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LECTURE 2
HOMOGENIZATION IN POROUS MEDIA

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The goal of this lecture is to show that homogenization is a very efficient tool in the modeling of complex phenomena in heterogeneous media. In the first lecture we considered a model problem of diffusion for which the homogenized operator was of the same type (still a diffusion equation). In this context, homogenization is really a matter of defining and computing effective diffusion tensors. On the contrary, this second lecture will focus on models which have different homogenized limits (in the sense that the partial differential equations are of a different mathematical nature). For example, we shall see that the Stokes equations for a viscous fluid in a porous medium yield the Darcy's law as an homogenized model. Therefore, in this context, homogenization is a modeling tool which can justify new models arising as homogenized limits of complex microscopic equations.

Chapter 1

Homogenization of Stokes equations

1.1 Derivation of Darcy's Law

1.1.1 Setting and Results

This section is devoted to the derivation of Darcy's law for an incompressible viscous fluid flowing in a porous medium. Starting from the steady Stokes equations in a periodic porous medium, with a no-slip (Dirichlet) boundary condition on the solid pores, Darcy's law is rigorously obtained by periodic homogenization using the two-scale convergence method. The assumption on the periodicity of the porous medium is by no means realistic, but it allows to cast this problem in a very simple framework and to prove theorems without too much effort. We denote by ϵ the ratio of the period to the overall size of the porous medium: it is the small parameter of our asymptotic analysis since the pore size is usually much smaller than the characteristic length of the reservoir. The porous medium is contained in a domain Ω, and its fluid part is denoted by Ω_ϵ. From a mathematical point of view, Ω_ϵ is a periodically perforated domain, i.e., it has many small holes of size ϵ which represent solid obstacles that the fluid cannot penetrate.

The motion of the fluid in Ω_ϵ is governed by the steady Stokes equations, complemented with a Dirichlet boundary condition. We denote by u_ϵ and p_ϵ the velocity and pressure of the fluid, and f the density of forces acting on the fluid (u_ϵ and f are vector-valued functions, while p_ϵ is scalar). The fluid viscosity is a positive constant μ that we scale by a factor ϵ^2 (where ϵ is the period). The Stokes equations are

$$\begin{cases} \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{1.1}$$

The above scaling for the viscosity is such that the velocity u_ϵ has a non-trivial limit as ϵ goes to zero. Physically speaking, the very small viscosity, of order ϵ^2, balances exactly the friction of the fluid on the solid pore boundaries due to the no-slip boundary condition. Remark that this scaling is perfectly legitimate since by linearity of the equations one can always replace u_ϵ by $\epsilon^2 u_\epsilon$. To obtain an existence and uniqueness result for (1.1), the forcing term is assumed to have the usual regularity: $f(x) \in L^2(\Omega)^N$. Then, as is well-known (see e.g. [24]), the Stokes equations (1.1) admits a unique solution

$$u_\epsilon \in H_0^1(\Omega_\epsilon)^N, \; p_\epsilon \in L^2(\Omega_\epsilon)/\mathbb{R}, \tag{1.2}$$

the pressure being uniquely defined up to an additive constant. The homogenization problem for (1.1) is to find the effective equation satisfied by the limits of u_ϵ, p_ϵ. From the point of view of homogenization, the mathematical originality of system (1.1) is that the periodic oscillations are not in the coefficients of the operator but in the geometry of the porous medium Ω_ϵ.

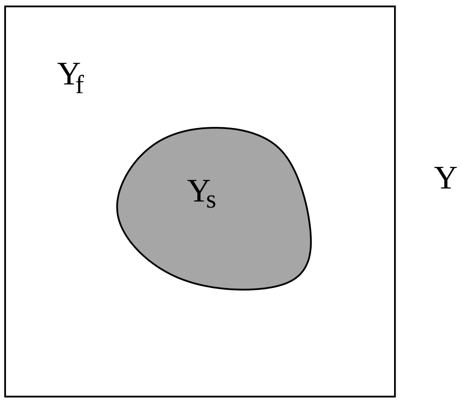

Figure 1.1: Unit cell of a porous medium.

Before stating the main result, let us describe more precisely the assumptions on the porous domain Ω_ϵ. As usual in periodic homogenization, a periodic structure is defined by a domain Ω and an associated microstructure, or periodic cell $Y = (0,1)^N$, which is made of two complementary parts : the fluid part Y_f and the solid part Y_b, satisfying $Y_f \cup Y_b = Y$ and $Y_f \cap Y_b = \emptyset$ (see Figure 1.1). We assume that Ω is a smooth, bounded, connected set in $\mathbb{R}^N$, and that Y_f is a smooth and connected open subset of Y, identified with the unit torus (i.e. Y_f, repeated by Y-periodicity in $\mathbb{R}^N$, is a smooth and connected open set of $\mathbb{R}^N$). The domain Ω is covered by a regular mesh of size ϵ: each cell Y_i^ϵ is of the type $(0,\epsilon)^N$, and is divided in a fluid part $Y_{f,i}^\epsilon$ and a solid part $Y_{s,i}^\epsilon$, i.e. is similar to the unit cell Y rescaled to size ϵ. The fluid part Ω_ϵ of a porous medium is defined by

$$\Omega_\epsilon = \Omega \setminus \bigcup_{i=1}^{N(\epsilon)} Y_{s,i}^\epsilon = \Omega \cap \bigcup_{i=1}^{N(\epsilon)} Y_{f,i}^\epsilon \tag{1.3}$$

where the number of cells is $N(\epsilon) = |\Omega|\epsilon^{-N}(1 + o(1))$.

A final word of caution is in order : the sequence of solutions (u_ϵ, p_ϵ) is not defined in a fixed domain *independent of* ϵ but rather in a varying set, Ω_ϵ. To state the homogenization theorem, convergences in fixed Sobolev spaces (defined on Ω) are used which requires first that (u_ϵ, p_ϵ) be extended to the whole domain Ω. Recall that, by definition, an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of (u_ϵ, p_ϵ) is defined on Ω and coincides with (u_ϵ, p_ϵ) on Ω_ϵ.

Theorem 1.1.1 *There exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) of (1.1) such that the velocity $\tilde{u}_\epsilon$ converges weakly in $L^2(\Omega)^N$ to u, and the pressure $\tilde{p}_\epsilon$ converges strongly in $L^2(\Omega)/\mathbb{R}$ to p, where (u, p) is the unique solution of the homogenized problem, a Darcy's law,*

$$\begin{cases} u(x) = \frac{1}{\mu} A\left(f(x) - \nabla p(x)\right) & \text{in } \Omega \\ \text{div} u(x) = 0 & \text{in } \Omega \\ u(x) \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where A is a symmetric, positive definite, tensor (the so-called permeability tensor) defined by its entries

$$A_{ij} = \int_{Y_f} \nabla w_i(y) \cdot \nabla w_j(y) dy \tag{1.5}$$

where, $(e_i)_{1 \le i \le N}$ being the canonical basis of $\mathbb{R}^N$, $w_i(y)$ denotes the unique solution in $H^1_\#(Y_f)^N$ of the local, or unit cell, Stokes problem

$$\begin{cases} \nabla q_i - \Delta w_i = e_i & \text{in } Y_f \\ \text{div} w_i = 0 & \text{in } Y_f \\ w_i = 0 & \text{in } Y_b \\ y \to q_i, w_i & Y\text{-periodic.} \end{cases} \tag{1.6}$$

The weak convergence of the velocity can be further improved by the following *corrector* result.

Proposition 1.1.2 *With the same notations as in Theorem 1.1.1, the velocity satisfies*

$$\left(\tilde{u}_\epsilon(x) - \sum_{i=1}^{N} w_i\left(\frac{x}{\epsilon}\right) u_i(x)\right) \to 0 \text{ strongly in } L^2(\Omega)^N, \tag{1.7}$$

where $(w_i)_{1 \le i \le N}$ are the local, unit cell, velocities and $(u_i)_{1 \le i \le N}$ are the components of the homogenized velocity $u(x)$.

