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LECTURE 1
INTRODUCTION TO HOMOGENIZATION THEORY

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This lecture is devoted to a brief introduction to the mathematical theory of homogenization. For a more advanced presentation of homogenization, the reader is referred to the books [2], [5], [6], [11], [12], [21], [27], [32] and [33]. Roughly speaking, homogenization is a rigorous version of what is known as averaging. In other words, homogenization extracts homogeneous effective parameters from disordered or heterogeneous media.

Homogenization has first been developed for periodic structures. Indeed, in many fields of science and technology one has to solve boundary value problems in periodic media. Quite often the size of the period is small compared to the size of a sample of the medium, and, denoting by ϵ their ratio, an asymptotic analysis, as ϵ goes to zero, is called for. Starting from a microscopic description of a problem, we seek a macroscopic, or effective, description. This process of making an asymptotic analysis and seeking an averaged formulation is called homogenization. The first chapter will focus on the homogenization of periodic structures. The method of two-scale asymptotic expansions is presented, and its mathematical justification will be briefly discussed.

However we emphasize that homogenization is not restricted to the periodic case and can be applied to any kind of disordered media. This is the focus of the second chapter where the notion of G- or H-convergence is introduced. It allows to consider any possible geometrical situation without any specific assumptions like periodicity or randomness.

Chapter 1

Periodic homogenization

1.1 Setting of the problem.

We consider a model problem of diffusion or conductivity in a periodic medium (for example, an heterogeneous domain obtained by mixing periodically two different phases, one being the matrix and the other the inclusions; see Figure 1.1). To fix ideas, the periodic domain is called Ω (a bounded open set in $\mathbb{R}^N$ with $N \geq 1$ the space dimension), its period ϵ (a positive number which is assumed to be very small in comparison with the size of the domain), and the rescaled unit periodic cell $Y = (0,1)^N$. The conductivity in Ω is not constant, but varies periodically with period ϵ in each direction. It is a matrix (a second order tensor) $A(y)$, where $y = x/\epsilon \in Y$ is the fast periodic variable, while $x \in \Omega$ is the slow variable. Equivalently, x is also called the macroscopic variable, and y the microscopic variable. Since the component conductors do not need to be isotropic, the matrix A can be any second order tensor that is bounded and positive definite, i.e., there exist two positive constants $\beta \leq \alpha > 0$ such that, for any vector $\xi \in \mathbb{R}^N$ and at any point $y \in Y$,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2. \tag{1.1}$$

At this point, the matrix A is not necessarily symmetric (such is the case when some drift is taken into account in the diffusion process). The matrix $A(y)$ is a periodic function of y, with period Y, and it may be discontinuous in y (to model the discontinuity of conductivities from one phase to the other).

Denoting by $f(x)$ the source term (a scalar function defined in Ω), and enforcing a Dirichlet boundary condition (for simplicity), our model problem of conductivity reads

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

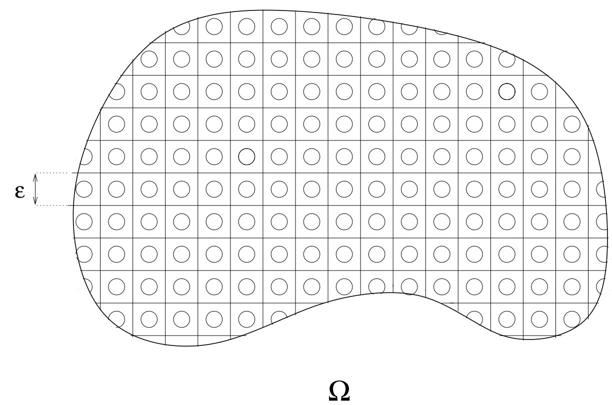

Figure 1.1: A periodic domain.

where $u_\epsilon(x)$ is the unknown function, modeling the electrical potential or the temperature.

Remark 1.1.1 *From a mathematical point of view, problem (1.2) is well posed in the sense that, if the source term $f(x)$ belongs to the space $L^2(\Omega)$ of square integrable functions on Ω, then the Lax-Milgram lemma implies existence and uniqueness of the solution u_ϵ in the Sobolev space $H_0^1(\Omega)$ of functions which belong to $L^2(\Omega)$ along with their first derivatives. Furthermore, the following energy estimate holds*

$$\|u_\epsilon\|_{L^2(\Omega)} + \|\nabla u_\epsilon\|_{L^2(\Omega)} \leq C,$$

where the constant C does not depend on ϵ.

The domain Ω, with its conductivity $A\left(\frac{x}{\epsilon}\right)$, is highly heterogeneous with periodic heterogeneities of lengthscale ϵ. Usually one does not need the full details of the variations of the potential or temperature u_ϵ, but rather some global of averaged behavior of the domain Ω considered as an homogeneous domain. In other words, an effective or equivalent macroscopic conductivity of Ω is sought. From a numerical point of view, solving equation (1.2) by any method will require too much effort if ϵ is small since the number of elements (or degrees of freedom) for a fixed level of accuracy grows like $1/\epsilon^N$. It is thus preferable to average or homogenize the properties of Ω and compute an approximation of u_ϵ on a coarse mesh. Averaging the solution of (1.2) and finding the effective properties of the domain Ω is what we call homogenization.

There is a difference of methodology between the traditional physical approach of homogenization and the mathematical theory of homogenization. In the mechanical literature, the so-called representative volume element (RVE) method is often used (see [8], or Chapter 1 in [17]). Roughly speaking, it consists in taking a sample of the heterogeneous medium of size much larger than the heterogeneities, but still much smaller than the medium, and averaging over it the gradient ∇u_ϵ and the flux $A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon$. Denoting by ξ the average of the gradient and by σ that of the flux, the effective tensor of conductivity A^* of this sample is defined by the linear relationship $\sigma = A^*\xi$. It turns out that the averaged stored energy $A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla u_\epsilon$ is also equal to the effective energy $A^*\xi \cdot \xi$. Although this type of definition is very intuitive, it is not clear whether it defines correctly an effective tensor A^*. In particular, it may depend on the choice of source term f, sample size, or boundary conditions.

The mathematical theory of homogenization works completely differently. Rather than considering a single heterogeneous medium with a fixed lengthscale, the problem is first embedded in a sequence of similar problems for which the lengthscale ϵ, becoming increasingly small, goes to zero. Then, an asymptotic analysis is performed as ϵ tends to zero, and the conductivity tensor of the limit problem is said to be the *effective* or *homogenized* conductivity. This seemingly more complex approach has the advantage of defining uniquely the homogenized properties. Further, the approximation made by using effective properties instead of the true microscopic coefficients can be rigorously justified by quantifying the resulting error.

In the case of a periodic medium Ω, this asymptotic analysis of equation (1.2), as the period ϵ goes to zero, is especially simple. The solution u_ϵ is written as a power series in ϵ

$$u_\epsilon = \sum_{i=0}^{+\infty} \epsilon^i u_i.$$

The first term u_0 of this series will be identified with the solution of the so-called homogenized equation whose effective conductivity A^* can be exactly computed. It turns out that A^* is a constant tensor, describing a homogeneous medium, which is independent of f and of the boundary conditions. Therefore, numerical computations on the homogenized equation do not require a fine mesh since the heterogeneities of size ϵ have been averaged out. This homogenized tensor A^* is almost never a usual average (arithmetic or harmonic) of $A(y)$. Various estimates will confirm this asymptotic analysis by telling in which sense u_ϵ is close to u_0 as ϵ tends to zero.

