HOMOGENIZATION OF THE STOKES FLOW
IN A CONNECTED POROUS MEDIUM

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Abstract. In this paper we prove the convergence of the homogenization process of the Stokes equations with Dirichlet boundary condition in a periodic porous medium. We consider here the case where the solid part of the porous medium is connected, and we generalize to this case the results obtained by Tartar (1980).

Introduction

We define a porous medium as the periodic repetition in a bounded domain $\Omega$ of an elementary $\varepsilon$-sized cell in which the solid part of the porous medium is also of size $\varepsilon$. A typical case of such a porous medium is a regular lattice of interconnected cylinders (see Fig. 1). Let $\Omega_\varepsilon$ be the fluid part contained in $\Omega$. The flow of an incompressible viscous fluid in $\Omega_\varepsilon$ under the action of an exterior force $f$ is ruled by the Stokes equations (S$_\varepsilon$) with Dirichlet boundary condition:

\[(S_\varepsilon): \nabla p_\varepsilon - \Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon, \quad \nabla \cdot u_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \text{ on } \partial \Omega_\varepsilon\]

$(u_\varepsilon, p_\varepsilon)$ being the velocity and the pressure of the fluid.

Fig. 1.
The goal of this paper is to prove the following result: let \( \bar{u}_\varepsilon \) and \( P_\varepsilon \) be extensions of \( u_\varepsilon \) and \( p_\varepsilon \) to the whole of \( \Omega \), defined in the following way:

\[
\bar{u}_\varepsilon = \begin{cases} 
  u_\varepsilon & \text{in } \Omega_\varepsilon, \\
  0 & \text{in } \Omega - \Omega_\varepsilon,
\end{cases} \quad P_\varepsilon = \begin{cases} 
  p_\varepsilon & \text{in } \Omega_\varepsilon, \\
  \frac{1}{|Y^e_i|} \int_{Y^e_i} p_\varepsilon & \text{in } Y^e_i \text{ for each } i;
\end{cases}
\]

where, for an \( \varepsilon \)-cell \( Y^e_i \), we denote by \( Y^e_i \) and \( Y^s_i \) the corresponding fluid and solid parts. Then

\[
u_\varepsilon/\varepsilon^2 \rightharpoonup u \quad \text{in } [L^2(\Omega)]^N \text{ weakly,} \quad P_\varepsilon \rightarrow p \quad \text{in } L^2_{\text{loc}}(\Omega)/\mathbb{R} \text{ strongly}
\]

where \((u, p)\) is the unique solution of Darcy’s law \((S)\):

\[(S): \quad u = A(f - \nabla p) \quad \text{in } \Omega, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad u \cdot n = 0 \quad \text{on } \partial \Omega.
\]

Here \( A \) is a constant, symmetric and positive definite matrix depending only on the elementary cell’s geometry.

It is worth noticing that if \( \Omega_\varepsilon \) is defined as the union of entire elementary cells, then the convergence of the sequence \( P_\varepsilon \) occurs in \( L^2(\Omega)/\mathbb{R} \) (there is no more “loc”). This is the case, for example, when \( \Omega \) is a cube.

The above result is the generalization of results obtained by Tartar, Lipton, and Avellaneda to another geometry. These authors considered the case of a porous medium the solid part of which is composed of disconnected grains, each included in a corresponding \( \varepsilon \)-sized cell. For that geometry, Tartar proved in [8] that there exists an extension \( P_\varepsilon \) of the pressure which allows to pass to the limit in \((S_\varepsilon)\) obtaining \((S)\). Lipton and Avellaneda recently noticed [4] that, actually, this extension \( P_\varepsilon \) is just obtained by taking the mean value of the pressure \( p_\varepsilon \) in the fluid part of each \( \varepsilon \)-sized cell as the value of \( P_\varepsilon \) in the solid part of the same cell. This remark illuminates the meaning of the pressure’s extension defined by Tartar using a transposition process. However, Tartar’s way of defining the extension seems to be necessary to prove that the sequence \( P_\varepsilon \) is bounded (and even relatively compact) in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \). Here we are concerned with the physically realistic case of a porous medium which has connected solid and fluid parts, and a diphasic boundary \( \partial \Omega \). So we follow the scheme of [8], except for the pressure’s extension which is new. In order to pass to the limit in the equations, we use the energy method introduced by Tartar [9] (see also [1, Chapter 1]). For more details about periodic homogenization of Stokes equations, see [7, Chapter 7]. In [6] (see also [2, Chapter 1, Section 5]) Polisevsky has already proved a similar result with a different method, but in his result the convergence of \( P_\varepsilon \) occurs in \( L^{5/2}(\Omega) \) instead of \( L^2(\Omega) \).

1. Modelization of the porous medium and formulation of the Stokes problem

As usual in periodic homogenization theory (see, e.g., [7,1]), we consider a porous medium obtained by the periodic repetition of an elementary cell of size \( \varepsilon \), in a bounded domain of \( \mathbb{R}^N \). We will first define the corresponding dimensionless elementary cell \( Y \).
Fig. 2. Forbidden situations. (a) The boundary of $E_S$ is not of class $C^1$ because $Y_S$ is not $Y$-periodic. (b) No contact between the fluid parts of two adjacent cells implies $E_F$ is not connected. (c) Although $E_F$ has a smooth boundary, $\partial Y_F$ is not locally Lipschitz at point $M$.

1.1. Definition of the elementary cell $Y$

Let $Y = \mathbb{Z}^N \subseteq \mathbb{R}^N$, $N \geq 2$. Let $Y_S$ be a closed subset of $\overline{Y}$. We define $Y_F$, open set of $\mathbb{R}^N$, by $Y_F = Y - Y_S$, where $Y_S$ represents the part of $Y$ occupied by the solid and $Y_F$ represents the part of $Y$ occupied by the fluid.

The closed set $Y_S$ is repeated by $Y$-periodicity and fills the entire space $\mathbb{R}^N$, in order to obtain a closed set of $\mathbb{R}^N$, noted $E_S$. Let the open set $E_F$ be the complementary of $E_S$ in $\mathbb{R}^N$, i.e.

$$E_S = \left\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid \exists (k_1, \ldots, k_N) \in \mathbb{Z}^N \text{ such that} \right. $$

$$\left. (x_1 - 2k_1, \ldots, x_N - 2k_N) \in Y_S \right\}, \quad E_F = \mathbb{R}^N - E_S.$$

We assume the following hypotheses on $Y_F$ and $E_F$:

(i) $Y_F$ and $Y_S$ have strictly positive measures in $\overline{Y}$;

(ii) $E_F$ and the interior of $E_S$ are open sets with boundary of class $C^1$, and are locally located on one side of their boundary. Moreover $E_F$ is connected; \hspace{1cm} (1.1)

(iii) $Y_F$ is an open connected set with a locally Lipschitz boundary.

What is the concrete meaning of those hypotheses?

(i) means that the elementary cell $Y$ contains fluid and solid together.

(ii) implies that $Y_F$ has some properties:

- $Y_F$ is "$Y$-periodic", because $E_F$ has a boundary of class $C^1$; for example, the situation of Fig. 2(a) is forbidden.

- $Y_F$ has an intersection with each face of the cube $\overline{Y}$ which has a strictly positive surface measure; if not, $E_F$ could not be connected when $E_F$ and $E_S$ are locally located on one side of their boundary; for example, the situation of Fig. 2(b) is forbidden.

(iii) is a technical assumption necessary for the proof of Lemma 3.4; for example, the situation of Fig. 2(c) is forbidden.

In Fig. 3 we give three typical situations which agree with assumptions (1.1). Note that in Figs. 3(b) and 3(c), $Y_F$ has a locally Lipschitz boundary which is not of class $C^1$. This motivates (iii) in (1.1).
Let \((\Sigma_K)_{K \in \mathcal{K}}\) be the 2N faces of the cube \(\bar{Y}\) with \(\mathcal{K} = \{-N; -(N - 1); \ldots; -1; +1; \ldots; N - 1; N\}\) such that \(\Sigma_K\) and \(\Sigma_{-K}\) are the two faces of \(\bar{Y}\) orthogonal to the \(K\)th unit vector \(e_K\). A first consequence of the hypotheses (1.1) is the existence of a family of functions \((\phi_K)_{K \in \mathcal{K}}\) such that

(i) \(\phi_K \in C^\infty(\bar{Y})\) and \(\phi_K \geq 0\);
(ii) \(\phi_K \not= 0\) on \(\Sigma_K\), \(\phi_K \equiv 0\) on \(Y_S\) and \(\Sigma_{K'}\) for each \(K' \not= K\);
(iii) let \(\Sigma_K\) and \(\Sigma_{-K}\) be two opposite faces of \(\bar{Y}\); then \(\phi_K|_{\Sigma_K} = \phi_{-K}|_{\Sigma_{-K}}\).