Remark 1.1.3 *The homogenized problem (1.4) is a Darcy's law, i.e., the flow rate u is proportional to the balance of forces including the pressure. The permeability tensor A depends only on the microstructure, Y_f of the porous media (and not on the exterior forces, nor on the physical properties of the fluid). Quite early, many papers have been devoted to* *the derivation of Darcy's law by homogenization, using formal asymptotic expansions (see for example [12], [14], [21]). The first rigorous proof (including the difficult construction of a pressure extension) appeared in [23]. Further extensions are to be found in [1], [15], and [18]. A good reference for physical aspects of this problem, as well as mathematical ones, is the book [11]. Of course, more complicated models than the incompressible Stokes equations can be homogenized to derive various variants of Darcy's law. The next sections investigate such more general microscopic models. Of course there are other methods, apart from periodic homogenization, which permit to derive Darcy's law. It can also be established by stochastic homogenization, representative volume averaging, and so on.*

1.1.2 Two-scale asymptotic expansions

We apply the method of two-scale asymptotic expansions to the previous Stokes equations. We start from the following *two-scale asymptotic expansion* (or *ansatz*) of the velocity u_ϵ and pressure p_ϵ

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right), \quad p_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i p_i\left(x, \frac{x}{\epsilon}\right), \tag{1.8}$$

where each term $u_i(x,y)$ or $p_i(x,y)$ is a function of both variables x and y, periodic in y with period $Y = (0,1)^N$. These series are plugged into equation (1.1), and the following derivation rule is used:

$$\nabla\left(u_i\left(x, \frac{x}{\epsilon}\right)\right) = \left(\epsilon^{-1}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right),$$

where ∇_x and ∇_y denote the partial derivative with respect to the first and second variable of $u_i(x,y)$. Equation (1.1) becomes a series in ϵ

$$\begin{cases} \epsilon^{-1}\nabla_y p_0\left(x, \frac{x}{\epsilon}\right) + \epsilon^0\left[\nabla_x p_0 + \nabla_y p_1 - \mu\Delta_{yy}u_0\right]\left(x, \frac{x}{\epsilon}\right) + \mathcal{O}(\epsilon) = f(x) \\ \epsilon^{-1}\mathrm{div_y} u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon^0\left[\mathrm{div_x} u_0 + \mathrm{div_y} u_1\right]\left(x, \frac{x}{\epsilon}\right) + \mathcal{O}(\epsilon) = 0. \end{cases} \tag{1.9}$$

Identifying each coefficient of (1.9) as an individual equation yields a cascade of equations (a series of the variable ϵ is zero for all values of ϵ if each coefficient is zero). Here, only the two first equations are enough for our purpose. The ϵ^{-1} equation for the pressure is

$$\nabla_y p_0(x,y) = 0,$$

which is nothing else than an equation in the unit cell Y with periodic boundary condition. This implies that p_0 does not depend on y, i.e. there exists a function $p(x)$ such that

$$p_0(x,y) \equiv p(x).$$

The ϵ^{-1} equation from the incompressibility condition and the ϵ^0 equation from the momentum equation are

$$\begin{cases} \nabla_y p_1 - \mu \Delta_{yy} u_0 = f(x) - \nabla_x p(x) \\ \\ \mathrm{div_y} u_0 = 0 \end{cases} \tag{1.10}$$

which is a Stokes equation for the velocity u_0 and pressure p_1 in the periodic unit cell Y. It is a well-posed problem, which admits a unique solution, as soon as the right hand side is known. Equation (1.10) allows one to compute u_0 in terms of f and $\nabla_x p$ which do not depend on y. By linearity we find

$$u_0(x,y) = \frac{1}{\mu}\sum_{i=1}^{N} w_i(y)\left(f - \frac{\partial p}{\partial x_i}\right)(x), \quad p_1(x,y) = \sum_{i=1}^{N} q_i(y)\left(f - \frac{\partial p}{\partial x_i}\right)(x),$$

where w_i is the cell velocity and q_i is the cell pressure, solutions of the cell Stokes problem (1.6).

Finally, the ϵ^0 equation from the incompressibility condition yields

$$\mathrm{div_x} u_0(x,y) + \mathrm{div_y} u_1(x,y) = 0. \tag{1.11}$$

We average equation (1.11) in the unit cell Y. Taking into account the periodicity condition and the no-slip condition on the solid part Y_b leads, by application of Stokes theorem, to

$$\int_{Y_f} \mathrm{div_y} u_1(x,y)\, dy = \int_{\partial Y} u_1 \cdot n\, ds + \int_{\partial Y_b} u_1 \cdot n\, ds = 0.$$

This implies that the first term of (1.11) must have zero average over Y, i.e.,

$$\int_Y \mathrm{div_x}\left[\sum_{i=1}^{N} w_i(y)\left(f - \frac{\partial p}{\partial x_i}\right)(x)\right] dy = 0,$$

which simplifies to

$$-\mathrm{div_x} A\left(\nabla_x p(x) - f(x)\right) = 0 \quad \text{ in } \Omega, \tag{1.12}$$

which is a second-order elliptic equation for the pressure p. The constant tensor A is defined by its columns

$$A e_i = \int_Y w_i(y)\, dy,$$

which is equivalent to the previous definition (1.5) by a simple integration by parts (multiply the Stokes cell problem (1.6) by w_j).

Of course, p is the homogenized pressure, and the homogenized velocity u is defined by

$$u(x) = \int_Y u_0(x,y)\, dy = \frac{1}{\mu} A\left(f - \nabla p\right)(x).$$

1.1.3 Proof of the Homogenization Theorem

This subsection is devoted to the proof of Theorem 1.1.1 by the method of two-scale convergence. We assume the existence of bounded extensions of the velocity and pressure of the fluid in the porous medium (see [1], [11], [23] for a proof).

Lemma 1.1.4 *There exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) satisfying the a priori estimates*

$$\|\tilde{u}_\epsilon\|_{L^2(\Omega)^N} + \epsilon\|\nabla\tilde{u}_\epsilon\|_{L^2(\Omega)^{N\times N}} \leq C \tag{1.13}$$

and

$$\|\tilde{p}_\epsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C, \tag{1.14}$$

where the constant C does not depend on ϵ.

We also take for granted the following generalization of Theorem 1.3.7 in the first lecture, the proof of which may be found in [4].

Proposition 1.1.5 *Let u_ϵ be a bounded sequence in $L^2(\Omega)$ such that $\epsilon\nabla u_\epsilon$ is also bounded in $L^2(\Omega)^N$. Then, there exists a two-scale limit $u_0(x,y) \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, u_ϵ two-scale converges to $u_0(x,y)$, and $\epsilon\nabla u_\epsilon$ to $\nabla_y u_0(x,y)$.*

Let u_ϵ be a bounded sequence of vector valued functions in $L^2(\Omega)^N$ such that its divergence $\mathrm{div} u_\epsilon$ is also bounded in $L^2(\Omega)$. Then, there exists a two-scale limit $u_0(x,y) \in L^2(\Omega \times Y)^N$ which is divergence-free with respect to y, i.e. $\mathrm{div_y} u_0 = 0$, has a divergence with respect to x, $\mathrm{div_x} u_0$, in $L^2(\Omega \times Y)$, and such that, up to a subsequence, u_ϵ two-scale converges to $u_0(x,y)$, and $\mathrm{div} u_\epsilon$ to $\mathrm{div_x} u_0(x,y)$.

By application of the two-scale convergence method, we firstly prove

Theorem 1.1.6 *The extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution of (1.1) two-scale converges to the unique solution $(u_0(x,y), p(x))$ of the two-scale homogenized problem*

$$\begin{cases} \nabla_y p_1(x,y) + \nabla_x p(x) - \mu\Delta_{yy} u_0(x,y) = f(x) & \text{in } \Omega \times Y_f \\ \mathrm{div_y} u_0(x,y) = 0 & \text{in } \Omega \times Y_f \\ \mathrm{div_x} \left(\int_Y u_0(x,y)dy\right) = 0 & \text{in } \Omega \\ u_0(x,y) = 0 & \text{in } \Omega \times Y_b \\ \left(\int_Y u_0(x,y)dy\right) \cdot n = 0 & \text{on } \partial\Omega \\ y \to u_0(x,y), p_1(x,y) & Y\text{-periodic.} \end{cases} \tag{1.15}$$

Remark 1.1.7 *The two-scale homogenized problem is also called a two-pressure Stokes system. It is a combination of the usual homogenized and cell problems (see chapter 1). By elimination of the y variable, the homogenized Darcy's law will be recovered in the end.*

Proof of Theorem 1.1.6. Applying Proposition 1.1.5 there exists a two-scale limit $u_0(x,y) \in L^2\left(\Omega; H^1_\#(Y)^N\right)$ such that, up to a subsequence, the sequences $\tilde{u}_\epsilon$ and $\epsilon\nabla\tilde{u}_\epsilon$ two-scale converge to u_0 and $\nabla_y u_0$ respectively. Furthermore, u_0 satisfies

$$\begin{cases} \mathrm{div_y} u_0(x,y) = 0 & \text{in } \Omega \times Y \\ \mathrm{div_x}\left(\int_Y u_0(x,y)dy\right) = 0 & \text{in } \Omega \\ u_0(x,y) = 0 & \text{in } \Omega \times Y_b \\ \left(\int_Y u_0(x,y)dy\right)\cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

By the compactness theorem of two-scale convergence, there exists a two-scale limit $p_0(x,y) \in L^2(\Omega \times Y)$ such that, up to a subsequence, $\tilde{p}_\epsilon$ two-scale converges to p_0. Multiplying the momentum equation in (1.1) by $\epsilon\psi\left(x, \frac{x}{\epsilon}\right)$, where $\psi(x,y)$ is a smooth, vector-valued, Y-periodic function, and integrating by parts, leads to

$$\lim_{\epsilon\to 0}\int_\Omega \tilde{p}_\epsilon \mathrm{div_y}\psi\left(x, \frac{x}{\epsilon}\right)dx = \int_\Omega\int_Y p_0(x,y)\mathrm{div_y}\psi(x,y)dxdy = 0. \tag{1.17}$$

Another integration by parts in (1.17) shows that $\nabla_y p_0(x,y)$ is 0. Thus, there exists $p(x) \in L^2(\Omega)/\mathbb{R}$ such that $p_0(x,y) = p(x)$.