Remark 1.1.2 *From a more theoretical point of view, homogenization can be interpreted as follows. Rather than studying a single problem (1.2) for the physically relevant value of ϵ, we consider a sequence of such problems indexed by the period ϵ, which is now regarded as a small parameter going to zero. The question is to find the limit of this sequence of problems. The notion of limit problem is defined by considering the convergence of the sequence $(u_\epsilon)_{\epsilon>0}$ of solutions of (1.2): Denoting by u its limit, the limit problem is defined as the problem for which u is a solution. Of course, u will turn out to coincide with u_0, the first term in the series defined above, and it is therefore the solution of the homogenized* *equation. Clearly the mathematical difficulty is to define an adequate topology for this notion of convergence of problems as ϵ goes to zero.*

1.2 Two-scale asymptotic expansions.

1.2.1 Ansatz

The method of two-scale asymptotic expansions is an heuristic method, which allows one to formally homogenize a great variety of models or equations posed in a periodic domain. We present it briefly and refer to the classical books [5], [6], and [27] for more detail. A mathematical justification of what follows is to be found in Section 1.3. As already stated, the starting point is to consider the following *two-scale asymptotic expansion* (also called an *ansatz*), for the solution u_ϵ of equation (1.2)

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right), \tag{1.3}$$

where each term $u_i(x,y)$ is a function of both variables x and y, periodic in y with period $Y = (0,1)^N$ (u_i is called a Y-periodic function with respect to y). This series is plugged into the equation, and the following derivation rule is used:

$$\nabla\left(u_i\left(x, \frac{x}{\epsilon}\right)\right) = \left(\epsilon^{-1}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right), \tag{1.4}$$

where ∇_x and ∇_y denote the partial derivative with respect to the first and second variable of $u_i(x,y)$. For example, one has

$$\nabla u_\epsilon(x) = \epsilon^{-1}\nabla_y u_0\left(x, \frac{x}{\epsilon}\right) + \sum_{i=0}^{+\infty} \epsilon^i \left(\nabla_y u_{i+1} + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right).$$

Equation (1.2) becomes a series in ϵ

$$\begin{aligned}
&-\epsilon^{-2}\left[\mathrm{div_y} A\nabla_y u_0\right]\left(x, \frac{x}{\epsilon}\right)\\
&-\epsilon^{-1}\left[\mathrm{div_y} A(\nabla_x u_0 + \nabla_y u_1) + \mathrm{div_x} A\nabla_y u_0\right]\left(x, \frac{x}{\epsilon}\right)\\
&-\epsilon^{0}\left[\mathrm{div_x} A(\nabla_x u_0 + \nabla_y u_1) + \mathrm{div_y} A(\nabla_x u_1 + \nabla_y u_2)\right]\left(x, \frac{x}{\epsilon}\right)\\
&-\sum_{i=1}^{+\infty}\epsilon^{i}\left[\mathrm{div_x} A(\nabla_x u_i + \nabla_y u_{i+1}) + \mathrm{div_y} A(\nabla_x u_{i+1} + \nabla_y u_{i+2})\right]\left(x, \frac{x}{\epsilon}\right)\\
&= f(x).
\end{aligned} \tag{1.5}$$

Identifying each coefficient of (1.5) as an individual equation yields a cascade of equations (a series of the variable ϵ is zero for all values of ϵ if each coefficient is zero). It turns out that the three first equations are enough for our purpose. The ϵ^{-2} equation is

$$-\mathrm{div_y} A(y) \nabla_y u_0(x, y) = 0,$$

which is nothing else than an equation in the unit cell Y with periodic boundary condition. In this equation, y is the variable, and x plays the role of a parameter. It can be checked (see Lemma 1.2.1) that there exists a unique solution of this equation up to a constant (i.e., a function of x independent of y since x is just a parameter). This implies that u_0 is a function that does not depend on y, i.e., there exists a function $u(x)$ such that

$$u_0(x, y) \equiv u(x).$$

Since $\nabla_y u_0 = 0$, the ϵ^{-1} equation is

$$-\mathrm{div_y} A(y) \nabla_y u_1(x, y) = \mathrm{div_y} A(y) \nabla_x u(x), \tag{1.6}$$

which is an equation for the unknown u_1 in the periodic unit cell Y. Again, it is a well-posed problem, which admits a unique solution up to a constant, as soon as the right hand side is known. Equation (1.6) allows one to compute u_1 in terms of u, and it is easily seen that $u_1(x, y)$ depends linearly on the first derivative $\nabla_x u(x)$.

Finally, the ϵ^0 equation is

$$\begin{aligned} -\mathrm{div_y} A(y) \nabla_y u_2(x, y) &= \mathrm{div_y} A(y) \nabla_x u_1 \\ &\quad + \mathrm{div_x} A(y) \left(\nabla_y u_1 + \nabla_x u \right) + f(x), \end{aligned} \tag{1.7}$$

which is an equation for the unknown u_2 in the periodic unit cell Y. Equation (1.7) admits a solution if a compatibility condition is satisfied (the so-called *Fredholm alternative*; see Lemma 1.2.1). Indeed, integrating the left hand side of (1.7) over Y, and using the periodic boundary condition for u_2, we obtain

$$\int_Y \mathrm{div_y} A(y) \nabla_y u_2(x, y) dy = \int_{\partial Y} \left[A(y) \nabla_y u_2(x, y) \right] \cdot n \, ds = 0,$$

which implies that the right hand side of (1.7) must have zero average over Y, i.e.,

$$\int_Y \left[\mathrm{div_y} A(y) \nabla_x u_1 + \mathrm{div_x} A(y) \left(\nabla_y u_1 + \nabla_x u \right) + f(x) \right] dy = 0,$$

which simplifies to

$$-\mathrm{div_x} \left(\int_Y A(y) \left(\nabla_y u_1 + \nabla_x u \right) dy \right) = f(x) \quad \text{in } \Omega. \tag{1.8}$$

Since $u_1(x, y)$ depends linearly on $\nabla_x u(x)$, equation (1.8) is simply an equation for $u(x)$ involving only the second order derivatives of u.

1.2.2 The cell and the homogenized problems.

The method of two-scale asymptotic expansions give rise to a couple of equations (1.6) (1.8) that have a mathematical, as well as physical, interpretation. In order to compute u_1 and to simplify (1.8), we introduce the so-called *cell problems*. We denote by $(e_i)_{1\le i\le N}$ the canonical basis of $\mathbb{R}^N$. For each unit vector e_i, consider the following conductivity problem in the periodic unit cell:

$$\begin{cases} -\mathrm{div_y} A(y)\,(e_i + \nabla_y w_i(y)) = 0 & \text{in } Y \\ y \to w_i(y) & Y\text{-periodic}, \end{cases} \tag{1.9}$$

where $w_i(y)$ is the local variation of potential or temperature created by an averaged (or macroscopic) gradient e_i. The existence of a solution w_i to equation (1.9) is guaranteed by the following result.