(\(\phi_K|_{\Sigma_K}\) denotes the restriction of \(\phi_K\) to \(\Sigma_K\)). Examples are shown in Figs. 4 and 5.

Remark 1.1. In [8], Tartar considered the case (corresponding to Fig. 3(a)) \(Y_S \subset Y\) (\(Y_S\) is strictly included in \(Y\)). This assumption is not physically realistic in three dimensions because the solid part \(E_S\) is not a connected body.

1.2. Definition of the open set \(\Omega_\varepsilon\)

Let \(\Omega\) be a bounded and connected open set of \(\mathbb{R}^N\) with a smooth boundary \(\partial \Omega\) of class \(C^1\) \((N \geq 2)\). Let \(\varepsilon > 0\). The set \(\Omega\) is covered with a regular mesh of size \(2\varepsilon\), each cell being a cube \(Y_\varepsilon^p\)
with $1 \leq i \leq N(\varepsilon)$. An elementary geometrical consideration gives

$$N(\varepsilon) = \left| \frac{\Omega}{(2\varepsilon)^N} \right| [1 + o(1)].$$

Let $\pi^i$ be the linear continuous invertible application, composed of a translation and an homothety of ratio $1/\varepsilon$, which maps $Y^i$ onto $Y$:

$$Y^i \xrightarrow{\pi^i} Y, \quad x \rightarrow y = x/\varepsilon + \text{translation}.$$  \hfill (1.3)

Now we define

$$Y_K^i = (\pi^i)^{-1}(Y_K), \quad Y_{F}^i = (\pi^i)^{-1}(Y_F), \quad \Sigma_K^i = (\pi^i)^{-1}(\Sigma_K).$$

We construct $\Omega_\varepsilon$ by picking out from $\Omega$ the solid parts $Y_K^i$: $\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} Y_K^i$. $\Omega_\varepsilon$ denotes the part of $\Omega$ occupied by the fluid.
By construction $\Omega_\epsilon$ is a bounded open set of $\mathbb{R}^N$. Unfortunately, $\Omega_\epsilon$ is not necessarily connected. Indeed, near the boundary $\partial \Omega$ there may be connected components of $\Omega$, which have a maximum size $\epsilon$ (see Fig. 6). In order to simplify the exposition and without loss of generality, we can suppress those $\epsilon$-sized connected components, and from now on we assume $\Omega_\epsilon$ to be connected (it will allow us to define a unique pressure in $\Omega_\epsilon$, up to a single additive constant in $\Omega$).

The set $\Omega$ represents the porous medium, and $\Omega_\epsilon$ its fluid part. Remark that $\Omega_\epsilon$ is supposed to be connected, but the solid part of the porous medium, which is represented by $\Omega - \Omega_\epsilon$, may be connected or not. Moreover, one can see that the boundary $\partial \Omega$ of the porous medium may be either diphasic (i.e., $\partial \Omega \cap \overline{\Omega} \neq \emptyset$ and $\partial \Omega \cap (\Omega - \Omega_\epsilon) \neq \emptyset$) or not. Obviously, the physically realistic case (a connected solid part, and a diphasic boundary) is taken into account in the present paper.

We now define $C_\epsilon$ and $\Omega'_\epsilon$, which approach respectively $\Omega$ and $\Omega_\epsilon$ in the following sense: let $C_\epsilon$ be the polygonal open set constituted by all the cells $\overline{Y}_i'$ entirely included in $\Omega$, and let $\Omega'_\epsilon$ be the fluid part of $C_\epsilon$ (see Fig. 7). More precisely,

$$\overline{C}_\epsilon = \bigcup_{i \in I(\epsilon)} \overline{Y}_i' \quad \text{with} \quad I(\epsilon) = \{ i \in [1; N(\epsilon)] \mid Y_i' \subset \Omega \}, \quad \Omega'_\epsilon = C_\epsilon \cap \Omega_\epsilon. \quad (1.4)$$

An elementary geometrical consideration gives $|\Omega - C_\epsilon| \leq C \epsilon \to 0$ and, for sufficiently small values of $\epsilon$, $C_\epsilon$ and $\Omega'_\epsilon$ are connected.

**Remark 1.2.** By definition of $\Omega'_\epsilon$, $\epsilon$ represents the characteristic length of the periodic elementary cell. In other words, $\epsilon$ is the ratio between the two spatial scales: the microscopic and macroscopic ones (see [7, Chapter 5]).

**Remark 1.3.** We assume that $\Omega$ and $E_F$ have smooth boundary of class $C^1$, but this has been done only for the convenience of the reader. In fact, all the following only requires the assumption that $\partial \Omega$ and $\partial E_F$ are locally Lipschitz.

**Remark 1.4.** In the general case we have no indications about the regularity of the boundary $\partial \Omega$, near the boundary $\partial \Omega$. For particular values of $\epsilon$, $\partial \Omega_\epsilon$ may be locally Lipschitz near $\partial \Omega$, but it is, in general, not true for all the values of $\epsilon$. That is why, in the sequel, the pressure's extension will
be defined only in $L^2_{\text{loc}}(\Omega)$ (instead of $L^2(\Omega)$). This is the price to pay in order to be able to handle the diphasic boundary $\partial \Omega$.

1.3. Formulation of the Stokes problem

The flow of an incompressible viscous fluid in the domain $\Omega_\varepsilon$ under the action of an exterior force $f$, and with a no-slip (Dirichlet) boundary condition, is described by the following Stokes equations, where $u_\varepsilon$ is the fluid velocity, $p_\varepsilon$ is the fluid pressure and $f$ given in $[L^2(\Omega)]^N$:

(S\varepsilon): \nabla p_\varepsilon - \Delta u_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon, \quad \nabla \cdot u_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega_\varepsilon. \tag{1.5-7}

(The viscosity and density of the fluid are supposed to be equal to 1.)

If $\Omega_\varepsilon$ is a bounded and connected open set with a locally Lipschitz boundary, a classical result asserts that there exists a unique solution $(u_\varepsilon, p_\varepsilon)$ of (S\varepsilon) belonging to $[H^1_0(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}]$. But in the present case the boundary $\partial \Omega_\varepsilon$ is not locally Lipschitz in the vicinity of $\partial \Omega$, hence the existence and uniqueness of the solution $(u_\varepsilon, p_\varepsilon)$ stand only in the space $V_\varepsilon \times L_\varepsilon$ with $V_\varepsilon$ being the closure in $[H^1_0(\Omega_\varepsilon)]^N$ of the space $\mathcal{D}_\varepsilon$, where

$$
\mathcal{D}_\varepsilon = \left\{ \phi \in \mathcal{D}(\Omega_\varepsilon)^N \mid \nabla \cdot \phi = 0 \quad \text{in} \quad \Omega_\varepsilon \right\},
$$

$$
L_\varepsilon = \left\{ q \in \mathcal{D}'(\Omega_\varepsilon) \mid \forall \omega \subset \subset \Omega, q \in L^2(\omega \cap \Omega_\varepsilon) \right\} / \mathbb{R}. \tag{1.8}
$$

We have the following inclusions which may be strict:

$$
V_\varepsilon \subset \left\{ \phi \in [H^1_0(\Omega_\varepsilon)]^N \mid \nabla \cdot \phi = 0 \quad \text{in} \quad \Omega_\varepsilon \right\}, \quad L_\varepsilon \supset L^2(\partial \Omega_\varepsilon)/\mathbb{R}.
$$

For more details about those existence and uniqueness results for Stokes equations, see, for example, [11, Chapter 1, Section 2].