The next step in the two-scale convergence method is to multiply system (1.1) by a test function having the form of the two-scale limit u_0, and to read off a variational formulation for the limit. Therefore, we choose a test function $\psi(x,y) \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right)$ with $\psi(x,y) = 0$ in $\Omega \times Y_b$ (thus, $\psi\left(x, \frac{x}{\epsilon}\right) \in H^1_0(\Omega_\epsilon)^N$). Furthermore, we assume that ψ satisfies the incompressibility conditions (1.16), i.e. $\mathrm{div_y}\psi(x,y) = 0$ and $\mathrm{div_x}\left(\int_Y \psi(x,y)dy\right) = 0$. Multiplying equation (1.1) by $\psi\left(x, \frac{x}{\epsilon}\right)$, and integrating by parts yields

$$-\int_{\Omega_\epsilon} p_\epsilon(x)\mathrm{div_x}\psi\left(x, \frac{x}{\epsilon}\right)dx + \mu\int_{\Omega_\epsilon}\epsilon\nabla u_\epsilon(x)\cdot\nabla_y\psi\left(x, \frac{x}{\epsilon}\right)dx = \int_{\Omega_\epsilon} f(x)\cdot\psi\left(x, \frac{x}{\epsilon}\right)dx + O(\epsilon) \tag{1.18}$$

where $O(\epsilon)$ stands for the the remaining terms of order ϵ. In (1.18) the domain of integration Ω_ϵ can be replaced by Ω since the test function is zero in $\Omega \setminus \Omega_\epsilon$. Then, passing to the two-scale limit, the first term in (1.18) contributes nothing because the two-scale limit of $\tilde{p}_\epsilon$ does not depend on y and ψ satisfies $\mathrm{div_x}\left(\int_Y \psi(x,y)dy\right) = 0$, while the other terms give

$$\mu\int_\Omega\int_{Y_f}\nabla_y u_0(x,y)\cdot\nabla_y\psi(x,y)dxdy = \int_\Omega\int_{Y_f} f(x)\cdot\psi(x,y)dxdy. \tag{1.19}$$

By density (1.19) holds for any function ψ in the Hilbert space V defined by

$$V = \left\{\psi(x,y) \in L^2\left(\Omega; H^1_\#(Y)^N\right) \text{ such that } \begin{array}{l} \mathrm{div_y}\psi(x,y) = 0 \text{ in } \Omega\times Y \\ \mathrm{div_x}\left(\int_Y \psi(x,y)dy\right) = 0 \text{ in } \Omega \end{array},\right.$$

$$\left. \text{and } \begin{array}{l} \psi(x,y) = 0 \text{ in } \Omega \times Y_b \\ \left(\int_Y \psi(x,y)dy\right)\cdot n = 0 \text{ on } \partial\Omega \end{array}\right\}. \tag{1.20}$$

It is not difficult to check that the hypothesis of the Lax-Milgram lemma holds for the variational formulation (1.19) in the Hilbert space V, which, by consequence, admits a unique solution u_0 in V. Furthermore, by Lemma 1.1.8 below, the orthogonal of V, a subset of $L^2\left(\Omega; H^{-1}_{\#}(Y)^N\right)$, is made of gradients of the form $\nabla_x q(x) + \nabla_y q_1(x,y)$ with $q(x) \in H^1(\Omega)/\mathbb{R}$ and $q_1(x,y) \in L^2\left(\Omega; L^2_{\#}(Y_f)/\mathbb{R}\right)$. Thus, by integration by parts, the variational formulation (1.19) is equivalent to the two-scale homogenized system (1.15). (There is a subtle point here; one must check that the pressure $p(x)$ arising as a Lagrange multiplier of the incompressibility constraint $\mathrm{div_x}\left(\int_Y u_0(x,y)dy\right) = 0$ is the same as the two-scale limit of the pressure $\tilde{p}_\epsilon$. This can easily be done by multiplying equation (1.1) by a test function ψ which is divergence free only in y, and identifying limits.) Since (1.15) admits a unique solution, then the entire sequence $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ converges to its unique solution $(u_0(x,y), p(x))$. This completes the proof of Theorem 1.1.6. □

We now arrive at the last step of the two-scale convergence method which amounts to eliminate, if possible, the microscopic variable y in the homogenized system. This allows to deduce Theorem 1.1.1 from Theorem 1.1.6.

Proof of Theorem 1.1.1. The derivation of the homogenized Darcy's law (1.4) from the two-scale homogenized problem (1.15) is just a matter of algebra. From the first equation of (1.15), the velocity $u_0(x,y)$ is computed in terms of the macroscopic forces and the local velocities

$$u_0(x,y) = \frac{1}{\mu}\sum_{i=1}^{N}\left(f_i(x) - \frac{\partial p}{\partial x_i}(x)\right) w_i(y). \tag{1.21}$$

Averaging (1.21) on Y, and denoting by u the average of u_0, i.e. $u(x) = \int_{Y_f} u_0(x,y)dy$, yields the Darcy relationship

$$u(x) = \frac{1}{\mu} A\Big(f(x) - \nabla p(x)\Big), \tag{1.22}$$

since the matrix A satisfies

$$A_{ij} = \int_{Y_f} \nabla w_i(y)\cdot\nabla w_j(y)dy = \int_{Y_f} w_i(y)\cdot e_j dy. \tag{1.23}$$

Equation (1.23) is obtained by multiplying the i^{th} local problem (1.6) by w_j and integrating by parts (the boundary integrals cancel out, thanks to the periodic boundary condition). Combining (1.22) with the divergence-free condition on u yields the homogenized Darcy's law. Note that it is a well-posed problem since it is just a second order elliptic equation for the pressure p, complemented by a Neumann boundary condition. To complete the proof of Theorem 1.1.1 it remains to show that the convergence of the pressure $\tilde{p}_\epsilon$ to p is not only weak, but also strong, in $L^2(\Omega)/\mathbb{R}$: this will be done in the next subsection. □

Lemma 1.1.8 *Let V be the subspace of $L^2(\Omega; H^1_\#(Y)^N)$ defined by (1.20). Its orthogonal $V^\perp$ (with respect to the usual scalar product in $L^2(\Omega \times Y)$) has the following characterization*

$$V^\perp = \left\{ \nabla_x \varphi(x) + \nabla_y \varphi_1(x,y) \text{ with } \varphi \in H^1(\Omega) \text{ and } \varphi_1 \in L^2\left(\Omega; L^2_\#(Y_f)/\mathbb{R}\right) \right\}. \quad (1.24)$$

Proof. Remark that $V = V_1 \cap V_2$ with

$$V_1 = \left\{ v(x,y) \in L^2(\Omega; H^1_\#(Y)^N) \text{ s.t. } \mathrm{div_y} v = 0 \text{ in } \Omega \times Y, v = 0 \text{ in } \Omega \times Y_b \right\}$$

$$V_2 = \left\{ v(x,y) \in L^2(\Omega; H^1_\#(Y)^N) \text{ s.t. } \mathrm{div_x} \left(\int_{Y_f} v dy \right) = 0 \text{ in } \Omega, \left(\int_{Y_f} v dy \right) \cdot n_x = 0 \text{ on } \partial\Omega \right\}$$

It is a well-known result (see, e.g., [24]) that

$$V_1^\perp = \left\{ \nabla_y \varphi_1(x,y) \text{ with } \varphi_1 \in L^2\left(\Omega; L^2_\#(Y_f)/\mathbb{R}\right) \right\}$$

$$V_2^\perp = \left\{ \nabla_x \varphi(x) \text{ with } \varphi \in H^1(\Omega) \right\}.$$