Lemma 1.2.1 *Let $f(y) \in L^2_\#(Y)$ be a periodic function. There exists a solution in $H^1_\#(Y)$ (unique up to an additive constant) of*

$$\begin{cases} -\mathrm{div} A(y)\nabla w(y) = f & \textit{in } Y \\ y \to w(y) & Y\textit{-periodic}, \end{cases} \tag{1.10}$$

if and only if $\int_Y f(y)dy = 0$ (this is called the Fredholm alternative).

By linearity, it is not difficult to compute $u_1(x, y)$, solution of (1.6), in terms of $u(x)$ and $w_i(y)$

$$u_1(x, y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y). \tag{1.11}$$

In truth, $u_1(x, y)$ is merely defined up to the addition of a function $\tilde{u}_1(x)$ (depending only on x), but this does not matter since only its gradient $\nabla_y u_1(x, y)$ is used in the homogenized equation. Inserting this expression in equation (1.8), we obtain the homogenized equation for u that we supplement with a Dirichlet boundary condition on $\partial\Omega$,

$$\begin{cases} -\mathrm{div_x} A^* \nabla_x u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

The homogenized conductivity A^* is defined by its entries

$$A^*_{ij} = \int_Y \left[(A(y)\nabla_y w_j) \cdot e_i + A_{ij}(y) \right] dy,$$

or equivalently, after a simple integration by parts in Y,

$$A^*_{ij} = \int_Y A(y)\,(e_j + \nabla_y w_j) \cdot (e_i + \nabla w_i)\,dy. \tag{1.13}$$

The constant tensor A^* describes the effective or homogenized properties of the heterogeneous material $A\left(\frac{x}{\epsilon}\right)$. Note that A^* does not depend on the choice of domain Ω, source term f, or boundary condition on $\partial\Omega$.

Remark 1.2.2 *This method of two-scale asymptotic expansions is unfortunately not rigorous from a mathematical point of view. In other words, it yields heuristically the homogenized equation, but it does not yield a correct proof of the homogenization process. The reason is that the ansatz (1.3) is usually not correct after the two first terms. For example, it does not include possible boundary layers in the vicinity of $\partial\Omega$ (for details, see, e.g., [20]). Nevertheless, it is possible to rigorously justify the above homogenization process (see Section 1.3).*

1.2.3 A variational characterization of the homogenized coefficients.

The homogenized conductivity A^* is defined in terms of the solutions of the cell problems by equation (1.13). When the conductivity tensor $A(y)$ is symmetric, it is convenient to give another definition of A^* involving standard variational principles. From now on we assume that $A(y)$ is indeed symmetric. Therefore, by (1.13), A^* is symmetric too, and is completely determined by the knowledge of the quadratic form $A^*\xi \cdot \xi$ where ξ is any constant vector in $\mathbb{R}^N$. From definition (1.13) it is not difficult to check that

$$A^*\xi \cdot \xi = \int_Y A(y)\,(\xi + \nabla_y w_\xi) \cdot (\xi + \nabla_y w_\xi)\,dy, \tag{1.14}$$

where w_ξ is the solution of the following cell problem:

$$\begin{cases} -\mathrm{div}_y A(y)\,(\xi + \nabla_y w_\xi(y)) = 0 & \text{in } Y, \\ y \to w_\xi(y) & Y\text{-periodic.} \end{cases} \tag{1.15}$$

It is well-known that equation (1.15) is the Euler-Lagrange equation of the following variational principle: Find $w(y)$ that minimizes

$$\int_Y A(y)\,(\xi + \nabla_y w) \cdot (\xi + \nabla_y w)\,dy$$

over all periodic functions w. In other words, $A^*\xi \cdot \xi$ is given by the minimization of the potential energy

$$A^*\xi \cdot \xi = \min_{w(y) \in H^1_\#(Y)} \int_Y A(y)\,(\xi + \nabla_y w) \cdot (\xi + \nabla_y w)\,dy, \tag{1.16}$$

where $H^1_\#(Y)$ is the Sobolev space of Y-periodic functions w with finite energy, namely,

$$\int_Y \left(w^2 + |\nabla_y w|^2\right) dy < +\infty.$$

Remark that all the above equivalent definitions of A^* are not simple algebraic formulas, but rather they deliver the value of A^* at the price of a non-explicit computation of the solutions of the cell problems. However, in practice one is not always interested in the precise value of A^*, but rather in lower or upper estimates of its value. In this respect, the variational characterization (1.16) of A^* is useful since it provides an upper bound by choosing a specific test function $w(y)$. The simplest choice is to take $w(y) = 0$, which yields the so-called *arithmetic mean upper bound*

$$A^*\xi \cdot \xi \leq \left(\int_Y A(y)dy \right) \xi \cdot \xi. \tag{1.17}$$

A lower bound can also be obtained from (1.16) if the space of admissible fields in the minimization is enlarged. Indeed, remarking that the gradient $\nabla_y w(y)$ has zero-average over Y because of the periodicity of $w(y)$, this gradient can be replaced by any zero-average vector field

$$A^*\xi \cdot \xi \geq \min_{\substack{\zeta(y)\in L^2_\#(Y)^N \\ \int_Y \zeta(y)dy=0}} \int_Y A(y)\left(\xi + \zeta(y)\right) \cdot \left(\xi + \zeta(y)\right) dy, \tag{1.18}$$

where $L^2_\#(Y)$ is the space of square summable Y-periodic functions. The minimum in the right hand side of (1.18) is easy to compute: The optimal vector $\zeta_\xi(y)$ satisfies the following Euler-Lagrange equation

$$A(y)\left(\xi + \zeta_\xi(y)\right) = C,$$

where C is a constant (a Lagrange multiplier for the constraint $\int_Y \zeta_\xi(y)dy = 0$). After some algebra, one can compute explicitly the optimal ζ_ξ, as well as the minimal value that delivers the so-called *harmonic mean lower bound*

$$A^*\xi \cdot \xi \geq \left(\int_Y A^{-1}(y)dy \right)^{-1} \xi \cdot \xi. \tag{1.19}$$

From a physical point of view, the harmonic mean in (1.19) corresponds to an overall conductivity obtained by assuming that the values of the conductivity $A(y)$ are placed in series, while the arithmetic mean in (1.17) corresponds to an overall conductivity obtained by assuming that the values of the conductivity $A(y)$ are placed in parallel. These estimates hold true in great generality, but usually are not optimal and can be improved (see [2] in the case of two-phase composites). Actually, improving the harmonic and arithmetic mean bounds is one of the main problems of homogenization theory applied to the modeling of composite materials (see [21] for more details on this issue).