In order to predict the “homogenized” limit of equations (S\varepsilon), one can apply the asymptotic expansion method to the system (S\varepsilon). Assume that

$$
\begin{bmatrix}
0^\prime(x) = \varepsilon^2 \left[ u^0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots \right], \\
p^\prime(x) = p_0(x, x/\varepsilon) + \varepsilon p_1(x, x/\varepsilon) + \varepsilon^2 p_2(x, x/\varepsilon) + \cdots
\end{bmatrix} \tag{1.9}
$$

where $u_0(x, y)$ and $p_0(x, y)$ are $Y$-periodic in the variable $y$. Then one can heuristically obtain (see [7, Chapter 7]) that $p_0(x, x/\varepsilon) = p(x)$, where $p \in H^1(\Omega)$ is the unique solution of

(S): \nabla \cdot [\overline{A} (f - \nabla p)] = 0 \quad \text{in} \quad \Omega, \quad \overline{A} (f - \nabla p) \cdot n = 0 \quad \text{on} \quad \partial \Omega, \tag{1.10}

and that $u_0(x, x/\varepsilon) = A(x/\varepsilon) \cdot (f(x) - \nabla p(x))$. Here, $\overline{A} = (1/|Y|) \int_Y A(y) \, dy$, a symmetric, positive definite matrix and $A(y)$ is the matrix composed of the column vectors $\overline{v}_K(y)$ for $1 \leq K \leq N$, defined as the unique solutions $(v_\varepsilon, q_\varepsilon) \in [H^1(Y_\varepsilon)]^N \times [L^2(Y_\varepsilon)/\mathbb{R}]$ of the following systems (S\varepsilon):

\begin{align*}
\nabla q_\varepsilon - \Delta v_\varepsilon &= e_K \quad \text{in} \quad Y_\varepsilon, \\
\nabla \cdot v_\varepsilon &= 0 \quad \text{in} \quad Y_\varepsilon, \\
v_\varepsilon &= 0 \quad \text{on} \quad \partial Y_\varepsilon, \quad v_\varepsilon \text{ and } q_\varepsilon \text{ are } Y\text{-periodic} \tag{1.11}
\end{align*}

where $e_K$ is the $K$th unit vector of $\mathbb{R}^N$.

Remark 1.5. (1.10) is a Darcy’s law for the pressure $p$ in a medium of permeability $\overline{A}$. (For more details about those results, see [7, Chapter 7].)
2. Convergence of the homogenization process

Notation. Let \( \tilde{\gamma} \) denote the extension by zero operator in \( \Omega - \Omega_\varepsilon \). More precisely, for each \( \phi \in H^1_0(\Omega) \), we define \( \tilde{\phi} \in H^1_0(\Omega) \) by

\[
\tilde{\phi} = \begin{cases} 
\phi & \text{in } \Omega_\varepsilon, \\
0 & \text{in } \Omega - \Omega_\varepsilon.
\end{cases}
\]

2.1. Statement of the main results

The purpose of this paper is to prove the following theorem.

Theorem 2.1. Consider the unique solution \((u_\varepsilon, p_\varepsilon)\) of \((S)\). Under the assumptions \((1.1)\) on the elementary cell, there exists \( P_\varepsilon \in L^2_{\text{loc}}(\Omega) \), which extends the pressure \( p_\varepsilon \) to the whole of \( \Omega \) (i.e. \( P_\varepsilon = p_\varepsilon \) in \( \Omega_\varepsilon \)), such that

\[
\tilde{u}_\varepsilon / \varepsilon^2 \rightarrow u \quad \text{in} \quad [L^2(\Omega)]^N \text{ weakly}, \quad P_\varepsilon \rightarrow p \quad \text{in} \quad L^2_{\text{loc}}(\Omega)/\mathbb{R} \text{ strongly}
\]

where \( u = \tilde{A}(f - \nabla p) \) and \( p \) is the unique solution of \((S)\) (see \((1.10)\)).

Remark 2.2. In the statement of Theorem 2.1 the convergence of the pressure's extension occurs in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \), instead of \( L^2(\Omega)/\mathbb{R} \) as one could expect. Nevertheless, if \( \Omega \) is such that \( \Omega = \Omega_\varepsilon \) for a sequence of \( \varepsilon \) values which tends to zero (e.g., this is the case if \( \Omega \) is a cube), then the pressure's convergence really occurs in \( L^2(\Omega)/\mathbb{R} \) (without the “loc”) for this sequence. This “local” result is imposed only because there are “cutted” cubes \( Y_\varepsilon \) in the vicinity of \( \partial \Omega \) (see also Remark 3.2).

The main difficulty in the proof of Theorem 2.1 is to extend the pressure to the whole of \( \Omega \). For this purpose we need the following crucial theorem.

Theorem 2.3. Assume that the hypotheses \((1.1)\) on the elementary cell \( Y \) hold true. Then there exists a linear continuous operator \( R_\varepsilon \), such that

(i) \( R_\varepsilon \in \mathcal{L}([H^1_0(C_\varepsilon)]^N; [H^1_0(\Omega_\varepsilon')]^N) \),

(ii) \( u \in [H^1_0(\Omega_\varepsilon')]^N \) implies \( R_\varepsilon u = u \) in \( \Omega_\varepsilon' \),

(iii) \( \nabla \cdot u = 0 \) in \( C_\varepsilon \) implies \( \nabla \cdot (R_\varepsilon u) = 0 \) in \( \Omega_\varepsilon' \),

(iv) there exists a constant \( C \), which does not depend on \( \varepsilon \), such that, for each \( u \in [H^1_0(C_\varepsilon)]^N \), we have

\[
\| R_\varepsilon u \|_{L^2(\Omega_\varepsilon')} + \varepsilon \| \nabla (R_\varepsilon u) \|_{L^2(\Omega_\varepsilon')} \leq C \left[ \| u \|_{L^2(\Omega_\varepsilon')} + \varepsilon \| \nabla u \|_{L^2(\Omega_\varepsilon')} \right]
\]

(see \((1.4)\) for the definitions of \( C_\varepsilon \) and \( \Omega_\varepsilon' \)).

Remark 2.4. In order to construct an extension \( P_\varepsilon \) of the pressure \( p_\varepsilon \), we will proceed as follows. In fact, we construct an extension of \( \nabla p_\varepsilon \) in \([H^{-1}(\Omega)]^N\) which is defined as the “dual” operator of \( R_\varepsilon \). Because \( R_\varepsilon \) is some kind of “restriction” operator from the set of the divergence-free vectors of \([H^1_0(\Omega)]^N\) into the set of the divergence-free vectors of \([H^1_0(\Omega_\varepsilon')]^N\), the extension of
\( \nabla p_e \) is also the gradient of a function \( P_e \). And, thanks to the properties of \( R_e \), \( P_e \) is just the bounded pressure's extension we were looking for.

**Remark 2.5.** Theorems 2.1 and 2.3 have already been proved by Tartar in [8], in the particular case where \( Y_S \) is strictly included in \( Y \) (see Remark 1.1 and Fig. 3(a)). In this case, the pressure's extension converges strongly in \( L^2(\Omega) / \mathbb{R} \) (without the "loc") if we get rid of the solid parts \( Y_S^e \) which cut the boundary \( \partial \Omega \).

The originality of the present paper comes from the proof of Theorem 2.3 which follows the lines of Tartar's proof but needs additional ideas, due to the more complex geometry. Another original aspect is the "local" convergence of the pressure's extension in Theorem 2.1 (this is a consequence of the diphasic boundary \( \partial \Omega \)). Poliseevsky has also proved theorems similar to Theorems 2.1 and 2.3 (see [6] or [2, Chapter 1, Section 5]), but his pressure's extension converges in \( L^{6/5}(\Omega) \) instead of \( L^2(\Omega) \).

Whereas \( R_e \) is explicitly constructed (see the proof of Theorem 2.3 in Section 3), the extension \( P_e \) of the pressure \( p_e \) is not explicitly constructed but is derived from \( R_e \) through a theoretical "duality" argument. An interesting problem is then to find the explicit values of \( P_e \) in the "solid part" \( \Omega - \Omega e \). Lipton and Avellaneda [4] work out this problem, and we reproduce their important result.

**Theorem 2.6.** Let \( R_e \) be the operator of \( \mathcal{L}(\{ [H^1_0(C_e)]^N; [H^1_0(\Omega)]^N \}) \) which is explicitly constructed in the proof of Theorem 2.3. Then the extension \( P_e \) of the pressure \( p_e \), which is derived from \( R_e \) in Theorem 2.1, satisfies the following:

(i) \( P_e = p_e \) in the fluid part \( \Omega e \)
(ii) in each cell \( Y^e \) included in \( C_e \), \( P_e \) is a constant in the solid part \( Y^e_S \), which is explicitly given by

\[
P_e = \frac{1}{|Y^e_F|} \int_{Y^e_F} p_e \text{ in } Y^e_S.
\]

The proofs of Theorems 2.3 and 2.6 can be found in Section 3.

### 2.2. Some technical lemmas

**Lemma 2.7** (Poincaré's inequality in \( \Omega e \)). There exists a constant \( C \) which depends only on \( Y_F \), and not on \( \Omega e \), such that, for each \( u \in H^1_0(\Omega e) \), one has \( \| u \|_{L^2(\Omega e)} \leq C \| \nabla u \|_{L^2(\Omega e)} \).

**Proof.** See [8].

**Lemma 2.8.** Let \( \omega \) be a bounded, connected, open set of \( \mathbb{R}^N \), with a locally Lipschitz boundary. Let \( p \) be a distribution in \( \omega \). If \( \nabla p \in [H^{-1}(\omega)]^N \), then \( p \in L^2(\omega) / \mathbb{R} \) and one has \( \| p \|_{L^2(\omega) / \mathbb{R}} \leq C \| \nabla p \|_{H^{-1}(\omega)} \) where the constant \( C \) depends only on \( \omega \) (and not on \( p \)).

