocabulary of homogenization).

Theorem 3.4.

Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega\times Y)$.

(i) Then u_ε converges also to $u(x) = \int_Y u_0(x,y) \, dy$ in $L^2(\Omega)$ weakly, and we have

$$\lim_{\varepsilon\to 0} \| u_\varepsilon \|_{L^2(\Omega)} \geq \| u_0 \|_{L^2(\Omega\times Y)} \geq \| u \|_{L^2(\Omega)} . \tag{3.8}$$

(ii) Assume further that $u_0(x,y)$ is smooth (for example, belongs to $L^2[\Omega;C_\#(Y)]$), and that

$$\lim_{\varepsilon\to 0} \| u_\varepsilon \|_{L^2(\Omega)} = \| u_0 \|_{L^2(\Omega\times Y)} . \tag{3.9}$$

Then, we have

$$\lim_{\varepsilon\to 0} \| u_\varepsilon(x) - u_0(x,\frac{x}{\varepsilon}) \|_{L^2(\Omega)} = 0. \tag{3.10}$$

Proof.

By taking test functions $\psi(x)$, which depends only on x, in the definition of two-scale convergence, we immediately obtain that u_ε weakly converges to $u(x) = \int_Y u_0(x,y)\, dy$ in $L^2(\Omega)$. To obtain (3.8), we take a smooth and Y-periodic function $\psi(x,y)$ and we compute

$$\int_\Omega [u_\varepsilon(x) - \psi(x,\frac{x}{\varepsilon})]^2\, dx = \int_\Omega u_\varepsilon(x)^2\, dx - 2\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx + \int_\Omega \psi(x,\frac{x}{\varepsilon})^2\, dx \geq 0. \tag{3.11}$$

Passing to the limit as ε goes to zero yields

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)^2\, dx \geq 2\int_\Omega\int_Y u_0(x,y)\psi(x,y)\, dxdy - \int_\Omega\int_Y \psi(x,y)^2\, dxdy\ .$$

Then, using a sequence of smooth functions which converges strongly to u_0 in $L^2(\Omega\times Y)$ leads to

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)^2\, dx \geq \int_\Omega\int_Y u_0(x,y)^2\, dxdy\ .$$

On the other hand, the Cauchy-Schwarz inequality in Y gives the other inequality in (3.8). To obtain (3.10) we use assumption (3.9) when passing to the limit in the right hand side of (3.11). This yields

$$\lim_{\varepsilon\to 0}\int_\Omega [u_\varepsilon(x) - \psi(x,\frac{x}{\varepsilon})]^2\, dx = \int_\Omega\int_Y [u_0(x,y) - \psi(x,y)]^2\, dxdy\ . \tag{3.12}$$

Now, if u_0 is smooth enough as to ensure that $u_0(x,\frac{x}{\varepsilon})$ is measurable and belongs to $L^2(\Omega)$, we can replace ψ by u_0 in (3.12) to obtain (3.10). □

We have just seen that the smoothness assumption on u_0 in part (ii) of theorem 3.4 is needed only to achieve the measurability of $u_0(x,\frac{x}{\varepsilon})$ (which otherwise is not guaranteed for a function of $L^2(\Omega\times Y)$). However, one could wonder if all two-scale limits automatically satisfy this property. Unfortunately, this is not true, and it can be shown that any function in $L^2(\Omega\times Y)$ is attained as a two-scale limit (see lemma 1.13 in [2]).

So far we have only considered bounded sequences in $L^2(\Omega)$. The next proposition investigates the case of a bounded sequence in $H^1(\Omega)$.

Proposition 3.5.

Let u_ε be a bounded sequence in $H^1(\Omega)$. Then, there exist $u(x) \in H^1(\Omega)$ and $u_1(x,y) \in L^2[\Omega;H^1_\#(Y)/\mathbb{R}]$ such that, up to a subsequence, u_ε two-scale converges to $u(x)$, and ∇u_ε two-scale converges to $\nabla_x u(x)+\nabla_y u_1(x,y)$.

Proof.

Since u_ε (resp. ∇u_ε) is bounded in $L^2(\Omega)$ (resp. $[L^2(\Omega)]^N$), up to a subsequence, it two-scale converges to a limit $u_0(x,y) \in L^2(\Omega\times Y)$ (resp. $\chi_0(x,y) \in [L^2(\Omega\times Y)]^N$). Thus for any $\Psi(x,y) \in D[\Omega;C^\infty_\#(Y)]^N$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega \nabla u_\varepsilon(x).\Psi(x,\frac{x}{\varepsilon})\,dx = \iint_{\Omega Y} \chi_0(x,y).\Psi(x,y)\,dxdy. \tag{3.13}$$

Integrating by parts the left hand side of (3.13) gives

$$\varepsilon \int_\Omega \nabla u_\varepsilon(x).\Psi(x,\frac{x}{\varepsilon})\,dx = -\int_\Omega u_\varepsilon(x)[div_y \Psi(x,\frac{x}{\varepsilon}) + \varepsilon div_x \Psi(x,\frac{x}{\varepsilon})]\,dx\ .$$