1.2.4 Evolution problem

The previous analysis extends easily to evolution problems. Let us consider first a parabolic equation modeling, for example, a diffusion process. For a final time $T > 0$, a source term $f(t,x) \in L^2((0,T)\times\Omega)$, and an initial data $a \in L^2(\Omega)$, the Cauchy problem is

$$\begin{cases} c\left(\frac{x}{\epsilon}\right)\frac{\partial u_\epsilon}{\partial t} - \operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega\times(0,T) \\ u_\epsilon = 0 & \text{on } \partial\Omega\times(0,T) \\ u_\epsilon(0,x) = a(x) & \text{in } \Omega. \end{cases} \tag{1.20}$$

where A satisfies the coercivity assumption (1.1), and c is a bounded positive Y-periodic function

$$0 < c^- \leq c(y) \leq c^+ < +\infty \quad \forall\, y \in Y.$$

Remark 1.2.3 *It is a well-known result that there exists a unique solution u_ϵ of (1.20) in the space $L^2\left((0,T); H_0^1(\Omega)\right) \cap C([0,T]; L^2(\Omega))$ which, furthermore, satisfies the energy estimate*

$$\|u_\epsilon\|_{C([0,T];L^2(\Omega))} + \|\nabla u_\epsilon\|_{L^2((0,T);L^2(\Omega))} \leq C, \tag{1.21}$$

where the constant C does not depend on ϵ.

One can perform the same two-scale asymptotic expansion on (1.20). The ansatz is

$$u_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(t, x, \frac{x}{\epsilon}\right),$$

where each term $u_i(t,x,y)$ is a function of time t and both space variables x and y. It is clear that the time derivative yield no contribution in the two first equations of the cascade of equations (1.5). However it gives a contribution for the third one. In other words the cell problem is the same as in the steady case, but the homogenized equation is changed. The reader will check easily that

$$u_0(t,x,y) \equiv u(t,x), \quad u_1(t,x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(t,x) w_i(y),$$

and the homogenized equation is

$$\begin{cases} c^* \dfrac{\partial u}{\partial t} - \operatorname{div}\left(A^* \nabla u\right) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times]0,T[\\ u(0) = a & \text{in } \Omega, \end{cases} \tag{1.22}$$

where the homogenized tensor is still given by (1.13) and

$$c^* = \int_Y c(y)\, dy. \tag{1.23}$$

We now consider an hyperbolic equation modeling, for example, the propagation of waves. For a final time $T > 0$, a source term $f(t,x) \in L^2((0,T) \times \Omega)$, a pair of initial data $a \in H_0^1(\Omega)$ and $b \in L^2(\Omega)$, the Cauchy problem is

$$\begin{cases} c\left(\dfrac{x}{\epsilon}\right) \dfrac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right) \nabla u_\epsilon\right) = f & \text{in } \Omega \times (0,T) \\ u_\epsilon = 0 & \text{on } \partial\Omega \times]0,T[\\ u_\epsilon(0,x) = a(x) & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial t}(0,x) = b(x) & \text{in } \Omega. \end{cases} \tag{1.24}$$

where A satisfies the coercivity assumption (1.1), and c is a bounded positive Y-periodic function.

Remark 1.2.4 *It is a well-known result that there exists a unique solution u_ϵ of (1.24) in the space $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ which, furthermore, satisfies the energy estimate*

$$\|\frac{\partial u_\epsilon}{\partial t}\|_{C([0,T];L^2(\Omega))} + \|\nabla u_\epsilon\|_{C([0,T];L^2(\Omega))} \leq C, \tag{1.25}$$

where the constant C does not depend on ϵ.

Again one can perform a two-scale asymptotic expansion on (1.24) with the ansatz

$$u_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i u_i \left(t, x, \frac{x}{\epsilon}\right),$$

where each term $u_i(t,x,y)$ is a function of time t and both space variables x and y. As in the parabolic case, the time derivative yield no contribution in the two first equations of the cascade of equations (1.5). However it gives a contribution for the third one. In other words the cell problem is the same as in the steady case, but the homogenized equation is changed. The reader will check easily that

$$u_0(t,x,y) \equiv u(t,x), \quad u_1(t,x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(t,x) w_i(y),$$

and the homogenized equation is

$$\begin{cases} c^* \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(A^* \nabla u) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times]0,T[\\ u(0) = a & \text{in } \Omega \\ \frac{\partial u}{\partial t}(0) = b & \text{in } \Omega, \end{cases} \tag{1.26}$$

where the homogenized tensor is still given by (1.13) and c^* is given by (1.23).

1.3 Mathematical justification of homogenization

This section is devoted to a brief introduction to the mathematical methods that justify the previous heuristic analysis of homogenization. We consider only two methods out of many more available.

1.3.1 The oscillating test function method

The oscillating test function method is a very elegant and efficient method for rigorously homogenizing partial differential equations which was devised by Tartar [30], [23] (sometimes it is also called the *energy method*). This method is very general and does not require any geometric assumptions on the behavior of the p.d.e. coefficients: neither periodicity nor statistical properties like stationarity or ergodicity. However, for the sake of clarity we present the oscillating test function method only in the periodic setting. Let us also mention that this method works for many models, and not only diffusion equations.

Recall that our model problem of diffusion reads

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.27}$$

where the source term $f(x)$ belongs to $L^2(\Omega)$. By application of Lax-Milgram lemma, equation (1.27) admits a unique solution u_ϵ in the space $H_0^1(\Omega)$ which satisfies the a priori estimate

$$\|u_\epsilon\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{1.28}$$

where C is a positive constant which does not depend on ϵ. Estimate (1.28) is obtained by multiplying equation (1.27) by u_ϵ, integrating by parts, and using Poincaré inequality. It implies that the sequence u_ϵ, indexed by a sequence of periods ϵ which goes to 0, is bounded in the Sobolev space $H_0^1(\Omega)$. Therefore, up to a subsequence, it converges weakly to a limit u in $H_0^1(\Omega)$. The goal is to find the homogenized equation satisfied by u.

Theorem 1.3.1 *The sequence $u_\epsilon(x)$ of solutions of (1.27) converges weakly in $H_0^1(\Omega)$ to a limit $u(x)$ which is the unique solution of the homogenized problem*

$$\begin{cases} -\operatorname{div}\left(A^*\nabla u(x)\right) = f(x) & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.29}$$

where the homogenized diffusion tensor, A^*, is defined by (1.13).

In order to shed some light on the principles of the energy method, let us begin with a naive attempt to prove Theorem 1.3.1 by passing to the limit in the variational formulation. The original problem (1.27) admits the following variational formulation

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon(x)\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \tag{1.30}$$

for any test function $\varphi \in H_0^1(\Omega)$. By estimate (1.28), we can extract a subsequence, still denoted by ϵ, such that u_ϵ converges weakly in $H_0^1(\Omega)$ to a limit u. Unfortunately, the left hand side of (1.30) involves the product of two weakly converging sequences in $L^2(\Omega)$,

$A\left(\frac{x}{\epsilon}\right)$ and $\nabla u_\epsilon(x)$, and it is not true that it converges to the product of the weak limits. Therefore, we cannot pass to the limit in (1.30) without any further argument.

The main idea of the energy method is to replace in (1.30) the fixed test function φ by a weakly converging sequence φ_ϵ (the so-called *oscillating test function*), chosen in such a way that the left hand side of (1.30) miraculously passes to the limit. This phenomenon is an example of the *compensated compactness* theory, developed by Murat and Tartar [22] [31], which under additional conditions permits to pass to the limit in some products of weak convergences.