The lemma is proved if one can check that $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$. Since V_1 and V_2 are two closed subspaces, this equality is equivalent to $V_1 + V_2 = \overline{V_1 + V_2}$. This is indeed true because we are going to prove that $V_1 + V_2$ is equal to $L^2(\Omega; H^1_\#(Y_f)^N)$. Introducing the divergence-free solutions $(w_i(y))_{1 \le i \le N}$ of the cell Stokes problem (1.6), for any given $v(x,y) \in L^2(\Omega; H^1_\#(Y_f)^N)$, we define a unique solution $q(x)$ in $H^1(\Omega)/\mathbb{R}$ of the Neumann problem

$$\begin{cases} -\mathrm{div_x} \left(A\nabla q(x) - \int_{Y_f} v(x,y) dy \right) = 0 & \text{in } \Omega \\ \left(A\nabla q(x) - \int_{Y_f} v(x,y) dy \right) \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.25)$$

where A is the matrix A defined by (1.5). This allows us to decompose v as

$$v(x,y) = \sum_{i=1}^N w_i(y) \frac{\partial q}{\partial x_i}(x) + \left(v(x,y) - \sum_{i=1}^N w_i(y) \frac{\partial q}{\partial x_i}(x) \right),$$

where the first term belongs to V_1, while the second one belongs to V_2. □

1.2 Inertia Effects

This section is devoted to a generalization of the previous model when inertial effects are added to the Stokes equations which then become the Navier-Stokes equations. To simplify the exposition we shall consider successively and separately the different inertial terms arising in the equations. A first subsection is concerned with the linear, evolutionary Stokes equations. A second one focuses on non-linear, steady Navier-Stokes equations. Of course, these two cases could be combined together with no special difficulties, but at the price of unnecessary and lengthy technical details.

The geometrical situation is the same as that of section 1.1. Namely, a periodic porous domain Ω and its fluid part Ω_ϵ are considered, with period ϵ and unit cell Y. For a precise description of Ω_ϵ, the reader is referred to definition (1.3) above.

1.2.1 Darcy's Law with Memory

We consider the unsteady Stokes equations in the fluid domain Ω_ϵ with a no-slip (Dirichlet) boundary condition. We denote by u_ϵ and p_ϵ the velocity and pressure of the fluid, f the density of forces acting on the fluid, and u_ϵ^0 an initial condition for the velocity. We assume that the density of the fluid is equal to 1, while its viscosity is very small, and indeed is exactly $\mu\epsilon^2$ (where ϵ is the pore size). The system of equations is

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } (0,T)\times\Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } (0,T)\times\Omega_\epsilon \\ u_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega_\epsilon \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \Omega_\epsilon \text{ at time } t=0. \end{cases} \tag{1.26}$$

The scaling ϵ^2 of the viscosity is the same as that in section 1.1. However, here it is not a simple change of variable since the density in front of the inertial term has been scaled to 1. The scalings in system (1.26) are precisely those which give a non-zero limit for the velocity u_ϵ and a limit problem depending on time. In particular, (1.26) is not equivalent to the following system (which gives rise to a different homogenized system with no inertial term in the limit)

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + \nabla p_\epsilon - \mu \Delta u_\epsilon = f & \text{in } (0,T)\times\Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } (0,T)\times\Omega_\epsilon \\ u_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega_\epsilon \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \Omega_\epsilon \text{ at time } t=0. \end{cases} \tag{1.27}$$

In some sense, the scaling of the viscosity in (1.26) can also be interpreted as a choice of the time scale of the order of the pore size squared. System (1.26) has been first studied by J.-L. Lions [14], using formal asymptotic expansions. Rigorous homogenization results have been proved later in [5]. A study of the different system (1.27) may be found in [18].

To obtain an existence result and convenient a priori estimates for the solution of (1.26), the force $f(t,x)$ is assumed to belong to $L^2\left((0,T)\times\Omega\right)^N$, and the initial condition $u_\epsilon^0(x)$ to $H_0^1(\Omega_\epsilon)^N$. Furthermore, denoting by $\tilde{u}_\epsilon^0$ the extension by zero in the solid part $\Omega\setminus\Omega_\epsilon$ of the initial condition, we assume that it satisfies

$$\begin{cases} \|\tilde{u}_\epsilon^0\|_{L^2(\Omega)} + \epsilon\|\nabla\tilde{u}_\epsilon^0\|_{L^2(\Omega)} \le C \\ \text{div}\tilde{u}_\epsilon^0 = 0 \text{ in } \Omega \\ \tilde{u}_\epsilon^0(x) \text{ two-scale converges to a unique limit } v^0(x,y). \end{cases} \tag{1.28}$$

Then, standard theory (see e.g. [24]) yields the following

Proposition 1.2.1 *Under assumption (1.28) on the initial condition, the Stokes equations (1.26) admits a unique solution* $u_\epsilon \in L^2\left((0,T); H_0^1(\Omega_\epsilon)\right)^N$, *and* $p_\epsilon \in L^2\left((0,T); L^2(\Omega_\epsilon)/\mathbb{R}\right)$. *Furthermore, the extension by zero in the solid part* $\Omega \setminus \Omega_\epsilon$ *of the velocity* $\tilde{u}_\epsilon$ *satisfies the a priori estimates*

$$\|\tilde{u}_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \epsilon\|\nabla\tilde{u}_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} \leq C \text{ and } \|\frac{\partial\tilde{u}_\epsilon}{\partial t}\|_{L^2((0,T)\times\Omega)} \leq C, \tag{1.29}$$

where the constant C *does not depend on* ϵ.

The following homogenization theorem states that the limit problem is a Darcy's law with memory (due to the convolution in time) which generalizes the usual Darcy's law.

Theorem 1.2.2 *There exists an extension* $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ *of the solution* (u_ϵ, p_ϵ) *of (1.26) which converges weakly in* $L^2\left((0,T); L^2(\Omega)^N\right) \times L^2\left((0,T); L^2(\Omega)/\mathbb{R}\right)$ *to the unique solution* (u,p) *of the homogenized problem*

$$\begin{cases} u(t,x) = v(t,x) + \frac{1}{\mu}\int_0^t A(t-s)\left(f - \nabla p\right)(s,x)ds & \text{in } (0,T)\times\Omega \\ \mathrm{div}u(t,x) = 0 & \text{in } (0,T)\times\Omega \\ u(t,x)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.30}$$

where $v(t,x)$ *is an initial condition which depends only on the sequence* u_ϵ^0 *and on the microstructure* Y_f, *and* $A(t)$ *is a symmetric, positive definite, time-dependent, permeability tensor which depends only on the microstructure* Y_f *(their precise form is to be found in the proof of the present theorem).*

The complicated form of the homogenized problem (1.30), which is not a parabolic p.d.e. but an integro-differential equation, is due to the elimination of a *hidden* microscopic variable. Actually, to prove Theorem 1.2.2 we first prove a result on the corresponding two-scale homogenized system which includes this hidden variable and has a much nicer form.

Theorem 1.2.3 *Under assumption (1.28) on the initial condition, there exists an extension* $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ *of the solution of (1.26) which two-scale converges to the unique solution* $(u_0(x,y), p(x))$ *of the two-scale homogenized problem*

$$\begin{cases} \frac{\partial u_0}{\partial t}(x,y) + \nabla_y p_1(x,y) + \nabla_x p(x) - \mu\Delta_{yy}u_0(x,y) = f(x) & \text{in } (0,T)\times\Omega\times Y_f \\ \mathrm{div_y} u_0(x,y) = 0 & \text{in } (0,T)\times\Omega\times Y_f \\ \mathrm{div_x}\left(\int_Y u_0(x,y)dy\right) = 0 & \text{in } (0,T)\times\Omega \\ u_0(x,y) = 0 & \text{in } (0,T)\times\Omega\times Y_b \\ \left(\int_Y u_0(x,y)dy\right)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega \\ y \to u_0, p_1 & Y\text{-periodic} \\ u_0(0,x,y) = v^0(x,y) & \text{at time } t=0. \end{cases} \tag{1.31}$$

Remark 1.2.4 *The two-scale homogenized problem (1.31) is also called a two-pressures Stokes system (see [14]). Eliminating y in (1.31) yields the Darcy's law with memory (1.30). It is not difficult to check that both $v(t,x)$ and $A(t,x)$ decay exponentially in time. Thus, the permeability keeps track mainly of the recent history. If the force f is steady (i.e. does not depend on t), asymptotically, for large time t, the usual steady Darcy's law for u and p is recovered. As we shall see, the two-scale homogenized problem (1.31) is equivalent to (1.30) complemented with the cell problems (1.32)-(1.33).*

Proof of theorem 1.2.3. The proof is completely parallel to that of Theorem 1.1.6 in section 1.1 (see [5]). The form of the two-scale homogenized problem (1.31) can also be obtained by using two-scale asymptotic expansions as in Subsection 1.1.2. The only difference is that the time derivative $\frac{\partial u_0}{\partial t}$ has to be added in the ϵ^0 equation (1.10) which yields (1.31). □