Passing to the limit yields

$$0 = -\iint_{\Omega Y} u_0(x,y) div_y \Psi(x,y)\,dxdy\ .$$

This implies that $u_0(x,y)$ does not depend on y. Thus there exists $u(x) \in L^2(\Omega)$, such that $u_0 \equiv u$. Next, in (3.13) we choose a function Ψ such that $div_y \Psi(x,y) = 0$. Integrating by parts we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x) div_x \Psi(x,\frac{x}{\varepsilon})\,dx &= -\iint_{\Omega Y} \chi_0(x,y).\Psi(x,y)\,dxdy \\ &= \iint_{\Omega Y} u(x) div_x \Psi(x,y)\,dxdy. \end{aligned} \tag{3.14}$$

If Ψ does not depend on y, (3.14) proves that $u(x)$ belongs to $H^1(\Omega)$. Furthermore, we deduce from (3.14) that

$$\iint_{\Omega Y} [\chi_0(x,y) - \nabla u(x)].\Psi(x,y)\,dxdy = 0$$

for any function $\Psi(x,y) \in D[\Omega;C^\infty_\#(Y)]^N$ with $div_y \Psi(x,y) = 0$. Recall that the orthogonal of divergence-free functions are exactly the gradients (this well-known result can be very easily proved in the present context by means of Fourier analysis in Y). Thus, there exists a unique function $u_1(x,y)$ in $L^2[\Omega;H^1_\#(Y)/\mathbb{R}]$ such that

$$\chi_0(x,y) = \nabla u(x) + \nabla_y u_1(x,y)\ . \ \square$$

For more results about two-scale convergence (including generalizations to the L^p case or to the multi-scale case) the reader is referred to [2].

4. Application to a model problem.

We go back to the model problem introduced in the first section :

$$\begin{cases} -\, div\left[A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right] = f \quad in\ \Omega \\ u_\varepsilon = 0 \quad on\ \partial\Omega \end{cases} \tag{4.1}$$

where $A(y)$ is a Y-periodic matrix satisfying the coercitivity hypothesis (1.2). We recall that equation (4.1) admits a unique solution u_ε in $H_0^1(\Omega)$ which satisfies the a priori estimate

$$\| u_\varepsilon \|_{H_0^1(\Omega)} \leq C \, \| f \|_{L^2(\Omega)} \tag{4.2}$$

where C is a positive constant which does not depend on ε.

We now describe what we call the "two-scale convergence method" for homogenizing problem (4.1). In a **first step,** we deduce from the a priori estimate (4.2) the precise form of the two-scale limit of the sequence u_ε. Applying proposition 3.5, we know that there exists two functions, $u(x) \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$, such that, up to a subsequence, u_ε two-scale converges to $u(x)$, and ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$. In view of these limits, u_ε is expected to behave as $u(x) + \varepsilon u_1(x, x/\varepsilon)$.

Thus, in a **second step,** we multiply equation (4.1) by a test function similar to the limit of u_ε, namely $\phi(x) + \varepsilon\phi_1(x, x/\varepsilon)$, where $\phi(x) \in D(\Omega)$ and $\phi_1(x,y) \in D[\Omega; C_\#^\infty(Y)]$. This yields

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon . \left[\nabla\phi(x) + \nabla_y \phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x \phi_1\left(x, \frac{x}{\varepsilon}\right) \right] dx = \int_\Omega f(x) \left[\phi(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)\right] dx. \tag{4.3}$$

Regarding ${}^t A(x/\varepsilon)[\nabla\phi(x) + \nabla_y \phi_1(x, x/\varepsilon)]$ as a test function for the two-scale convergence (cf. definition 2.1), we pass to the two-scale limit in (4.3) for the sequence ∇u_ε. (Although this test function is not necessarily very smooth, it belongs at least to $L_\#^2[Y; C(\overline{\Omega})]$ which is enough for the two-scale convergence theorem 3.2 to hold.) Thus, the two-scale limit of (4.3) is

$$\int_\Omega \int_Y A(y) [\nabla u(x) + \nabla_y u_1(x,y)] . [\nabla\phi(x) + \nabla_y \phi_1(x,y)] \, dx dy = \int_\Omega f(x)\phi(x) \, dx. \tag{4.4}$$

In a **third step,** we read off a variational formulation for (u, u_1) in (4.4). By density, (4.4) holds true for any (ϕ, ϕ_1) in the Hilbert space $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$. Endowing this space with the norm $||\nabla u(x)||_{L^2(\Omega)} + ||\nabla_y u_1(x,y)||_{L^2(\Omega \times Y)}$, we check the conditions of the Lax-Milgram lemma for (4.4). Let us focus on the coercivity of the bilinear form defined by the left hand side of (4.4)

$$\int_\Omega \int_Y A(y)[\nabla\phi(x) + \nabla_y \phi_1(x,y)].[\nabla\phi(x) + \nabla_y \phi_1(x,y)]\, dxdy \geq$$

$$\alpha \int_\Omega \int_Y |\nabla\phi(x) + \nabla_y \phi_1(x,y)|^2\, dxdy = \alpha \int_\Omega |\nabla\phi(x)|^2\, dx + \alpha \int_\Omega \int_Y |\nabla_y \phi_1(x,y)|^2\, dxdy .$$