Proof of Theorem 1.3.1. The key idea is the choice of an oscillating test function $\varphi_\epsilon(x)$. Let $\varphi(x) \in \mathcal{D}(\Omega)$ be a smooth function with compact support in Ω. Copying the two first terms of the asymptotic expansion of u_ϵ, the oscillating test function φ_ϵ is defined by

$$\varphi_\epsilon(x) = \varphi(x) + \epsilon \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) w_i^*\left(\frac{x}{\epsilon}\right), \tag{1.31}$$

where $w_i^*(y)$ are not the solutions of the cell problems, defined in (1.9), but that of the *dual cell problems*

$$\begin{cases} -\mathrm{div_y}\left(A^t(y)\left(e_i + \nabla_y w_i^*(y)\right)\right) = 0 & \text{in } Y \\ y \to w_i^*(y) & Y\text{-periodic.} \end{cases} \tag{1.32}$$

The difference between (1.9) and (1.32) is that the matrix $A(y)$ has been replaced by its *transpose* $A^t(y)$. By periodicity in y of w_i^*, it is easily seen that $\epsilon w_i^*\left(\frac{x}{\epsilon}\right)$ is a bounded sequence in $H^1(\Omega)$ which converges weakly to 0 (see Lemma 1.3.2 below if necessary).

The next step is to insert this oscillating test function φ_ϵ in the variational formulation (1.30)

$$\int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi_\epsilon(x) dx = \int_\Omega f(x)\varphi_\epsilon(x) dx. \tag{1.33}$$

To take advantage of our knowledge of equation (1.32), we develop and integrate by parts in (1.33). Remarking that

$$\nabla \varphi_\epsilon = \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x)\left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right) + \epsilon \sum_{i=1}^{N} \frac{\partial \nabla \varphi}{\partial x_i}(x) w_i^*\left(\frac{x}{\epsilon}\right),$$

yields

$$\int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi_\epsilon(x) dx = \int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x)\left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right) dx$$

$$+\epsilon \int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \sum_{i=1}^{N} \frac{\partial \nabla \varphi}{\partial x_i}(x) w_i^*\left(\frac{x}{\epsilon}\right). \tag{1.34}$$

The last term in (1.34) is easily seen to be bounded by a constant time ϵ, and thus cancels out in the limit. In the first term of (1.34), an integration by parts gives

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon(x)\cdot\sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\left(e_i+\nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right)dx = \tag{1.35}$$

$$-\int_\Omega u_\epsilon(x)\mathrm{div}\left(A^t\left(\frac{x}{\epsilon}\right)\sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\left(e_i+\nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right)\right)dx.$$

Let us compute the divergence in the right hand side of (1.35) which is actually a function of x and $y = x/\epsilon$

$$d_\epsilon(x) = \mathrm{div}\left(A^t\left(\frac{x}{\epsilon}\right)\sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\left(e_i+\nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right)\right)$$

$$= \sum_{i=1}^N \frac{\partial\nabla\varphi}{\partial x_i}(x)\cdot A^t(y)\left(e_i+\nabla_y w_i^*(y)\right) + \frac{1}{\epsilon}\sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\mathrm{div_y}\left(A^t(y)\left(e_i+\nabla_y w_i^*(y)\right)\right). \tag{1.36}$$

The last term of order ϵ^{-1} in the right hand side of (1.36) is simply zero by definition (1.32) of w_i^*. Therefore, $d_\epsilon(x)$ is bounded in $L^2(\Omega)$, and, since it is a periodically oscillating function, it converges weakly to its average by virtue of Lemma 1.3.2.

The main point of this simplification is that we are now able to pass to the limit in the right hand side of (1.35). Recall that u_ϵ is bounded in $H_0^1(\Omega)$: by application of Rellich theorem, there exists a subsequence (still indexed by ϵ for simplicity) and a limit $u \in H_0^1(\Omega)$ such that u_ϵ converges *strongly* to u in $L^2(\Omega)$. The right hand side of (1.35) is the product of a weak convergence (d_ϵ) and a strong one (u_ϵ), and thus its limit is the product of the two limits. In other words,

$$\lim_{\epsilon\to 0}\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon(x)\cdot\nabla\varphi_\epsilon(x)dx =$$

$$-\int_\Omega u(x)\mathrm{div_x}\left(\int_Y A^t(y)\sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\left(e_i+\nabla_y w_i^*(y)\right)dy\right)dx. \tag{1.37}$$

By definition (1.13) of A^*, a simple computation (based on multiplying (1.32) by w_j and (1.9) for the index j by w_i^*) shows that the right hand side of (1.37) is nothing else than

$$-\int_\Omega u(x)\mathrm{div_x}\left(A^{*t}\nabla\varphi(x)\right)dx.$$

Finally, a last integration by parts yields the limit variational formulation of (1.33)

$$\int_\Omega A^*\nabla u(x)\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx. \tag{1.38}$$

By density of smooth functions in $H_0^1(\Omega)$, (1.38) is valid for any test function $\varphi \in H_0^1(\Omega)$. Since A^* satisfies the same coercivity condition as A, Lax-Milgram lemma shows that (1.38) admits a unique solution in $H_0^1(\Omega)$. This last result proves that any subsequence of u_ϵ converges to the same limit u. Therefore, the entire sequence u_ϵ, and not only a subsequence, converges to the homogenized solution u. This concludes the proof of Theorem 1.3.1. $\square$

In the course of the proof of Theorem 1.3.1, the following lemma on periodically oscillating functions was used several times. Its proof is elementary (see if necessary [1], [11]), at least for smooth functions, by using a covering of the domain Ω in small cubes of size ϵ and the notion of Riemann integration (approximation of integrals by discrete sums).

Lemma 1.3.2 *Let $w(x,y)$ be a continuous function in x, square integrable and Y-periodic in y, i.e. $w(x,y) \in L^2_\#(Y; C(\Omega))$. Then, the sequence $w\left(x, \frac{x}{\epsilon}\right)$ converges weakly in $L^2(\Omega)$ to $\int_Y w(x,y)dy$.*

1.3.2 Two-Scale Convergence

Unlike the oscillating test function method, the two-scale convergence method is devoted only to periodic homogenization problems. It is therefore a less general method, but it is rather more efficient and simple in this context. Two-scale convergence has been introduced by Nguetseng [24] and Allaire [1] to which we refer for most proofs.

We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^N$ which are periodic of period Y, and by $C_\#(Y)$ the Banach space of continuous and Y-periodic functions. Eventually, $\mathcal{D}(\Omega; C^\infty_\#(Y))$ denotes the space of infinitely smooth and compactly supported functions in Ω with values in the space $C^\infty_\#(Y)$.

Definition 1.3.3 *A sequence of functions u_ϵ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^2(\Omega \times Y)$ if, for any function $\varphi(x,y)$ in $\mathcal{D}(\Omega; C^\infty_\#(Y))$, it satisfies*

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \varphi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y u_0(x,y)\varphi(x,y) dx dy.$$

In the above definition we use the fact that Y is the unit cube. Otherwise, the right hand side, which is an average over the periodicity cell Y, should be divied by its measure $|Y|$.

Theorem 1.3.4 *From each bounded sequence u_ϵ in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.*

Here are some examples of two-scale convergence.

1. Any sequence u_ϵ which converges strongly in $L^2(\Omega)$ to a limit $u(x)$, two-scale converges to the same limit $u(x)$.