Proof of theorem 1.2.2. The only thing to prove is that eliminating the microscopic variable y in (1.31) leads to the Darcy's law with memory (1.30). The solution u_0 is decomposed in two parts $u_1 + u_2$ where u_1 is just the evolution (without any forcing term) of the initial condition v^0. Thus, at each point $x \in \Omega$, u_1 is the unique solution of an equation posed solely in Y_f

$$\begin{cases} \frac{\partial u_1}{\partial t}(x,y) + \nabla_y q(x,y) - \mu\Delta_{yy} u_1(x,y) = 0 & \text{in } (0,T)\times Y_f \\ \mathrm{div_y} u_1(x,y) = 0 & \text{in } (0,T)\times Y_f \\ u_1(x,y) = 0 & \text{in } (0,T)\times Y_b \\ y \to u_1, q & Y\text{-periodic} \\ u_1(0,x,y) = v^0(x,y) & \text{at time } t=0. \end{cases} \tag{1.32}$$

The average of u_1 in y is just $v(t,x)$ (the initial condition in the homogenized system (1.30)). On the other hand, u_2 is given by

$$u_2(t,x,y) = \int_0^t \sum_{i=1}^N \left(f_i - \frac{\partial p}{\partial x_i} \right)(s,x) \frac{\partial w_i}{\partial t}(t-s,y)ds$$

where, for $1 \le i \le N$, w_i is the unique solution of the cell problem, which does not depend on the macroscopic variable x. The cell problem is defined by

$$\begin{cases} \frac{\partial w_i}{\partial t}(y) + \nabla_y q_i(y) - \mu\Delta_{yy} w_i(y) = e_i & \text{in } (0,T)\times Y_f \\ \mathrm{div_y} w_i(y) = 0 & \text{in } (0,T)\times Y_f \\ w_i(y) = 0 & \text{in } (0,T)\times Y_b \\ y \to w_i, q_i & Y\text{-periodic} \\ w_i(0,y) = 0 & \text{at time } t=0. \end{cases} \tag{1.33}$$

Introducing the matrix A defined by

$$A_{ij}(t) = \mu \int_{Y_f} \frac{\partial w_i}{\partial t}(t,y) e_j dy, \tag{1.34}$$

the Darcy's law with memory is easily deduced from the two-scale homogenized problem by averaging u_1 and u_2 with respect to y. Eventually, using semi-group theory and integrating by parts in the cell problem (1.34), one can prove that A is symmetric, positive definite, and decays exponentially in time. □

1.2.2 Non-linear Darcy's Law

We consider the steady Navier-Stokes equations

$$\begin{cases} \epsilon^\gamma u_\epsilon \cdot \nabla u_\epsilon + \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \mathrm{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{1.35}$$

As before, the fluid viscosity μ has been scaled by a factor ϵ^2, which implies precisely that the velocity u_ϵ has a non-zero limit. The non-linear convective term has also been scaled by a factor ϵ^γ, where γ is a positive constant such that $\gamma \geq 1$. The limit case $\gamma = 1$ corresponds exactly to the scaling which yields a non-linear homogenized problem. The case $\gamma = 4$ allows to replace $\epsilon^2 u_\epsilon$ by a new velocity v_ϵ which satisfies the usual *unscaled* Navier-Stokes equations. Intermediate values of γ are analyzed below. For larger values, the convective terms are much smaller than the viscous ones, and the Navier-Stokes equations are just a small perturbation of the Stokes ones. For values smaller than 1, the opposite situation arises: convective terms dominate viscous ones. Unfortunately, in this last case, the homogenized limit is unclear.

We begin with a result of Mikelić [18] which states that, for $\gamma > 1$, the convective term of the Navier-Stokes equations disappears in the limit and the homogenized system is the usual Darcy's law (as in section 1.1). The only price to pay is a weaker convergence of the pressure: the closer γ to 1, the weaker the estimate on the pressure.

Theorem 1.2.5 *Let $\gamma > 1$. Define a constant $\beta > 1$ by*

$$\beta = \min\left(2, \frac{N}{N-2}, \frac{N}{N+2-2\gamma}\right). \tag{1.36}$$

There exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) of (1.35) such that the velocity $\tilde{u}_\epsilon$ converges weakly in $L^2(\Omega)^N$ to u, and the pressure $\tilde{p}_\epsilon$ converges strongly in $L^{q'}(\Omega)/\mathbb{R}$ to p, for any $1 < q' < \beta$, where (u, p) is the unique solution of the homogenized problem, a linear Darcy's law,

$$\begin{cases} u(x) = \frac{1}{\mu} A \left(f(x) - \nabla p(x)\right) & \text{in } \Omega \\ \mathrm{div} u(x) = 0 & \text{in } \Omega \\ u(x) \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.37}$$

In (1.37), the permeability tensor A is the usual homogenized matrix for Darcy's law, defined by (1.5) in Theorem 1.1.1.

We consider the limit case $\gamma = 1$ which yields a non-linear Darcy-type law (sometime called a Dupuit-Forchheimer-Ergun law). The non-linear convective term does not disappear in the homogenized problem which indicates a non-linear behavior of Darcy's law.

Theorem 1.2.6 *Let $\gamma = 1$. Let f be a smooth function such that its norm in $C^{1,\alpha}(\overline{\Omega})$, with $0 < \alpha < 1$, is sufficiently small. Then, there exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) of (1.35) and a unique solution (u_0, p_1, p) of the homogenized system*

$$\begin{cases} \nabla_y p_1 + u_0 \cdot \nabla_y u_0 - \mu \Delta_{yy} u_0 = f(x) - \nabla p(x) & \textit{in } Y_f \times \Omega \\ \mathrm{div_y} u_0 = 0 & \textit{in } Y_f \times \Omega \\ \mathrm{div_x} \left(\int_Y u_0 dy \right) = 0 & \textit{in } \Omega \\ u_0 = 0 & \textit{in } Y_b \times \Omega, \\ \left(\int_Y u_0 dy \right) \cdot n = 0 & \textit{on } \partial\Omega, \\ y \to (u_0, p_1) & Y\textit{-periodic,} \end{cases} \tag{1.38}$$

such that $\tilde{p}_\epsilon$ converges strongly in $L^q(\Omega)/\mathbb{R}$ to p, for any $1 < q < 2$, and

$$\|\tilde{u}_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right)\|_{L^2(\Omega)^2} \leq C\epsilon^l \quad \textit{with} \quad 0 < l < 1/6. \tag{1.39}$$

The homogenized system (1.38), called a two-pressure Navier-Stokes system, is very similar to the two-scale homogenized system (1.15). It is not possible to eliminate the y variable to obtain an explicit macroscopic effective law. Therefore, the non-linear Darcy's law is not a local, explicit, partial differential equation. Such a homogenized problem has formally been derived in [21] and [14]. A rigorous proof of convergence has recently been given in [17] (see also [19] in the two-dimensional case). The proof of Theorem 1.2.6 is very technical and is not reproduced here (the key argument is to prove an existence and uniqueness result for the homogenized system (1.38) by using a monotonicity argument).

1.3 Derivation of Brinkman's Law

1.3.1 Setting of the Problem

This section is devoted to the derivation of Brinkman's law for an incompressible viscous fluid flowing in a porous medium. As in the previous sections, we start from the steady Stokes equations in a periodic porous medium, with a no-slip (Dirichlet) boundary condition on the solid pores. We assume that the solid part of the porous medium is a collection of periodically distributed obstacles. We denote by ϵ the period, or the inter-obstacles distance. The major difference with the previous sections is that the solid obstacles are assumed to be much smaller than the period ϵ. Their size is denoted by $a_\epsilon \ll \epsilon$. There are now two small parameters in our asymptotic analysis which means that two-scale asymptotic expansions cannot be used in the sequel. The assumption on the periodicity of the porous medium allows to simplify greatly the results, although it is not strictly necessary (and not very realistic). As before, the porous medium is denoted by Ω, and its fluid part by Ω_ϵ.

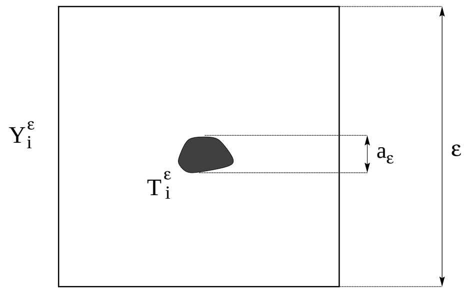

Figure 1.2: Scaling of the periodicity cell of a porous medium.