Thus, by application of the Lax-Milgram lemma, there exists a unique solution (u, u_1) of the variational formulation (4.4) in $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$. Consequently, the entire sequences u_ε and ∇u_ε converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x,y)$. An easy integration by parts shows that (4.4) is a variational formulation associated to the following system of equations that we call the "two-scale homogenized problem"

$$\begin{cases} -\,div_y \left[A(y)[\nabla u(x) + \nabla_y u_1(x,y)]\right] = 0 & in\ \Omega \times Y \\ -\,div_x \left[\int_Y A(y)[\nabla u(x) + \nabla_y u_1(x,y)]\, dy\right] = f & in\ \Omega \\ u(x) = 0 \quad on\ \partial\Omega & \\ y \to u_1(x,y) \quad Y\!-\!periodic. & \end{cases} \tag{4.5}$$

It is easily seen that (4.5) is equivalent to the usual homogenized and cell equations (2.6)-(2.8) through the relation

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y).$$

At this point, the homogenization process could be considered as achieved since the entire sequence of solutions u_ε converges to the solution of a well-posed limit problem, namely the two-scale homogenized problem (4.5). However, it is usually preferable, from a physical or numerical point of view, to eliminate the microscopic variable y (one doesn't want to solve the small scale structure).

Thus, in a **fourth (and optional) step,** we can eliminate from (4.5) the y variable and the u_1 unknown. This is an easy algebra exercise (left to the reader) to derive from (4.5) the usual homogenized and cell equations (2.6)-(2.8). Due to the simple form of our model problem the two equations of (4.5) can be decoupled in a macroscopic and microscopic equations, but we emphasize that it is not always possible, and sometimes it leads to very complicate forms of the homogenized equation, including integro-differential operators and non-explicit equations. Thus, the homogenized equation does not always belong to a class for which an existence and uniqueness theory is easily available, on the contrary of the two-scale homogenized system, which is, in most cases, of the same type as the original problem, but with twice more variables (x and y) and unknowns (u and u_1). The supplementary, microscopic, variable and unknown play the role of "hidden" variables in the vocabulary of mechanics. Although their presence doubles the size of the limit problem, it greatly simplifies its structure (which could be useful for numerical purposes too), while eliminating them introduces "strange" effects (like memory or non-local effects) in the usual homogenized problem. In short, both formulations ("usual" or two-scale) of the homogenized problem have their pros and cons, and none should be eliminated without second thoughts. Particulary striking examples of the above discussion may be found in [1], [2], [3] .

Corrector results are easily obtained with the two-scale convergence method. By application of theorem 3.4, we are going to prove that

$$\left[u_\varepsilon(x) - u(x) - \varepsilon\, u_1(x, \frac{x}{\varepsilon}) \right] \to 0 \quad \textit{in } H^1(\Omega) \textit{ strongly.} \tag{4.6}$$

This rigorously justifies the two first term in the usual asymptotic expansion (2.1) of the solution u_ε. Let us first remark that, by standard regularity results for the solutions $w_i(y)$ of the cell problem (2.6), the term $u_1(x, x/\varepsilon) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(x/\varepsilon)$ does actually belong to $L^2(\Omega)$ and can be seen as a test function for the two-scale convergence. Bearing this in mind, we write

$$\int_\Omega A(\frac{x}{\varepsilon})[\nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1(x, \frac{x}{\varepsilon})]^2\, dx = \int_\Omega f(x) u_\varepsilon(x)\, dx$$

$$+ \int_\Omega A(\frac{x}{\varepsilon})[\nabla u(x) + \nabla_y u_1(x, \frac{x}{\varepsilon})]^2\, dx - 2\int_\Omega A(\frac{x}{\varepsilon}) \nabla u_\varepsilon(x) . [\nabla u(x) + \nabla_y u_1(x, \frac{x}{\varepsilon})]\, dx.$$

Using the coercivity condition for A, and passing to the two-scale limit yields

$$\alpha \lim_{\varepsilon \to 0} \| \nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1(x, \frac{x}{\varepsilon}) \|^2_{L^2(\Omega)} \leq \int_\Omega f(x) u(x) dx$$

$$- \int_\Omega \int_Y A(y)[\nabla u(x) + \nabla_y u_1(x,y)]^2 dx dy. \tag{4.7}$$

In view of (4.5), the right hand side of (4.7) is equal to zero, which gives the desired result (4.6).

We conclude this short presentation of the two-scale convergence method by saying that it is a very general method which can handle all possible difficulties in periodic homogenization, as perforated domains, non-linear (monotone) equations, memory or non-local effects, highly heterogeneous coefficients, etc. : see [1], [2], [3],

[4] (where a so-called dilation operator, similar to two-scale convergence, is introduced), and [10].

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Grégoire ALLAIRE
Commissariat à l'Energie Atomique
DRN/DMT/SERMA
C.E.N. Saclay, F-91191 GIF sur YVETTE