2. For any smooth function $u_0(x,y)$, being Y-periodic in y, the associated sequence $u_\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right)$ two-scale converges to $u_0(x,y)$.

3. For the same smooth and Y-periodic function $u_0(x,y)$ the sequence defined by $v_\epsilon(x) = u_0(x, \frac{x}{\epsilon^2})$ has the same two-scale limit and weak-L^2 limit, namely $\int_Y u_0(x,y)dy$ (this is a consequence of the difference of orders in the speed of oscillations for v_ϵ and the test functions $\varphi\left(x, \frac{x}{\epsilon}\right)$). Clearly the two-scale limit captures only the oscillations which are in resonance with those of the test functions $\varphi\left(x, \frac{x}{\epsilon}\right)$.

4. Any sequence u_ϵ which admits an asymptotic expansion of the type $u_\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \cdots$, where the functions $u_i(x,y)$ are smooth and Y-periodic in y, two-scale converges to the first term of the expansion, namely $u_0(x,y)$.

The next theorem shows that more information is contained in a two-scale limit than in a weak-L^2 limit ; some of the oscillations of a sequence are contained in its two-scale limit. When all of them are captured by the two-scale limit (condition (1.40) below), one can even obtain a strong convergence (a corrector result in the vocabulary of homogenization).

Theorem 1.3.5 *Let u_ϵ be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$.*

1. Then, u_ϵ converges weakly in $L^2(\Omega)$ to $u(x) = \int_Y u_0(x,y)dy$, and we have

$$\lim_{\epsilon \to 0} \|u_\epsilon\|^2_{L^2(\Omega)} \geq \|u_0\|^2_{L^2(\Omega \times Y)} \geq \|u\|^2_{L^2(\Omega)}. \tag{1.39}$$

2. Assume further that $u_0(x,y)$ is smooth and that

$$\lim_{\epsilon \to 0} \|u_\epsilon\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega \times Y)}. \tag{1.40}$$

Then, we have

$$\|u_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right)\|^2_{L^2(\Omega)} \to 0. \tag{1.41}$$

Proof. By taking test functions depending only on x in Definition 1.3.3, the weak convergence in $L^2(\Omega)$ of the sequence u_ϵ is established. Then, developing the inequality

$$\int_\Omega |u_\epsilon(x) - \varphi\left(x, \frac{x}{\epsilon}\right)|^2 dx \geq 0,$$

yields easily formula (1.39). Furthermore, under assumption (1.40), it is easily obtained that

$$\lim_{\epsilon \to 0} \int_\Omega |u_\epsilon(x) - \varphi\left(x, \frac{x}{\epsilon}\right)|^2 dx = \int_\Omega \int_Y |u_0(x,y) - \varphi(x,y)|^2 dx dy.$$

If u_0 is smooth enough to be a test function φ, it yields (1.41). □

Theorem 1.3.6 *Let u_ϵ be a bounded sequence in $H^1(\Omega)$. Then, up to a subsequence, u_ϵ two-scale converges to a limit $u(x) \in H^1(\Omega)$, and ∇u_ϵ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$, where the function $u_1(x,y)$ belongs to $L^2(\Omega; H^1_\#(Y)/\mathbb{R})$.*

Proof. Since u_ϵ (resp. ∇u_ϵ) is bounded in $L^2(\Omega)$ (resp. $L^2(\Omega)^N$), up to a subsequence, it two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$ (resp. $\xi_0(x,y) \in L^2(\Omega \times Y)^N$). Thus for any $\psi(x,y) \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right)$, we have

$$\lim_{\epsilon\to 0} \int_\Omega \nabla u_\epsilon(x) \cdot \psi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y \xi_0(x,y) \cdot \psi(x,y) dx dy. \tag{1.42}$$

Integrating by parts the left hand side of (1.42) gives

$$\epsilon \int_\Omega \nabla u_\epsilon(x) \cdot \psi\left(x, \frac{x}{\epsilon}\right) dx = -\int_\Omega u_\epsilon(x) \left(\mathrm{div_y}\psi\left(x, \frac{x}{\epsilon}\right) + \epsilon \mathrm{div_x}\psi\left(x, \frac{x}{\epsilon}\right)\right) dx. \tag{1.43}$$

Passing to the limit yields

$$0 = -\int_\Omega \int_Y u_0(x,y) \mathrm{div_y}\psi(x,y) dx dy. \tag{1.44}$$

This implies that $u_0(x,y)$ does not depend on y. Thus there exists $u(x) \in L^2(\Omega)$, such that $u_0 = u$. Next, in (1.42) we choose a function ψ such that $\mathrm{div_y}\psi(x,y) = 0$. Integrating by parts we obtain

$$\begin{aligned} \lim_{\epsilon\to 0} \int_\Omega u_\epsilon(x) \mathrm{div_x}\psi\left(x, \frac{x}{\epsilon}\right) dx &= -\int_\Omega \int_Y \xi_0(x,y) \cdot \psi(x,y) dx dy \\ &= \int_\Omega \int_Y u(x) \mathrm{div_x}\psi(x,y) dx dy. \end{aligned} \tag{1.45}$$

If ψ does not depend on y, (1.45) proves that $u(x)$ belongs to $H^1(\Omega)$. Furthermore, we deduce from (1.45) that

$$\int_\Omega \int_Y (\xi_0(x,y) - \nabla u(x)) \cdot \psi(x,y) dx dy = 0 \tag{1.46}$$

for any function $\psi(x,y) \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right)$ with $\mathrm{div_y}\psi(x,y) = 0$. Recall that the orthogonal of divergence-free functions are exactly the gradients (this well-known result can be very easily proved in the present context by means of Fourier analysis in Y). Thus, there exists a unique function $u_1(x,y)$ in $L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that

$$\xi_0(x,y) = \nabla u(x) + \nabla_y u_1(x,y). \quad \Box \tag{1.47}$$

Application to the model problem (1.27). We now describe how the "two-scale convergence method" can justify the homogenization of (1.27). In a **first step**, we deduce from the a priori estimate (1.27) the precise form of the two-scale limit of the sequence u_ϵ. By application of Theorem 1.3.6, there exist two functions, $u(x) \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$, such that, up to a subsequence, u_ϵ two-scale converges to $u(x)$, and ∇u_ϵ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$. In view of these limits, u_ϵ is expected to behave as $u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right)$.