**Proof.** See [5].
Lemma 2.9. Let $\omega$ be a bounded, connected, open set of $\mathbb{R}^N$, with a locally Lipschitz boundary. Let $f \in \{H^{-1}(\omega)\}^N$ such that, for each $u \in \{H_0^1(\omega)\}^N$ with $\nabla \cdot u = 0$ in $\omega$, one has $\langle f, u \rangle_{H^{-1}(\omega), H_0^1(\omega)} = 0$; then there exists $p \in L^2(\omega)/\mathbb{R}$ such that $f = \nabla p$ in $\omega$.

Proof. See [11, Chapter I, Remark 1.9]. □

Lemma 2.10. Let $\omega$ be a bounded, connected, open set of $\mathbb{R}^N$, with a locally Lipschitz boundary. For each $f \in L^2(\omega)$ with $\int_\omega f = 0$, there exists $u \in \{H_0^1(\omega)\}^N$ such that $\nabla \cdot u = f$ in $\omega$. Moreover, one can choose $u$ in such a way that the application $f \mapsto u$ is linear and continuous, with $\|u\|_{H_0^1(\omega)} \leq C \|f\|_{L^2(\omega)}$ where the constant $C$ depends only on $\omega$.

Proof. See [11, Chapter I, Lemma 2.4]. □

Remark 2.11. Lemmas 2.8 through 2.10 are strongly connected. For example, Temam proved Lemmas 2.9 and 2.10 in [11] with the help of Lemma 2.8 proved by Necas in [5]. If $\omega$ has a boundary of class $C^1$, Tartar gives a self-contained proof of Lemmas 2.8–2.10 in [10, pp. 26-31] (see also [3, Chapter 1, Section 2] which reproduces the proof of Tartar).

2.3. Proof of convergence

This section is devoted to the proof of Theorem 2.1 under the assumption that Theorem 2.3 holds true. The proof is divided into two parts:

(2.3.1) extending the pressure,
(2.3.2) passing to the limit in the equations.

Part (2.3.1) follows [8] with slight modifications due to the “local” character of convergence of the pressure’s extension. Part (2.3.2) reproduces [8] and is given here in order for the present paper to be self-contained.

2.3.1. Extending the pressure

Multiplying the following equation by $u_\epsilon$

$$\nabla p_\epsilon - \Delta u_\epsilon = f \quad \text{in} \quad \Omega_\epsilon$$

and integrating by parts on $\Omega_\epsilon$, we obtain

$$\|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 = \int_{\Omega_\epsilon} f \cdot u_\epsilon.$$  \hspace{1cm} (2.1)

Using Lemma 2.7 we find that

$$\|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C \varepsilon \|f\|_{L^2(\Omega_\epsilon)}$$ \hspace{1cm} (2.2)

implying

$$\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C \varepsilon \|f\|_{L^2(\Omega_\epsilon)}.$$ \hspace{1cm} (2.3)

Thus $u_\epsilon/\varepsilon^2$ is a bounded sequence in $[L^2(\Omega)]^N$, and therefore we can extract a subsequence such that there exists $u \in [L^2(\Omega)]^N$ with

$$\tilde{u}_\epsilon/\varepsilon^2 \rightharpoonup u \quad \text{in} \quad [L^2(\Omega)]^N$$ \hspace{1cm} (2.4)
While the velocity \( u_\varepsilon \) can be naturally continued by zero in \( \Omega - \Omega_\varepsilon \), it is not obvious to construct an extension to \( \Omega \) of the pressure \( p_\varepsilon \) (which is defined only on \( \Omega_\varepsilon \)). With the help of Theorem 2.3 this will be achieved.

Let \( F_\varepsilon \) be defined in \([H^{-1}(\Omega_\varepsilon)]^N\) by the following formula (see (1.4) for the definitions of \( C_\varepsilon \) and \( \Omega'_\varepsilon \)):

\[
\text{for each } v \in \left[ H_0^1(C_\varepsilon) \right]^N \quad \langle F_\varepsilon, v \rangle_{H^{-1},H_0^1(C_\varepsilon)} = \langle \nabla p_\varepsilon, R v \rangle_{H^{-1},H_0^1(\Omega'_\varepsilon)}
\]  

(2.5)

where \( R_\varepsilon \) is given by Theorem 2.3 (we also denote by \( \nabla p_\varepsilon \) the restriction of \( \nabla p_\varepsilon \) to \( \Omega'_\varepsilon \)).

Because \( R_\varepsilon \) is linear, \( F_\varepsilon \) is a linear functional on \([H_0^1(C_\varepsilon)]^N\). In order to estimate its norm, we write

\[
\langle F_\varepsilon, v \rangle_{H^{-1},H_0^1(C_\varepsilon)} = \langle f + \Delta u_\varepsilon, R v \rangle_{H^{-1},H_0^1(\Omega'_\varepsilon)}.
\]

Integrating by parts, we obtain

\[
\langle F_\varepsilon, v \rangle_{H^{-1},H_0^1(C_\varepsilon)} = \int_{\Omega'_\varepsilon} f \cdot R v - \int_{\Omega'_\varepsilon} \nabla u_\varepsilon \cdot \nabla (R v).
\]

With the help of property (iv) of Theorem 2.3 and of inequality (2.2) we can majorate the functional and we obtain

\[
|\langle F_\varepsilon, v \rangle_{H^{-1},H_0^1(C_\varepsilon)}| \leq C \| f \|_{L^2(\Omega)} \left[ \| v \|_{L^2(C_\varepsilon)} + \varepsilon \| \nabla v \|_{L^2(C_\varepsilon)} \right].
\]  

(2.6)

Thus, if \( \varepsilon < 1 \), \( \| F_\varepsilon \|_{H^{-1}(C_\varepsilon)} \leq C \| f \|_{L^2(\Omega)} \).

Moreover property (iii) of Theorem 2.3 implies that, for each \( v \in [H_0^1(C_\varepsilon)]^N \) with \( \nabla \cdot v = 0 \) in \( C_\varepsilon \), we have \( \langle F_\varepsilon, v \rangle_{H^{-1},H_0^1(C_\varepsilon)} = 0 \). Applying Lemma 2.9 we deduce the existence of \( P_\varepsilon \in L^2(C_\varepsilon) \) such that \( F_\varepsilon = \nabla P_\varepsilon \) in \( C_\varepsilon \). On the other hand, Lemma 2.8 provides an estimate of \( P_\varepsilon \) depending on the norm of its gradient:

\[
\| P_\varepsilon \|_{L^2(C_\varepsilon) / \mathbb{R}} \leq C \| F_\varepsilon \|_{H^{-1}(C_\varepsilon)} \leq C \| f \|_{L^2(\Omega)}.
\]  

(2.7)

Thus we have defined \( P_\varepsilon \) on \( C_\varepsilon \), and we complete this definition on \( \Omega - C_\varepsilon \) with

\[
P_\varepsilon = p_\varepsilon \quad \text{on} \quad (\Omega - C_\varepsilon) \cap \Omega_\varepsilon, \quad P_\varepsilon = 0 \quad \text{on} \quad (\Omega - C_\varepsilon) - \Omega_\varepsilon.
\]  

(2.8)

Now we prove that \( P_\varepsilon \) is an extension of \( p_\varepsilon \) with the help of property (ii) of Theorem 2.3: for each \( v \in [H_0^1(\Omega'_\varepsilon)]^N \) we have \( \nabla \tilde{v} = v \) in \( \Omega'_\varepsilon \), implying

\[
\langle \nabla P_\varepsilon, \tilde{v} \rangle_{H^{-1},H_0^1(C_\varepsilon)} = \langle \nabla p_\varepsilon, v \rangle_{H^{-1},H_0^1(\Omega'_\varepsilon)}
\]

which is equivalent to

\[
\int_{C_\varepsilon} P_\varepsilon \nabla \cdot \tilde{v} = \int_{\Omega'_\varepsilon} P_\varepsilon \nabla \cdot v = \int_{\Omega'_\varepsilon} p_\varepsilon \nabla \cdot v.
\]