The motion of the fluid in Ω_ϵ is governed by the steady Stokes equations, complemented with a Dirichlet boundary condition. We denote by u_ϵ and p_ϵ the velocity and pressure of the fluid, μ its viscosity (a positive constant), and f the density of forces acting on the fluid (u_ϵ and f are vector-valued functions, while p_ϵ is scalar). The microscopic model is

$$\begin{cases} \nabla p_\epsilon - \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{1.40}$$

which admits a unique solution (u_ϵ, p_ϵ) in $H_0^1(\Omega_\epsilon)^N \times L^2(\Omega_\epsilon)/\mathbb{R}$ if $f(x) \in L^2(\Omega)^N$ (see e.g. [24]).

Let us describe more precisely the assumptions on the porous domain. It is contained in a bounded domain $\Omega \subset \mathbb{R}^N$, and its fluid part is denoted by Ω_ϵ. The set Ω is covered by a regular periodic mesh of period ϵ. At the center of each cell Y_i^ϵ (equal to $(0, \epsilon)^N$, up to a translation), a solid obstacle T_i^ϵ is placed which is obtained by rescaling a unit obstacle T to the size a_ϵ (i.e. $T_i^\epsilon = a_\epsilon T$, up to a translation, see Figure 1.2). The unit obstacle T is a non-empty, smooth, closed set included in the unit cell and such that $Y \setminus T$ is a smooth connected open set. The fluid part Ω_ϵ of the porous medium is defined by

$$\Omega_\epsilon = \Omega \setminus \bigcup_{i=1}^{N(\epsilon)} T_i^\epsilon, \tag{1.41}$$

where the number of obstacles is $N(\epsilon) = |\Omega| \epsilon^{-N} (1 + o(1))$. A fundamental assumption is that the obstacles are *much smaller* than the period,

$$\lim_{\epsilon \to 0} \frac{a_\epsilon}{\epsilon} = 0. \tag{1.42}$$

1.3.2 Main Results

According to the various scaling of the obstacle size a_ϵ in terms of the inter-obstacle distance ϵ, different limit problems arise. To sort these different regimes, we introduce a ratio σ_ϵ defined by

$$\sigma_\epsilon = \begin{cases} \left(\frac{\epsilon^N}{a_\epsilon^{N-2}}\right)^{1/2} & \text{for } N \geq 3, \\ \epsilon \left|\log\left(\frac{a_\epsilon}{\epsilon}\right)\right|^{1/2} & \text{for } N = 2. \end{cases} \tag{1.43}$$

As usual in perforated domains like Ω_ϵ, the sequence of solutions (u_ϵ, p_ϵ), being not defined in a fixed Sobolev space, *independent of* ϵ, needs to be extended to the whole domain Ω. We denote by $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ such an extension, which coincides, by definition, with (u_ϵ, p_ϵ) on Ω_ϵ.

Theorem 1.3.1 *According to the scaling of the obstacle size, there are three different flow regimes.*

1. *If the obstacles are too small, i.e.* $\lim_{\epsilon\to 0} \sigma_\epsilon = +\infty$, *then the extended solution* $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ *of (1.40) converges strongly in* $H_0^1(\Omega)^N \times L^2(\Omega)/\mathbb{R}$ *to* (u, p), *the unique solution of the homogenized Stokes equations*

$$\begin{cases} \nabla p - \mu\Delta u = f & \text{in } \Omega \\ \text{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.44}$$

2. *If the obstacles have a critical size, i.e.* $\lim_{\epsilon\to 0} \sigma_\epsilon = \sigma > 0$, *then the extended solution* $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ *of (1.40) converges weakly in* $H_0^1(\Omega)^N \times L^2(\Omega)/\mathbb{R}$ *to* (u, p), *the unique solution of the Brinkman law*

$$\begin{cases} \nabla p - \mu\Delta u + \frac{\mu}{\sigma^2} M u = f & \text{in } \Omega \\ \text{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.45}$$

3. *If the obstacles are too big, i.e.* $\lim_{\epsilon\to 0} \sigma_\epsilon = 0$, *then the rescaled solution* $(\frac{\tilde{u}_\epsilon}{\sigma_\epsilon^2}, \tilde{p}_\epsilon)$ *of (1.40) converges strongly in* $L^2(\Omega)^N \times L^2(\Omega)/\mathbb{R}$ *to* (u, p), *the unique solution of Darcy's law*

$$\begin{cases} u = \frac{1}{\mu} M^{-1} (f - \nabla p) & \text{in } \Omega \\ \text{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.46}$$

In all regimes, M is the same $N \times N$ symmetric matrix, which depends only on the model obstacle T (its inverse, M^{-1}, plays the role of a permeability tensor).

The following proposition gives the precise definition of M in terms of *local* problems around the unit obstacle T. Mathematically speaking, M can be interpreted in terms of the *hydrodynamic capacity* of the set T. From a physical point of view, the i^{th} column of M is the drag force of the local Stokes flow around T in the i^{th} direction. Thus, M may be interpreted as the slowing effect of the obstacles in the homogenized limit.

Proposition 1.3.2 *According to the space dimension N, the local Stokes problem and the matrix M are defined as follows.*

1. *If $N \geq 3$, for $1 \leq i \leq N$, the cell problems are*

$$\begin{cases} \nabla q_i - \Delta w_i = 0 & \text{in } \mathbb{R}^N \setminus T \\ \text{div} w_i = 0 & \text{in } \mathbb{R}^N \setminus T \\ w_i = 0 & \text{in } T \\ w_i \to e_i & \text{at } \infty. \end{cases} \tag{1.47}$$

   The matrix M is defined by its entries

$$M_{ij} = \int_{\mathbb{R}^N \setminus T} \nabla w_i \cdot \nabla w_j dx, \tag{1.48}$$

   or equivalently by its columns, equal to the drag forces applied on T by the local Stokes flows

$$M e_i = \int_{\partial T} \left(\frac{\partial w_i}{\partial n} - q_i n \right). \tag{1.49}$$

2. *If $N = 2$, for $1 \leq i \leq 2$, the cell problems are*

$$\begin{cases} \nabla q_i - \Delta w_i = 0 & \text{in } \mathbb{R}^2 \setminus T \\ \text{div} w_i = 0 & \text{in } \mathbb{R}^2 \setminus T \\ w_i = 0 & \text{in } T \\ w_i(x) \sim e_i \log(|x|) & \text{as } |x| \to \infty. \end{cases} \tag{1.50}$$

   The matrix M is defined by its columns, equal to the drag forces applied on T by the local Stokes flows

$$M e_i = \int_{\partial T} \left(\frac{\partial w_i}{\partial n} - q_i n \right). \tag{1.51}$$

   Furthermore, whatever the shape or the size of the obstacle T, in two space dimension the matrix M is always the same

$$M = 4\pi Id. \tag{1.52}$$

In the super-critical case, the homogenized problem is a Darcy's law with a permeability tensor M^{-1} (see (1.46)). This tensor has nothing to do with that, denoted by A, obtained in a two-scale periodic setting (see Theorem 1.1.1 in section 1.1). Recall that here the obstacles are much smaller than the period (see assumption (1.42)). The two matrices are not computed with the same local problems which are posed in a single periodicity cell in section 1.1 and in the whole space $\mathbb{R}^N$ here. However, it has been proved in [3] that M^{-1} is the rescaled limit of A, when the obstacle size goes to zero in the unit periodic cell Y.

Remark 1.3.3 *The surprising result in 2-D, that M is always equal to $4\pi Id$, is actually a consequence of the well-known* Stokes paradox*. This paradox asserts that there exist no solution of the local problem (1.47) in 2-D (it explains why the growth condition at infinity in (1.50) is different from the higher dimensional cases). Notice also that the critical size yielding Brinkman's law changes drastically from 2-D, $a_\epsilon = e^{-\sigma^2/\epsilon^2}$, to 3-D, $a_\epsilon = \sigma^2\epsilon^3$.*