Thus, in a **second step**, we multiply equation (1.27) by a test function similar to the limit of u_ϵ, namely $\varphi(x) + \epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right)$, where $\varphi(x) \in \mathcal{D}(\Omega)$ and $\varphi_1(x,y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. This yields

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \left(\nabla\varphi(x) + \nabla_y\varphi_1\left(x,\frac{x}{\epsilon}\right) + \epsilon\nabla_x\varphi_1\left(x,\frac{x}{\epsilon}\right)\right)dx = \int_\Omega f(x)\left(\varphi(x) + \epsilon\varphi_1\left(x,\frac{x}{\epsilon}\right)\right)dx. \tag{1.48}$$

Regarding $A^t\left(\frac{x}{\epsilon}\right)\left(\nabla\varphi(x) + \nabla_y\varphi_1\left(x,\frac{x}{\epsilon}\right)\right)$ as a test function for the two-scale convergence (see Definition 1.3.3), we pass to the two-scale limit in (1.48) for the sequence ∇u_ϵ. Although this test function is not necessarily very smooth, as required by Definition 1.3.3, it belongs at least to $C\left(\bar{\Omega}; L^2_\#(Y)\right)$ which can be shown to be enough for the two-scale convergence Theorem 1.3.4 to hold (see [1] for details). Thus, the two-scale limit of equation (1.48) is

$$\int_\Omega\int_Y A(y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)\cdot\left(\nabla\varphi(x) + \nabla_y\varphi_1(x,y)\right)dxdy = \int_\Omega f(x)\varphi(x)dx. \tag{1.49}$$

In a **third step**, we read off a variational formulation for (u, u_1) in (1.49). Remark that (1.49) holds true for any (φ, φ_1) in the Hilbert space $H_0^1(\Omega) \times L^2\left(\Omega; H^1_\#(Y)/\mathbb{R}\right)$ by density of smooth functions in this space. Endowing it with the norm $\sqrt{(}\|\nabla u(x)\|^2_{L^2(\Omega)} + \|\nabla_y u_1(x,y)\|^2_{L^2(\Omega\times Y)})$, the assumptions of the Lax-Milgram lemma are easily checked for the variational formulation (1.49). The main point is the coercivity of the bilinear form defined by the left hand side of (1.49): the coercivity of A yields

$$\int_\Omega\int_Y A(y)\left(\nabla\varphi(x) + \nabla_y\varphi_1(x,y)\right)\cdot\left(\nabla\varphi(x) + \nabla_y\varphi_1(x,y)\right)dxdy \geq$$

$$\alpha\int_\Omega\int_Y |\nabla\varphi(x) + \nabla_y\varphi_1(x,y)|^2 dxdy = \alpha\int_\Omega |\nabla\varphi(x)|^2 dx + \alpha\int_\Omega\int_Y |\nabla_y\varphi_1(x,y)|^2 dxdy.$$

By application of the Lax-Milgram lemma, we conclude that there exists a unique solution (u, u_1) of the variational formulation (1.49) in $H_0^1(\Omega) \times L^2\left(\Omega; H^1_\#(Y)/\mathbb{R}\right)$. Consequently, the entire sequences u_ϵ and ∇u_ϵ converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x,y)$. An easy integration by parts shows that (1.49) is a variational formulation associated to the following system of equations, the so-called “two-scale homogenized problem”,

$$\begin{cases} -\mathrm{div_y}\left(A(y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)\right) = 0 & \text{in } \Omega\times Y \\ -\mathrm{div_x}\left(\int_Y A(y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)dy\right) = f(x) & \text{in } \Omega \\ y \to u_1(x,y) & Y\text{-periodic} \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.50}$$

At this point, the homogenization process could be considered as achieved since the entire sequence of solutions u_ϵ converges to the solution of a well-posed limit problem, namely the two-scale homogenized problem (1.50). However, it is usually preferable, from a physical or numerical point of view, to eliminate the microscopic variable y (one does not want to solve the small scale structure). In other words, we want to extract and decouple the usual homogenized and local (or cell) equations from the two-scale homogenized problem.

Thus, in a **fourth (and optional) step**, the y variable and the u_1 unknown are eliminated from (1.50). It is an easy exercise of algebra to prove that u_1 can be computed in terms of the gradient of u through the relationship

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y), \tag{1.51}$$

where $w_i(y)$ are defined as the solutions of the cell problems (1.9). Then, plugging formula (1.51) in (1.50) yields the usual homogenized problem (1.12) with the homogenized diffusion tensor defined by (1.13).

Due to the simple form of our model problem the two equations of (1.50) can be decoupled in a microscopic and a macroscopic equation, (1.9) and (1.12) respectively, but we emphasize that it is not always possible, and sometimes it leads to very complicate forms of the homogenized equation, including integro-differential operators. Thus, the homogenized equation does not always belong to a class for which an existence and uniqueness theory is easily available, on the contrary of the two-scale homogenized system, which is in most cases of the same type as the original problem, but with a double number of variables (x and y) and unknowns (u and u_1). The supplementary microscopic variable and unknown play the role of "hidden" variables in the vocabulary of mechanics. Although their presence doubles the size of the limit problem, it greatly simplifies its structure (which could be useful for numerical purposes too), while eliminating them introduces "strange" effects (like memory or non-local effects) in the usual homogenized problem.

Remark 1.3.7 *It is often very useful to obtain so-called "corrector" results which permit to obtain strong (or pointwise) convergences instead of just weak ones by adding some extra information stemming from the local equations. Typically, in the above example we simply proved that the sequence u_ϵ converges weakly to the homogenized solution u in $H_0^1(\Omega)$. Introducing the local solution u_1, this weak convergence can be improved as follows*

$$\left(u_\epsilon(x) - u(x) - \epsilon u_1\left(x, \frac{x}{\epsilon}\right)\right) \to 0 \textit{ in } H_0^1(\Omega) \textit{ strongly.} \tag{1.52}$$

This type of result is easily obtained with the two-scale convergence method. This rigorously justifies the two first term in the usual asymptotic expansion of the sequence u_ϵ. Indeed we can develop

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\left(\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right)\cdot\left(\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right) dx.$$

After some algebra and passing to the two-scale limit, we deduce that $(\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right))$ *goes to zero in* $L^2(\Omega)^N$.

Remark 1.3.8 *There is a variant of the method of two-scale convergence which is called the periodic unfolding method [9], [10]. The main idea of the periodic unfolding method is to introduce a so-called unfolding operator* E_ϵ *from* $L^2(\Omega)$ *into* $L^2\left(\Omega; L^2_\#(Y)\right)$ *defined, for any* $u(x) \in L^2(\Omega)$, *by*

$$E_\epsilon(u)(x,y) = \sum_{i=1}^{N(\epsilon)} u(x_i^\epsilon + \epsilon y)\chi_{Y_i^\epsilon}(x),$$

where $(Y_i^\epsilon)_{1\leq i\leq N(\epsilon)}$ *is a tiling of* Ω *with cubic cells* Y_i^ϵ, *which are equal to* ϵY *translated to its origin* x_i^ϵ, *and* $\chi_{Y_i^\epsilon}(x)$ *is the* i*-th cell characteristic function. The unfolding operator is easily seen to be linear and bounded. The main result of [9], [10] is then to prove that a sequence* $u_\epsilon(x)$ *of* $L^2(\Omega)$ *two-scale converges to a limit* $u_0(x,y)$ *if and only if the sequence* $E_\epsilon(u_\epsilon)(x,y)$ *weakly converges to* $u_0(x,y)$ *in* $L^2\left(\Omega; L^2_\#(Y)\right)$. *Note that some of the ideas of the periodic unfolding method were anticipated in [3], [4], [19].*

Chapter 2

General theory of homogenization

2.1 Introduction.

The first chapter was devoted to a brief presentation of homogenization in a periodic setting. This second chapter focus on the general setting of homogenization when no geometric assumptions are available (like periodicity, or ergodicity in a probabilistic framework). It turns out that homogenization can be applied to any kind of disordered media, and is definitely not restricted to the periodic case (although the nice "explicit" formulae of the periodic setting for the homogenized conductivity tensor have no analogue). We introduce the notion of G- or H-convergence which is due to DeGiorgi and Spagnolo [16], [28], [29], and has been further generalized by Murat and Tartar [23], [30] (see also the textbooks [25], [32], [33]). It allows to consider any possible geometrical situation without any specific assumptions like periodicity or randomness. The G- or H-convergence turns out to be the adequate notion of convergence for effective properties that will be the key tool in the study of optimal shape design problems.