Thanks to Lemma 2.10 we can assert that, for each \( f \in L^2(\Omega'_\varepsilon) \) with \( \int_{\Omega'_\varepsilon} f = 0 \), there exists \( v \in [H_0^1(\Omega'_\varepsilon)]^N \) such that \( \nabla \cdot v = f \) in \( \Omega'_\varepsilon \). Then \( \int_{\Omega'_\varepsilon} f (P_\varepsilon - p_\varepsilon) = 0 \) for each \( f \in L^2(\Omega'_\varepsilon) \) with \( \int_{\Omega'_\varepsilon} f = 0 \) implying \( (P_\varepsilon - p_\varepsilon) = 0 \) in \( L^2(\Omega'_\varepsilon) / \mathbb{R} \). Moreover, by definition (2.8) we have \( P_\varepsilon = p_\varepsilon \) on \( (\Omega - C_\varepsilon) \cap \Omega_\varepsilon \), so we can conclude \( P_\varepsilon = p_\varepsilon \) in \( \Omega'_\varepsilon \) (up to an additive constant). Thus \( P_\varepsilon \) is strictly an extension of the pressure \( p_\varepsilon \).
Now we prove the existence of \( p \in L^2(\Omega)/\mathbb{R} \) such that \( P_\varepsilon \to p \) in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \) strongly. First recall the estimate (2.7): \( \| P_\varepsilon \|_{L^2(\Omega)/\mathbb{R}} \leq C \| f \|_{L^1(\Omega)} \). For each open set \( \omega \), strictly included in \( \Omega \), choose sufficiently small values of \( \varepsilon \) such that \( \omega \subset C_\varepsilon \). Thus one can deduce from (2.7) that \( \| P_\varepsilon \|_{L^2(\omega)/\mathbb{R}} \leq C \| f \|_{L^1(\Omega)} \). This means that \( P_\varepsilon \) is bounded in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \). So we can extract a subsequence, still denoted by \( P_\varepsilon \), and there exists \( p \in L^2_{\text{loc}}(\Omega)/\mathbb{R} \) such that \( P_\varepsilon \to p \) in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \) weakly. Because of the weak convergence of \( P_\varepsilon \) in \( L^2(\omega)/\mathbb{R} \), we have, for any open set \( \omega \) strictly included in \( \Omega \),

\[
\| p \|_{L^2(\omega)/\mathbb{R}} \leq \liminf_{\varepsilon \to 0} \| P_\varepsilon \|_{L^2(\omega)/\mathbb{R}}.
\]

By combining this estimate with inequality (2.7), we obtain \( \| p \|_{L^2(\omega)/\mathbb{R}} \leq C \| f \|_{L^1(\Omega)} \) for each \( \omega \subset \Omega \) and \( C \) does not depend on \( \omega \), implying \( p \in L^2(\Omega)/\mathbb{R} \) (there is no more “loc”).

To show that the convergence of \( P_\varepsilon \) to \( p \) is, in fact, strong in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \), we consider a sequence \( w_\varepsilon \) such that \( w_\varepsilon \to w \) in \( [H^1_0(\omega)]^N \) weakly, where \( \omega \) is a smooth open set, strictly included in \( \Omega \). Consider the following inequality:

\[
|\langle \nabla P_\varepsilon, w_\varepsilon \rangle_{H^{-1}, H^1_0(\omega)} - \langle \nabla P, w \rangle_{H^{-1}, H^1_0(\omega)}| \leq |\langle \nabla P_\varepsilon, w_\varepsilon - w \rangle| + |\langle \nabla P - \nabla p, w \rangle|.
\]  

(2.9)

Integrating by parts on \( \omega \) the second right-hand side member of (2.9), we obtain

\[
\langle \nabla P_\varepsilon - \nabla p, w \rangle = -\int_\omega (P_\varepsilon - p) \nabla \cdot w \quad \varepsilon \to 0
\]

because \( P_\varepsilon \to p \) in \( L^2(\omega)/\mathbb{R} \) weakly. Using inequality (2.6) for the first right-hand side member of (2.9), we obtain

\[
|\langle \nabla P_\varepsilon, w_\varepsilon - w \rangle| \leq C \| f \|_{L^1(\Omega)} \left[ \| w_\varepsilon - w \|_{L^2(\omega)} + \varepsilon \| \nabla w_\varepsilon - \nabla w \|_{L^2(\omega)} \right].
\]

(2.10)

By the definition of \( w_\varepsilon \), we have that \( \langle \nabla w_\varepsilon \rangle \) is bounded in \( [L^2(\omega)]^N \). By virtue of Rellich’s Theorem, \( w_\varepsilon \to w \) in \( [L^2(\omega)]^N \) strongly. Recalling inequality (2.9), we have

\[
|\langle \nabla P_\varepsilon, w_\varepsilon \rangle - \langle \nabla p, w \rangle| \quad \varepsilon \to 0
\]

for each sequence \( w_\varepsilon \) which converges weakly in \( [H^1_0(\omega)]^N \). This is just the definition of the strong convergence of \( \nabla P_\varepsilon \) in \( [H^{-1}(\omega)]^N \). Because \( \omega \) has a smooth boundary, we can apply Lemma 2.8, and we obtain \( P_\varepsilon \to p \) in \( L^2(\omega)/\mathbb{R} \) strongly and this is true for each smooth open set \( \omega \) strictly included in \( \Omega \). Thus we can conclude \( P_\varepsilon \to p \) in \( L^2_{\text{loc}}(\Omega)/\mathbb{R} \) strongly.

2.3.2. Passing to the limit in the equations

We apply the energy method introduced by Tartar in [9] (see also [1]). First we multiply the equation \( \nabla \cdot u_\varepsilon = 0 \) by \( \phi \in H^1(\Omega) \), and we integrate by parts on \( \Omega \). We obtain

\[
\int_\Omega (\nabla \cdot \bar{\eta}_\varepsilon) \phi = 0 \iff -\int_\Omega \bar{\eta}_\varepsilon \cdot \nabla \phi + \int_\Omega \phi \bar{\eta}_\varepsilon \cdot n = 0
\]

and \( \int_\Omega \phi \eta - 0 \) because \( \bar{\eta}_\varepsilon \in [H^1_0(\Omega)]^N \). We pass to the limit \( \varepsilon \to 0 \), and we get

\[
\int_\Omega (\nabla \cdot \phi) = 0 \iff -\int_\Omega \phi \nabla \cdot u + \int_\Omega \phi u \cdot n = 0
\]
and this is true for each $\phi \in H^1(\Omega)$. Thus,

$$
\begin{align*}
\nabla \cdot u &= 0 \quad \text{in } \Omega \quad (\text{in } H^{-1}(\Omega)), \\
u \cdot n &= 0 \quad \text{on } \partial \Omega \quad (\text{in } H^{-1/2}(\partial \Omega)),
\end{align*}
$$

(2.11)

On the other hand, using definition (1.11), we define the following functions: let $(v'_k, q'_k) \in [H^1(\Omega)]^N \times [L^2(\Omega)]$ such that

$$
\begin{align*}
v'_k(x) &= v_k(x/\varepsilon), \\
q'_k(x) &= q_k(x/\varepsilon)
\end{align*}
$$

(extended in $\Omega$ by $\varepsilon Y$-periodicity).

(2.12)

$(v'_k, q'_k)$ satisfy the following system:

$$
\varepsilon \nabla q'_k - \varepsilon^2 \Delta v'_k = e_K \quad \text{in } \Omega, \quad \nabla \cdot v'_k = 0 \quad \text{in } \Omega, \quad v'_k = 0 \quad \text{in } \Omega - \Omega
$$

(2.13–15)

with the following estimates:

$$
\| q'_k \|_{L^2(\Omega)} \leq C, \quad \| v'_k \|_{L^2(\Omega)} \leq C; \quad \| \nabla v'_k \|_{L^2(\Omega)} \leq C/\varepsilon
$$

(2.16; 17)

where $C$ does not depend on $\varepsilon$. Moreover, classically we know that

$$
q'_k \to 0 \quad \text{in } [L^2(\Omega)/\mathbb{R}] \quad \text{weakly},
$$

$$
v'_k \to \frac{1}{\varepsilon} \int_Y u_K(y) \, dy = \overline{A}e_K \quad \text{in } [L^2(\Omega)]^N \quad \text{weakly}.
$$

Let $\phi \in \mathcal{D}(\Omega)$. We multiply the equations (2.13) by $\phi \tilde{u}_e$, and (1.5) by $\phi v'_k$, and we integrate by parts on $\Omega$. We obtain

$$
\int_{\Omega} \nabla v'_k \cdot \nabla (\phi \tilde{u}_e) = \frac{1}{\varepsilon} \int_{\Omega} q'_k \tilde{u}_e \cdot \nabla \phi + \frac{1}{\varepsilon^2} \int_{\Omega} e_K \cdot \tilde{u}_e \phi
$$

(2.18)

$$
\int_{\Omega} \nabla \tilde{u}_e \cdot \nabla (\phi v'_k) = \int_{\Omega} f \cdot v'_k \phi + \int_{\Omega} P_{v'_k} \cdot \nabla \phi
$$

(2.19)