The proof of Theorem 1.3.1 relies on the oscillating test function method of Tartar as adapted to the present framework by Cioranescu and Murat [10]. Let us sketch the main idea of this method. It consists in multiplying the Stokes equation (1.40) by a test function, integrating by parts, and passing to the limit, as $\epsilon \to 0$, in order to obtain the variational formulation of the homogenized problem. The key difficulty here is that the test function must belong to $H_0^1(\Omega_\epsilon)^N$, namely it has to vanish on the obstacles for any value of ϵ. Of course it is not the case for a non-zero fixed test function φ. Therefore, *boundary layers* $(w_i^\epsilon)_{1\le i\le N}$ have to be constructed such that, φ being a smooth vector-field, $\sum_{i=1}^N \varphi_i w_i^\epsilon$ belongs to $H_0^1(\Omega_\epsilon)^N$ and converges to φ as ϵ goes to zero. These boundary layers $(w_i^\epsilon)_{1\le i\le N}$ are built with the help of the solutions $(w_i)_{1\le i\le N}$ of the local Stokes problems from Proposition 1.3.2 by rescaling them to the size a_ϵ around each obstacle and pasting each contribution at the cell boundary ∂Y_i^ϵ. Loosely speaking, w_i^ϵ is a divergence-free vector field which vanishes on the obstacles and is almost equal to the unit vector e_i far from the obstacles. Using this oscillating test function, $\sum_{i=1}^N \varphi_i w_i^\epsilon$ yields the desired result (see [2] for details). Another interest of the boundary layers $(w_i^\epsilon)_{1\le i\le N}$ is that they permit to obtain *corrector results*. For example, in the critical case and under a mild smoothness assumption on the Brinkman velocity u, the weak convergence in $H_0^1(\Omega)^N$ of $\tilde{u}_\epsilon$ can be improved in

$$\left(\tilde{u}_\epsilon - \sum_{i=1}^N v_i w_i^\epsilon\right) \to 0 \text{ strongly in } H_0^1(\Omega)^N.$$

Finally, as in section 1.1 on the derivation of Darcy's law, a technical difficulty is the construction of a bounded extension of the pressure p_ϵ. As usual, extending the velocity is easier: it suffices to take it equal to 0 inside the obstacles,

$$\begin{cases} \tilde{u}_\epsilon = u_\epsilon & \text{in } \Omega_\epsilon, \\ \tilde{u}_\epsilon = 0 & \text{in } \Omega \setminus \Omega_\epsilon. \end{cases}$$

Remark 1.3.4 *In space dimension $N = 2, 3$, Theorem 1.3.1 can be generalized easily to the non-linear Navier-Stokes equations (see Remark 1.1.10 in [2]). The microscopic equations in the porous medium are*

$$\begin{cases} \nabla p_\epsilon + u_\epsilon \cdot \nabla u_\epsilon - \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{1.53}$$

Then, there are still three limit flow regimes, corresponding to the same obstacle sizes, and the definitions of the local problems and of the matrix M are still given by Proposition 1.3.2. In the critical case, the homogenized problem is a non-linear Brinkman's law

$$\begin{cases} \nabla p + u \cdot \nabla u - \mu \Delta u + \frac{\mu}{\sigma^2} M u = f & \text{in } \Omega \\ \text{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.54}$$

while in the super-critical case it is still the same linear Darcy's law (1.46).

The rigorous derivation of Brinkman's law by homogenization of Stokes equations in a periodic porous medium has first been established by Marchenko and Khruslov [16]. The description of all limit regimes and the two-dimensional paradoxical result of Proposition 1.3.2 are due to Allaire [2]. Brinkman's law has also been obtained formally by *three-scale* asymptotic expansion by Lévy [13] and Sanchez-Palencia [22].

Chapter 2

Homogenization of diffusion equations

2.1 Double Permeability

2.1.1 Setting and results

This section is devoted to the derivation of the so-called *double porosity* model for describing single-phase flows in fractured porous media. This model is well known in the engineering literature [11]. It has rigorously been derived by means of homogenization techniques [9]. A fractured porous medium possesses two porous structures, one associated with the system of cracks or fractures, and the other with the matrix of porous rocks. In each of these structures, the fluid flow is assumed to be governed by Darcy's law. On the contrary of the previous sections where the starting model was a *microscopic* model (Stokes equations at the pore level), here the original model is already an averaged model (Darcy's law in both the matrix and the fractures). Therefore, we shall obtain a *macroscopic* model starting from a *mesoscopic* one. More precisely, we shall prove, under suitable assumptions, that the homogenization of Darcy's law in a periodic fractured porous medium yields a double porosity model.

As before, we denote by Ω the periodic porous medium with its period ϵ. The rescaled unit cell is $Y = (0,1)^N$, which is made of two complementary parts: the matrix block Y_b, and the fracture set Y_f ($Y_b \cup Y_f = Y$ and $Y_f \cap Y_b = \emptyset$). The matrix block Y_b is assumed to be completely surrounded by the fracture set Y_f, i.e. Y_b is strictly included in Y. We define the matrix and fracture parts of Ω by

$$Y_b^\epsilon = \Omega \cap \bigcup_{i=1}^{N(\epsilon)} Y_{b\,i}^\epsilon, \quad Y_f^\epsilon = \Omega \cap \bigcup_{i=1}^{N(\epsilon)} Y_{f\,i}^\epsilon, \tag{2.1}$$

where $Y_{b\,i}^\epsilon$ and $Y_{f\,i}^\epsilon$ are the ϵ-size copies of Y_b and Y_f covering Ω.

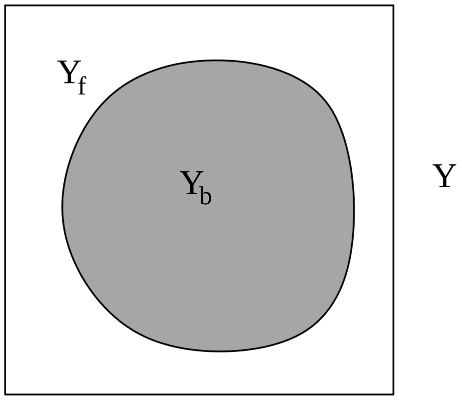

Figure 2.1: Unit cell of a fractured porous medium.

The reservoir Ω is periodic since its porosity ϕ_ϵ and permeability k_ϵ are periodic functions defined by

$$\begin{cases} k_\epsilon(x) = \epsilon^2 k_{Y_b}, \quad \phi_\epsilon(x) = \phi_{Y_b} & \text{in } Y_b^\epsilon \\ k_\epsilon(x) = k_{Y_f}, \quad \phi_\epsilon(x) = \phi_{Y_f} & \text{in } Y_f^\epsilon \end{cases} \tag{2.2}$$

where ϕ_{Y_b}, ϕ_{Y_f} are positive constants, and k_{Y_b}, k_{Y_f} are positive definite tensors (they could also depend on x and y). The fluid is assumed to be compressible, and the state law giving the relationship between its density ρ_ϵ and its pressure p_ϵ is linearized. Combining Darcy's law and the conservation of mass, and neglecting gravity effects, yields the following equation

$$\begin{cases} \phi_\epsilon \frac{\partial \rho_\epsilon}{\partial t} - \operatorname{div}(k_\epsilon \nabla \rho_\epsilon) = f & \text{in } (0,T)\times\Omega \\ (k_\epsilon \nabla \rho_\epsilon)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega \\ \rho_\epsilon(0,x) = \rho^{init}(x) & \text{at time } t = 0. \end{cases} \tag{2.3}$$

Remark 2.1.1 *We emphasize the particular scaling of the permeability defined in (2.2): the matrix is much less permeable than the fractures. Equivalently, the time scale of filtration inside the matrix is much smaller than that inside the fractures. On the other hand, the porosities have the same order of magnitude in both regions. Such scalings have been found to yield a homogenized double porosity model by Arbogast, Douglas, and Hornung in [9]. If the permeabilities of both phases (matrix and fractures) were of the same order, the homogenized system would easily be seen to be a single Darcy's law with effective coefficients computed with the usual rules of homogenization.*

To obtain an existence result and convenient a priori estimates for the solution of (1.40), the source term $f(t,x)$ is assumed to belong to $L^2((0,T)\times\Omega)$, and the initial condition $\rho^{init}(x)$ to $H^1(\Omega)$ (the initial condition could vary with ϵ as soon as it converges sufficiently smoothly). Then, standard theory yields the following

Proposition 2.1.2 *There exists a unique density $\rho_\epsilon \in L^2\left((0,T);H^1(\Omega)\right)$ solution of system (1.26). Furthermore, it satisfies the a priori estimates*

$$\begin{aligned} &\|\rho_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \|\nabla\rho_\epsilon\|_{L^\infty\left((0,T);L^2(Y_f^\epsilon)\right)} + \epsilon\|\nabla\rho_\epsilon\|_{L^\infty\left((0,T);L^2(Y_b^\epsilon)\right)} \leq C, \\ &\|\frac{\partial\rho_\epsilon}{\partial t}\|_{L^2((0,T)\times\Omega)} \leq C, \end{aligned} \tag{2.4}$$

where the constant C does not depend on ϵ.

The following homogenization theorem states that the homogenized problem is a double permeability model.