Finally, let us mention that there is also a stochastic theory of homogenization (see [18], [13], [26]) and a variational theory of homogenization (the Γ-convergence of De Giorgi, [14], [15], see also the books [7], [12]) that will not be described below.

2.2 Definition of G-, or H-convergence.

The G-convergence is a notion of convergence associated to sequences of symmetric operators (typically, these operators are applications giving the solution of a partial differential equation in terms of the right hand side). The G means Green since this type of convergence corresponds roughly to the convergence of the associated Green functions. The H-convergence is a generalization of the G-convergence to the case of non-symmetric operators (it provides also an easier mathematical framework, but we shall not dwell on that).

The H stands for Homogenization since it is an important tool of that theory. For the sake of simplicity, we restrict ourselves to the case of symmetric operators (i.e. diffusion equations with symmetric coefficients). In such a case, G- and H-convergence coincide. Therefore in the sequel, we use only the notation G-convergence.

The main result of the G-convergence is a compactness theorem in the homogenization theory which states that, for any bounded and uniformly coercive sequence of coefficients of a symmetric second order elliptic equation, there exist a subsequence and a *G-limit* (i.e. homogenized coefficients) such that, for any source term, the corresponding subsequence of solutions converges to the solution of the homogenized equation. In practical terms, it means that the mechanical properties of an heterogeneous medium (like its conductivity, or elastic moduli) can be well approximated by the properties of a homogeneous or homogenized medium if the size of the heterogeneities are small compared to the overall size of the medium.

The G-convergence can be seen as a mathematically rigorous version of the so-called *representative volume element* method [8], [17] for computing effective or averaged parameters of heterogeneous media.

We introduce the notion of G-convergence for the specific case of a diffusion equation with a Dirichlet boundary condition, but all the results hold for a larger class of second order elliptic operators and boundary conditions. Let Ω be a bounded open set in $\mathbb{R}^N$, and let α, β be two positive constants such that $0 < \alpha \leq \beta$. We introduce the set $\mathcal{M}(\alpha, \beta, \Omega)$ of all possible symmetric matrices defined on Ω with uniform coercivity constant α and $L^\infty(\Omega)$-bound β. In other words, $A \in \mathcal{M}(\alpha, \beta, \Omega)$ if $A(x)$ satisfies

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2.$$

We consider a sequence $A_\epsilon(x)$ of conductivity tensors in $\mathcal{M}(\alpha, \beta, \Omega)$, indexed by a sequence of positive numbers ϵ going to 0. Here, ϵ is not associated to any specific length-scale or statistical property of the elastic medium. In other words, no special assumptions (like periodicity or stationarity) are placed on the sequence A_ϵ.

For a given source term $f(x) \in L^2(\Omega)$, there exists a unique solution u_ϵ in the Sobolev space $H_0^1(\Omega)$ of the following diffusion equation

$$\begin{cases} -\mathrm{div}\,(A_\epsilon(x)\nabla u_\epsilon) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The G-convergence of the sequence A_ϵ is defined below as the convergence of the corresponding solutions u_ϵ.

Definition 2.2.1 *The sequence of tensors $A_\epsilon(x)$ is said to G-converge to a limit $A^*(x)$, as ϵ goes to 0, if, for any $f \in L^2(\Omega)$ in (2.1), the sequence of solutions u_ϵ converges weakly in $H_0^1(\Omega)$ to a limit u which is the unique solution of the homogenized equation associated to A^*:*

$$\begin{cases} -\mathrm{div}\,(A^*(x)\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Remark that, by definition, the homogenized tensor A^* is independent of the source term f. We shall see that it is also independent of the boundary condition and of the domain.

This definition makes sense because of the compactness of the set $\mathcal{M}(\alpha, \beta, \Omega)$ with respect to the G-convergence, as stated in the following theorem.

Theorem 2.2.2 *For any sequence A_ϵ in $\mathcal{M}(\alpha, \beta, \Omega)$, there exist a subsequence (still denoted by ϵ) and a homogenized limit A^*, belonging to $\mathcal{M}(\alpha, \beta, \Omega)$, such that A_ϵ G-converges to A^*.*

The G-convergence of a general sequence A_ϵ is always stated *up to a subsequence* since A_ϵ can be the union of two sequences converging to two different limits. The G-convergence of A_ϵ is not equivalent to any other "classical" convergence. For example, if A_ϵ converges strongly in $L^\infty(\Omega)$ to a limit A (i.e. the convergence is pointwise), then its G-limit A^* coincides with A. But the converse is not true ! On the same token, the G-convergence has nothing to do with the usual weak convergence. Indeed, the G-limit A^* of a sequence A_ϵ is usually different of its weak-* $L^\infty(\Omega)$-limit. For example, a straightforward computation in one space dimension ($N = 1$) shows that the G-limit of a sequence A_ϵ is given as the inverse of the weak-* $L^\infty(\Omega)$-limit of A_ϵ^{-1} (the so-called harmonic limit). However, this last result holds true only in 1-D, and no such explicit formula is available in higher dimensions.

The G-convergence enjoys a few useful properties as enumerated in the following proposition.

Proposition 2.2.3 *Properties of G-convergence.*

1. *If a sequence A_ϵ G-converges, its G-limit is unique.*

2. *Let A_ϵ and B_ϵ be two sequences which G-converge to A^* and B^* respectively. Let $\omega \subset \Omega$ be a subset strictly included in Ω such that $A_\epsilon = B_\epsilon$ in ω. Then $A^* = B^*$ in ω (this property is called the locality of G-convergence).*

3. *The G-limit of a sequence A_ϵ is independent of the source term f and of the boundary condition on $\partial\Omega$.*

4. *Let A_ϵ be a sequence which G-converges to A^*. Then, the associated density of energy $A_\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon$ also converges to the homogenized density of energy $A^* \nabla u \cdot \nabla u$ in the sense of distributions in Ω.*

5. *If a sequence A_ϵ G-converges to a limit A^*, then the sequence of fluxes $A_\epsilon \nabla u_\epsilon$ converges weakly in $L^2(\Omega)^N$ to the homogenized flux $A^* \nabla u$.*

These properties of the G-convergence implies that the homogenized medium A^* approximates the heterogeneous medium A_ϵ in many different ways. First of all, by definition of G-convergence, the fields u, u_ϵ and their gradients are closed (this is the sense of the convergence of u_ϵ to u in the Sobolev space $H_0^1(\Omega)$). Then, by application of the above proposition, the fluxes and the energy densities are also closed.

Remark also that, by locality of the G-convergence, the homogenized tensor is defined at each point of the domain Ω independently of what may happen in other regions of Ω.

Of course, a particular example of G-convergent sequences A_ϵ is given by periodic media of the type $A\left(\frac{x}{\epsilon}\right)$ as in the previous section.

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