For sufficiently small values of $\varepsilon$, we have $\text{Support}(\phi) \subset C_e$. Consequently, the local but strong convergence of $P_{v'_k}$ in $L^2(\Omega)/\mathbb{R}$ is sufficient to pass to the limit in the following term:

$$
\int_{\Omega} P_{v'_k} \cdot \nabla \phi \xrightarrow{\varepsilon \to 0} \int_{\Omega} \overline{A} e_K \cdot \nabla \phi.
$$

Therefore we have

$$
\int_{\Omega} \nabla v'_k \cdot \nabla (\phi \tilde{u}_e) \xrightarrow{\varepsilon \to 0} \int_{\Omega} \phi u \cdot e_K,
$$

$$
\int_{\Omega} \nabla \tilde{u}_e \cdot \nabla (\phi v'_k) \xrightarrow{\varepsilon \to 0} \int_{\Omega} \phi f \cdot (\overline{A} e_K) - \int_{\Omega} \phi (\overline{A} e_K) \cdot \nabla p.
$$

On the other hand, using the estimates of $u_e$ and $v'_k$ we obtain

$$
\left| \int_{\Omega} \nabla \tilde{u}_e \cdot \nabla (\phi v'_k) - \int_{\Omega} \nabla v'_k \cdot \nabla (\phi \tilde{u}_e) \right| \leq C \varepsilon \to 0.
$$
Hence,

$$\int_{\Omega} \phi u \cdot e_K = \int_{\Omega} \phi f \cdot (\overline{A}e_K) = \int_{\Omega} \phi \nabla p \cdot (\overline{A}e_K)$$

and this is true for each $\phi \in \mathcal{D}(\Omega)$. Thus, $u = \overline{A}(f - \nabla p)$ in $\mathcal{D}'(\Omega)$. Recalling (2.11) we obtain (S):

(S): \quad \nabla \cdot [\overline{A}(f - \nabla p)] = 0 \quad \text{in} \quad \Omega, \quad [\overline{A}(f - \nabla p)] \cdot n = 0 \quad \text{on} \quad \partial \Omega. \quad (1.10)

A classical result asserts that (S) has a unique solution $p \in H^1(\Omega)/\mathbb{R}$; therefore, not only a subsequence, but the whole sequence $P_\varepsilon$ converges to $p$ in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$. The same result holds for the velocity: the whole sequence $\tilde{u}_\varepsilon$ converges to $u$ in $[L^2(\Omega)]^N$ weakly. And thus Theorem 2.1 has been proved.

### 3. Construction of the operator $R_\varepsilon$

The third section of this paper is devoted to the proofs of Theorems 2.3 and 2.6. First we establish two important lemmas in order to deal, just afterwards, with the proof of Theorem 2.3. Lastly we reproduce the proof of Theorem 2.6 following [4].

The idea of the construction of an operator $R_\varepsilon$, which satisfies the properties of Theorem 2.3 in order to extend the pressure, is due to Tartar [8]. Unfortunately, his construction of $R_\varepsilon$ applies only to the case where $Y_\varepsilon$ is strictly included in $Y$ and is explicitly local in each cube $Y_\varepsilon^p$. We propose a generalization of this result when $Y_\varepsilon$ is no longer strictly included in $Y$, and, in this case, the construction of $R_\varepsilon$ is partly global (because of the introduction of an operator $Q_\varepsilon$ which projects $H^1_0(\Omega)$ on $H^1_0(\Omega')$), and partly local (because $R_\varepsilon$ is still constructed in each cube $Y_\varepsilon^p$). In fact, $Q_\varepsilon u$ defines the boundary values on $\partial Y_\varepsilon^p$ of the function $R_\varepsilon u$ in each cell $Y_\varepsilon^p$.

**Remark 3.1.** Using several trace properties of the functions belonging to $W^{1,6}(Y)$. Polisivsky has constructed in [6] (see also [2, Chapter 1, Section 5]) an operator $R_\varepsilon$ in the case where $Y_\varepsilon$ is not strictly included in $Y$. But $R_\varepsilon$ operates from $[L^6_0(\Omega)]^3$, instead of $[H^1_0(\Omega)]^3$, into $[H^1_0(\Omega')]^3$.

**Remark 3.2.** Why do we choose to construct $R_\varepsilon \in \mathcal{L}([H^1_0(\Omega)]^N; [H^1_0(\Omega')]^N)$ and not, “more naturally”, $R_\varepsilon \in \mathcal{L}([H^1_0(\Omega)]^N; [H^1_0(\Omega')]^N)$? This difference between $\Omega$ and $\Omega'$, or $\Omega$ and $\Omega'$, is the reason why the pressure’s extension $P_\varepsilon$ converges “locally” in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$. (Recall that this local convergence is sufficient to pass to the limit in the equations). The answer is that the construction of $R_\varepsilon u$ in each cell $Y_\varepsilon^p$ is reduced, with the help of the translation–homothety $\tau_\varepsilon$, to the construction of $\tilde{R} u$ in the unit cell $Y$ (see Lemma 3.4). Now, if the cell $Y_\varepsilon^p$ is “cut” by the boundary $\partial \Omega$, the construction of $R_\varepsilon u$ in $Y_\varepsilon^p \cap \Omega$ is reduced to the construction of $\tilde{R} u$ in a part of $Y$ which can have a very small size compared with $Y$ (and we cannot “control” this size). Unfortunately, the various constants $C$ which appear in the estimates of Lemmas 2.10 and 3.4, depend strongly on the size of the considered open set. And because we cannot estimate those constants for any small subset of $Y$, we cannot obtain estimate (iv) of Theorem 2.3 if we define $R_\varepsilon u$ in $\Omega$ instead of $\Omega$. 
Lemma 3.3. There exists a linear continuous operator $Q_\varepsilon$ such that

(i) $Q_\varepsilon \in \mathcal{L}(L^2(\Omega_\varepsilon; \Omega_\varepsilon'))$, 
(ii) $u \in [H^1_0(\Omega_\varepsilon')]^N$ implies $Q_\varepsilon u = u$ in $\Omega_\varepsilon'$,
(iii) for each $u \in [H^1_0(\Omega_\varepsilon')]^N$ we have

$$
\| Q_\varepsilon u \|_{L^2(\Omega_\varepsilon')} + \varepsilon \| \nabla (Q_\varepsilon u) \|_{L^2(\Omega_\varepsilon')} \leq C \left[ \| u \|_{L^2(\Omega_\varepsilon)} + \varepsilon \| \nabla u \|_{L^2(\Omega_\varepsilon)} \right]
$$

where the constant $C$ depends only on $Y_\varepsilon$, and not on $\varepsilon$.

Proof. There are many ways to construct such an operator $Q_\varepsilon$. We choose the following one: let $u \in [H^1_0(\Omega_\varepsilon')]^N$. We consider the following problem:

$$
\text{find } v_\varepsilon \in [H^1_0(\Omega_\varepsilon')]^N \text{ such that } -\Delta v_\varepsilon = -\Delta u \text{ in } \Omega_\varepsilon'.
$$

Thanks to the Lax–Milgram Theorem, we know that this problem has a unique solution $v_\varepsilon$ belonging to $H^1_0(\Omega_\varepsilon')$. Then we define $Q_\varepsilon$ by

$$
Q_\varepsilon u = v_\varepsilon
$$

Properties (i) and (ii) of $Q_\varepsilon$ can be easily checked, and in order to obtain estimate (iii), we multiply equation (3.1) by $v_\varepsilon$ and we integrate by parts on $\Omega_\varepsilon'$:

$$
\int_{\Omega_\varepsilon'} \nabla v_\varepsilon \cdot \nabla u = \| \nabla v_\varepsilon \|_{L^2(\Omega_\varepsilon')} \leq \| \nabla u \|_{L^2(\Omega_\varepsilon')}.
$$

Using Lemma 2.7 (Poincaré’s inequality), we obtain estimate (iii) with $Q_\varepsilon u = v_\varepsilon$:

$$
\| v_\varepsilon \|_{L^2(\Omega_\varepsilon')} + \varepsilon \| \nabla v_\varepsilon \|_{L^2(\Omega_\varepsilon')} \leq C \left[ \| u \|_{L^2(\Omega_\varepsilon')} + \varepsilon \| \nabla u \|_{L^2(\Omega_\varepsilon')} \right].
$$

□

Lemma 3.4. Let $Q$ be a linear operator belonging to $\mathcal{L}(H^1(\Omega'; \Omega'; Y_\varepsilon')) satisfying

for each $u \in [H^1(\Omega')]^N$ $Qu = 0$ in $Y_\varepsilon$.

We consider the following problem:

$$
\begin{cases}
\text{find } (v, q) \in [H^1(\Omega_F')]^N \times \left[ L^2(\Omega_F)/\mathbb{R} \right] \text{ such that } \\
\nabla q - \Delta v = -\Delta u \\
\n\nabla \cdot v = \nabla \cdot u + \frac{1}{|Y_F|} \int_{Y_F} \nabla \cdot u \\
v = Qu + \frac{\phi_K}{\int_{\Sigma_K} (u - Qu) \cdot e_K} e_K \\
v = 0
\end{cases}
$$

on $\Sigma_K \cap \overline{Y_F}$,

$$
\begin{cases}
\text{in } Y_F, \\
\text{in } Y_F, \\
\text{on } \partial Y_\varepsilon.
\end{cases}
$$

Then we have the following results:

(i) (3.4) has a unique solution,
ii) the application $R$ defined by $Ru = v$ belongs to $\mathcal{L}([H^1(Y)]^N; [H^1(Y_F)]^N),$ 
(iii) for each $u \in [H^1(Y)]^N$ we have 
$$
\| Ru \|_{H^1(Y_F)} \leq C \left( \| u \|_{H^1(Y)} + \| Qu \|_{H^1(Y)} \right) 
$$
where $C$ depends only on $Y_F$.

N.B.: See (1.2) for the definitions of $\phi_K$ and $\Sigma_K$.

**Proof.** A necessary condition for the existence of a solution of (3.4) is the following compatibility condition:

$$
\int_{Y_F} \nabla \cdot v = \int_{\partial Y_F} v \cdot n. 
$$

(3.5)

If in this formula we substitute the functions by their assigned values given in (3.4), we obtain

$$
\int_{Y_F} \nabla \cdot v = \int_{Y_F} \left( \nabla \cdot u + \frac{1}{|Y_F|} \int_{Y_N} \nabla \cdot u \right) = \int_Y \nabla \cdot u = \int_{\partial Y} u \cdot n
$$

and

$$
\int_{\partial Y_F} v \cdot n = \sum_{K \in \mathcal{T}_h}^{+N} \int_{\Sigma_K} \left[ Qu + \int_{\Sigma_K} \phi_K \left( \int_{\Sigma_K} (u - Qu) \cdot e_K \right) e_K \right] \cdot e_K
$$

$$
= \sum_{K \in \mathcal{T}_h}^{+N} \int_{\Sigma_K} \left[ Qu \cdot e_K + \int_{\Sigma_K} (u - Qu) \cdot e_K \right] = \int_{\partial Y} u \cdot n.
$$

So (3.5) is satisfied.

In order to show that system (3.4) has a unique solution, we transform (3.4) into a divergence-free system with homogeneous boundary conditions. Let $u_1$ be defined in $[H^1(Y_F)]^N$ by

$$
u_1 = Qu + \sum_{K \in \mathcal{T}_h}^{+N} \phi_K \left( \int_{\Sigma_K} (u - Qu) \cdot e_K \right) e_K.
$$

(3.6)

It is easy to see that if $v$ is a solution of (3.4), we have $(v - u_1) \in [H^1_0(Y_F)]^N$. Moreover, because $Y_F$ is connected with a locally Lipschitz boundary (see assumption (1.1)) we can apply Lemma 2.10 (about the lift of divergence): there exists $u_2 \in [H^1_0(Y_F)]^N$ such that

$$
\nabla \cdot u_2 = \left( \nabla \cdot u - \nabla \cdot u_1 + \frac{1}{|Y_F|} \int_{Y_N} \nabla \cdot u \right) \text{ in } Y_F
$$

and

$$
\| u_2 \|_{H^1(Y_F)} \leq C \left( \| u \|_{H^1(Y)} + \| Qu \|_{H^1(Y)} \right).
$$

(3.7)

Consequently, finding a solution $(v, q)$ of (3.4) is equivalent to finding a solution $(u; q) = (v - u_1 - u_2; q) \in [H^1_0(Y_F)]^N \times [L^2(Y_F)/\mathbb{R}]$ of the following system:

$$
\nabla q - \Delta u_3 = -\Delta (u - u_1 - u_2) \text{ in } Y_F, \quad \nabla \cdot u_3 = 0 \text{ in } Y_F.
$$

(3.8)
Because $Y_F$ is connected with a locally Lipschitz boundary, classical results (see, for example, [11, Chapter 1, Section 2]) state that there exists a unique solution of (3.8), which satisfies
\[ \| \nabla u \|_{L^2(Y_F)} \leq \| \nabla (u - u_1 - u_2) \|_{L^2(Y_F)}. \]  (3.9)
Then (3.4) has also a unique solution $(v, q)$.

This allows us to define an application $R$ by the following formula:

\[ \text{for each } u \in [H^1(Y)]^N \quad R u = v \text{ in } Y_F. \]
It is easy to see that $R \in \mathcal{L}([H^1(Y)]^N; [H^1(Y_F)]^N)$, and recalling estimates (3.9), (3.7) and (3.6) we obtain
\[ \| Ru \|_{H^1(Y_F)} \leq C \left[ \| u \|_{H^1(Y)} + \| Q u \|_{H^1(Y)} \right] \]
where $C$ depends only on $Y_F$, because the functions $\phi_k$ depend only on $Y_F$ too. Note that $C$ does not depend on $Q$, provided (3.3) holds true. And thus Lemma 3.4 has been proved. \hfill \square

Proof of Theorem 2.3. Let $(Y^*_i)_{i=1}^{N(e)}$ be the cubes which cover $\Omega$. Let $C_e$ be the polygonal open set such that (see (1.4)) $\tilde{C}_e = \cup_{i \in \pi(e)} \tilde{Y}^*_{i'}$ with $I(e) = \{ i \in \{1, \ldots, N(e)\} \}$, where $Y^*_i \subset \Omega$. Let $u \in [H^1_0(C_e)]^N$. Recall that $\pi_e^*$ is the translation–homothety which maps $Y^*_e$ on $Y$ (see (1.3)).

We define $R_{*,u}$ by its values in each cell $Y^*_e \subset \Omega$, which are denoted by $R_{*,u} |_{Y_i}$:
\[ R_{*,u} |_{Y^*_i} = \left[ R \left( u \circ (\pi_e^*)^{-1} \right) \right] \circ \pi_e^* \text{ in } Y^*_i \]  (3.10)
where $R$ is defined by Lemma 3.4 in which $Q$ is defined by
\[ \left[ Q \left( u \circ (\pi_e^*)^{-1} \right) \right] \circ \pi_e^* = \begin{cases} Q_{*,u} & \text{in } Y^*_e, \\ 0 & \text{in } Y^*_S. \end{cases} \]  (3.11)
Definition (3.10) is meaningful, provided that the operator $Q_{*,u}$ defined by (3.11) satisfies the assumptions of Lemma 3.4. And it is clear that property (i) of $Q_{*,u}$ (defined by Lemma 3.3) implies condition (3.3) of Lemma 3.4.

By its very construction, $R_{*,u} |_{Y_i}$ satisfies
\[ \begin{cases} \text{there exists } q_{*,u} \in L^2(Y^*_F) / \mathbb{R} \text{ such that } \\ \nabla q_{*,u} - \Delta (R_{*,u} |_{Y^*_e}) = \nabla u \text{ in } Y^*_F, \\ \nabla \cdot (R_{*,u} |_{Y^*_e}) = \nabla \cdot u + \frac{1}{|Y^*_F|} \int_{Y^*_S} \nabla \cdot u \text{ in } Y^*_S, \\ R_{*,u} |_{Y^*_e} = Q_{*,u} + \frac{\phi_K \circ \pi_e^*}{\Sigma_K} \left[ \int_{\Sigma_K} (u - Q_{*,u}) \cdot e_K \right] e_K \text{ on } \Sigma_K, \\ R_{*,u} |_{Y^*_S} = 0 \text{ on } \partial Y^*_S. \end{cases} \]  (3.12)
Thus $R_{*,u}$ is defined on $\Omega^*_e = \cup_{i \in \pi(e)} Y^*_i$. It remains to prove properties (i) to (iv) of $R_{*,u}$.

(i) Do we have $R_{*,u} \in \mathcal{L}([H^1_0(C_e)]^N; [H^1_0(\Omega^*_e)]^N)$? By construction, $R_{*,u}$ is linear because $R$ is linear. Moreover, it is easy to see in (3.12) that $R_{*,u} = 0$ on $\partial \Omega^*_e = \partial C_e \cup \{ \cup_i \partial Y^*_i \}$.
But it is crucial to check that \( R_i u \in [H^1_0(\Omega^i_t)]^N \). Because \( R_i u \in [H^1(Y^i_t)]^N \) for each \( i \in I(\varepsilon) \), it remains to show that \( R_i u \) is continuous through the faces of \( Y^i_t \).

\[
(3.12) \quad R_i u = Q_i u + \frac{\phi_K \circ \tau^i_t}{\int_{\Sigma^i_k} \phi_K \circ \tau^i_t} \left[ \int_{\Sigma^i_k} (u - Q_i u) \cdot e_K \right] e_K \quad \text{on} \quad \Sigma^i_k.
\]

By construction (see (1.2)) the functions \( \phi_K \) are equal on opposite faces of the same cell \( Y \), and, by definition, \( Q_i u \in [H^1_0(\Omega^i_t)]^N \); then \( R_i u \) is continuous through \( \Sigma^i_k \) and \( R_i \in \mathcal{L}([H^1_0(\Omega^i)]^N; [H^1_0(\Omega^i)]^N) \).

(ii) If \( u \in [H^1_0(\Omega^i_t)]^N \), then Lemma 3.3 implies that \( Q_i \tilde{u} = u \) in \( \Omega^i_t \). Then, from (3.11), we have \( Q(\psi \circ \tau^i_t) = \psi \circ \tau^i_t \) in \( Y \). In this case, the system (3.4) has an obvious solution which is \( u \), and the uniqueness of the solution implies that \( R(\psi \circ \tau^i_t) = \psi \circ \tau^i_t \) in \( Y_t \). Thus, \( R_i \tilde{u} = u \) in \( \Omega^i_t \).

(iii) If \( \nabla \cdot u = 0 \in C_r \), then from (3.12) we deduce that \( \nabla \cdot (R_i u) = 0 \) in \( \Omega^i_t \).

(iv) Estimate of the norm of \( R_i u \): let \( \psi \in H^1(Y) \). The norm of \( \psi \circ \tau^i_t \) in \( H^1(Y^i_t) \) in terms of the norm of \( \psi \) in \( H^1(Y) \) is given by

\[
\int_Y \psi^2(y) \, dy = \int_{Y^i_t} \left[ \psi \circ \tau^i_t(x) \right]^2 \cdot |\text{Jac} \, \tau^i_t| \, dx = \frac{1}{\varepsilon^N} \int_{Y^i_t} \left[ \psi \circ \tau^i_t \right]^2 \, dx
\]

and

\[
\int_Y |\nabla \psi|^2 \, dy = \int_{Y^i_t} \left[ \frac{1}{|\tau^i_t(x)|} |\nabla (\psi \circ \tau^i_t)| \cdot |\text{Jac} \, \tau^i_t| \right]^2 \, dx = \frac{1}{\varepsilon^N} \int_{Y^i_t} |\nabla (\psi \circ \tau^i_t)|^2 \, dx.
\]

Consequently, the estimate of Lemma 3.4 becomes, after the change of variables \( \tau^i_t \),

\[
\| R_i u \|_{L^2(Y^i_t)} + \varepsilon \| \nabla (R_i u) \|_{L^2(Y^i_t)} \leq C \left[ \| u \|_{L^2(Y^i_t)} + \| Q_i u \|_{L^2(Y^i_t)} + \varepsilon \left( \| \nabla u \|_{L^2(Y^i_t)} + \| \nabla (Q_i u) \|_{L^2(Y^i_t)} \right) \right].
\]

(3.13)

After summation of the estimates (3.13) for each \( i \in I(\varepsilon) \), we obtain

\[
\| R u \|_{L^2(Y^i_t)} + \varepsilon \| \nabla (R u) \|_{L^2(Y^i_t)} \leq C \left[ \| u \|_{L^2(Y^i_t)} + \| Q u \|_{L^2(Y^i_t)} + \varepsilon \left( \| \nabla u \|_{L^2(Y^i_t)} + \| \nabla (Q u) \|_{L^2(Y^i_t)} \right) \right],
\]

and with the help of estimate (iii) of Lemma 3.3, we get

\[
\| R u \|_{L^2(Y^i_t)} + \varepsilon \| \nabla (R u) \|_{L^2(Y^i_t)} \leq C \left[ \| u \|_{L^2(Y^i_t)} + \varepsilon \| \nabla u \|_{L^2(Y^i_t)} \right]
\]

where \( C \) depends only on \( Y^i_t \) (and not on \( \varepsilon \)). This inequality is just property (iv) we were looking for. Thus, Theorem 2.3 has been proved.

\[ \Box \]

**Proof of Theorem 2.6.** Let us first show that the pressure's extension \( P_r \) is a constant in the solid part \( Y^i_t \) of each cell included in \( C_r \).

Let \( w \) be a function of \([C^\infty(\bar{Y}^i)]^N \) with compact support contained in \( \bar{Y}^i \) (i.e. \( w \in [\mathcal{D}(\bar{Y}^i)]^N \)). By its very construction in Lemma 3.3, the operator \( Q_i \) is such that \( Q_i w = 0 \) in \( \Omega^i_t \). Furthermore,
system (3.12), which is satisfied by $R_* w$ in $Y'_e$, reduces to
\begin{eqnarray}
 there exists $q_e \in L^2(Y'_e) \cap \mathbb{R}$ such that \\
$\nabla q_e - \Delta (R_* w) = - \Delta w = 0$ in $Y'_e$, \\
$\nabla \cdot (R_* w) = \frac{1}{|Y'_e|} \int_{Y'_e} \nabla \cdot w = 0$ in $Y'_e$, \\
$R_* w = 0$ on $\partial Y'_e$.
\end{eqnarray}
(3.14)

because $w$ has a compact support contained in $Y'_e$. Then, (3.14) implies that $R_* w = 0$ in $Y'_e$, which means that
\begin{eqnarray}
 R_* w = 0 \quad \text{in } \Omega'_e.
\end{eqnarray}
(3.15)
Recall definition (2.5) of $P_e$: $\langle \nabla P_e, w \rangle_{H^{-1}(\Omega'_e)} = \langle \nabla P_e, R_* w \rangle_{H^{-1}(\Omega'_e)}$. Then, for each $w \in [\mathcal{D}(Y'_e)]^N$:
\begin{equation}
\langle \nabla P_e, w \rangle = 0 \iff \int_{Y'_e} P_e \nabla \cdot w = 0
\end{equation}
(3.16)

Using Lemma 2.10 this implies that $P_e$ is a constant in each solid part $Y'_e$.

Let us now compute explicitly the value of this constant in each $Y'_e$. Let $v$ be a function of $[C^\infty(Y'_e)]^N$ with compact support contained in the cell $Y'_e$ (i.e., $v \in [\mathcal{D}(Y'_e)]^N$). Remark that $v$ is different from the previous function $w$ because the compact support of $v$ is contained in $Y'_e$, while the support of $w$ is in $Y'_e$ which is strictly included in $Y'_e$. Recalling the definition (2.5) of $P_e$, we obtain
\begin{equation}
\langle \nabla P_e, v \rangle_{H^{-1}(\Omega'_e)} = \langle \nabla P_e, R_* v \rangle_{H^{-1}(\Omega'_e)} \iff \int_{Y'_e} P_e \nabla \cdot v = \int_{Y'_e} \int_{Y'_e} P_e \nabla \cdot (R_* v).
\end{equation}
(3.17)

By the very construction of $R_*$ and $P_e$, we have
\begin{equation}
P_e = \begin{cases} 
 p_e & \text{in } Y'_e, \\
 \text{constant} & \text{in } Y'_e
\end{cases}
\end{equation}
(3.12)
and (see (3.12))
\begin{equation}
\nabla \cdot (R_* v) = \nabla \cdot v + \frac{1}{|Y'_e|} \int_{Y'_e} \nabla \cdot v \quad \text{in } Y'_e.
\end{equation}

We substitute in (3.17)
\begin{equation}
\int_{Y'_e} P_e \nabla \cdot v + \int_{Y'_e} (P_e |_{Y'_e}) \nabla \cdot v = \int_{Y'_e} P_e \left[ \nabla \cdot v + \frac{1}{|Y'_e|} \int_{Y'_e} \nabla \cdot v \right]
\end{equation}
\begin{equation}
\Rightarrow \left( P_e |_{Y'_e} \right) \int_{Y'_e} \nabla \cdot v = \left( \frac{1}{|Y'_e|} \int_{Y'_e} \nabla \cdot v \right) \int_{Y'_e} P_e
\end{equation}
(3.18)
and this is true for each $v \in [\mathcal{D}(Y'_e)]^N$. Thus,
\begin{equation}
P_e |_{Y'_e} = \frac{1}{|Y'_e|} \int_{Y'_e} P_e
\end{equation}
(3.19)
This proves the fact that the pressure's extension $P_e$ is constant in the solid part of each cell included in $C_s$, and (3.19) gives the value of the constant. □

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References

[8] L. Tartar, Convergence of the homogenization process, Appendix of [7].