Theorem 2.1.3 *Under assumptions (2.2), the density ρ_ϵ, solution of (2.3), two-scale converges to $\rho_0(x,y) \in L^2((0,T)\times\Omega\times Y)$ such that*

$$\begin{cases} \rho_0(x,y) = \rho_{Y_f}(x) & \text{if } (x,y) \in \Omega\times Y_f \\ \rho_0(x,y) = \rho_{Y_b}(x,y) & \text{if } (x,y) \in \Omega\times Y_b \end{cases}$$

where $\left(\rho_{Y_f}(x), \rho_{Y_b}(x,y)\right) \in L^2\left((0,T);H^1(\Omega)\right) \times L^2\left((0,T)\times\Omega;H^1(Y_b)\right)$ is the unique solution of the coupled homogenized problem

$$\begin{cases} \theta\phi_{Y_f}\frac{\partial\rho_{Y_f}}{\partial t} - \mathrm{div_x}\left(k^*\nabla_x\rho_{Y_f}\right) = f - \phi_{Y_b}\int_{Y_b}\frac{\partial\rho_{Y_b}}{\partial t}(x,y)dy & \text{in } (0,T)\times\Omega \\ \left(k^*\nabla\rho_{Y_f}\right)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega \\ \rho_{Y_f}(0,x) = \rho^{init}(x) & \text{at time } t=0 \\ \phi_{Y_b}\frac{\partial\rho_{Y_b}}{\partial t} - \mathrm{div_y}\left(k_{Y_b}\nabla_y\rho_{Y_b}\right) = f(t,x) & \text{in } (0,T)\times Y_b \\ \rho_{Y_b}(x,y) = \rho_{Y_f}(x) & \text{on } (0,T)\times\partial Y_b \\ \rho_{Y_b}(0,x,y) = \rho^{init}(x) & \text{at time } t=0, \end{cases} \tag{2.5}$$

where $\theta = \frac{|Y_f|}{|Y|}$ is the volume fraction of fractures, and k^* is the homogenized permeability tensor defined by its entries

$$k^*_{ij} = \int_{Y_f} k_{Y_f}\left(e_i + \nabla_y\chi_i\right)\cdot\left(e_j + \nabla_y\chi_j\right)dy,$$

where $\chi_i(y)$ are the unique solutions in $H^1_\#(Y_f)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\mathrm{div_y}k_{Y_f}\left(e_i + \nabla_y\chi_i(y)\right) = 0 & \text{in } Y_f \\ k_{Y_f}\left(e_i + \nabla_y\chi_i(y)\right)\cdot n = 0 & \text{on } \partial Y_b \\ y \to \chi_i(y) & Y\text{-periodic,} \end{cases}$$

with $(e_i)_{1\leq i\leq N}$ the canonical basis of $\mathbb{R}^N$.

The homogenized problem (2.5) is called a *double porosity model* since it couples a macroscopic equation for ρ_{Y_f} (the first line of (2.5)) and a microscopic equation for ρ_{Y_b} (the fourth line of (2.5)). Remark that the (weak) limit of ρ_ϵ is not ρ_{Y_f}, but rather a combination of ρ_{Y_f} and ρ_{Y_b}

$$\rho_\epsilon \rightharpoonup \theta\rho_{Y_f} + (1-\theta)\rho_{Y_b} \quad \text{as } \epsilon \to 0.$$

Therefore, one can not eliminate the microscopic equation in the homogenized problem.

2.1.2 Asymptotic expansions

We apply the method of two-scale asymptotic expansions to (2.3). We start from the following *two-scale asymptotic expansion* (or *ansatz*)

$$\rho_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i \rho_i\left(t, x, \frac{x}{\epsilon}\right), \tag{2.6}$$

where each term $\rho_i(t,x,y)$ is a Y-periodic function. Plugging (2.6) into (2.3) yields a cascade of equations. One must be careful because there is a difference of order in ϵ in the matrix block Y_b and in the fracture set Y_f.

The ϵ^{-2} equation holds true only in Y_f

$$-\mathrm{div_y}\left(k_{Y_b}\nabla_y \rho_0(t,x,y)\right) = 0 \quad \text{in } Y_f,$$

which implies that ρ_0 is a function that does not depend on y in Y_f, i.e., there exists a function $\rho_{Y_f}(t,x)$ such that

$$\rho_0(t,x,y) \equiv \rho_{Y_f}(t,x) \quad \text{for any } y \in Y_f.$$

The ϵ^{-1} equation in Y_f is

$$-\mathrm{div_y}\left(k_{Y_b}\nabla_y \rho_1(t,x,y)\right) = \mathrm{div_y}\left(k_{Y_b}\nabla_x \rho_{Y_f}(t,x)\right) \quad \text{in } Y_f,$$

with a Neumann boundary condition on ∂Y_b, which allows one to compute ρ_1 in terms of $\nabla_x \rho_{Y_f}$.

Finally, the ϵ^0 equation is

$$\begin{array}{ll} \phi_{Y_b}\frac{\partial \rho_0}{\partial t} - \mathrm{div_y}\left(k_{Y_b}\nabla_y \rho_0\right) = f(t,x) & \text{in } Y_b \\ \phi_{Y_f}\frac{\partial \rho_0}{\partial t} - \mathrm{div_y}\left(k_{Y_f}\nabla_y \rho_2(x,y)\right) = g(t,x,y) & \text{in } Y_f, \end{array} \tag{2.7}$$

with

$$g(t,x,y) = \mathrm{div_y}\left(k_{Y_f}\nabla_x \rho_1\right) + \mathrm{div_x}\left(k_{Y_f}\nabla_y \rho_1\right) + \mathrm{div_x}\left(k_{Y_f}\nabla_x \rho_0\right) + f(x).$$

The first line of (2.7) is a parabolic equation for ρ_0 in the block Y_b with Dirichlet boundary condition on ∂Y_b. Writing $\rho_0 = \rho_{Y_b}$ in Y_b, it is precisely the fourth line of the homogenized problem (2.5). The second line of (2.7) is an elliptic equation for the unknown ρ_2 in the fracture Y_f with Neumann boundary conditions on ∂Y_b. The compatibility condition of this equation (in order that it admits a solution) is

$$\int_Y \left[g(t,x,y) - \phi_{Y_f}\frac{\partial \rho_0}{\partial t} \right] dy = 0,$$

which is exactly the first line of the homogenized problem (2.5).

2.2 Long time behavior of a diffusion equation

2.2.1 Setting and main results

In this section we discuss the homogenized limit of a time-evolution diffusion equation. This is a parabolic equation for which we are interested in the long time behavior. As we shall see, the homogenized limit is quite different in this setting from the usual one as described in the first lecture. For an initial data $a \in L^2(\Omega)$, the equation is

$$\begin{cases} c\left(\frac{x}{\epsilon}\right)\frac{\partial u_\epsilon}{\partial t} - \epsilon^2 \mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = \sigma\left(\frac{x}{\epsilon}\right)u_\epsilon & \text{in } \Omega \times \mathbb{R}^+ \\ u_\epsilon = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u_\epsilon(0) = a & \text{in } \Omega. \end{cases} \tag{2.8}$$

where A is a symmetric matrix satisfying the coercivity assumption

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2, \quad \text{with} \quad 0 < \alpha \leq \beta,$$

and c is a bounded positive Y-periodic function

$$0 < c^- \leq c(y) \leq c^+ < +\infty \quad \forall\, y \in Y.$$

Equation (2.8) models a reaction-diffusion problem (there is no sign restriction on the reaction coefficient σ). It is frequently used also in nuclear physics or neutronics [7], [8] (see also the lecture notes [6] where additional references and physical motivation are given).

Remark the ϵ^2 scaling in front of the diffusion term. One possible justification of this scaling is that, upon the change of variables $y = x/\epsilon$, the ϵ^2 factor disappears in front of the diffusion, the periodicity cell $(0,1)^N$ is independent of ϵ while the domain size is of order $1/\epsilon$. Another interpretation of the ϵ^2 scaling is concerned with the *long time behavior* of this reaction-diffusion equation. Indeed, if we change the time scale by the change of variable $\tau = \epsilon^2 t$, we can divide all terms of the equation by ϵ^2 and suppress this scaling (except in front of the reaction term). Clearly, if the new time variable τ is of order 1, then the original time variable t is of order ϵ^{-2}, i.e. we investigate the asymptotic of (2.8) for very long times.

In order to state the main convergence result for (2.8) we need to introduce an auxiliary problem which is a *cell spectral problem*. Let $\lambda \in \mathbb{R}$ and $w(y) \in L^2_\#(Y)$ be the first eigenvalue and the first eigenfunction of the cell spectral problem

$$\begin{cases} -\lambda c(y)w - \mathrm{div_y}\left(A(y)\nabla_y w\right) = \sigma(y)w & \text{in } Y, \\ y \to w(y) & Y - \text{periodic}. \end{cases} \tag{2.9}$$

The existence of (λ, w) is standard, and the Krein-Rutman theorem implies furthermore that the first eigenvalue is simple and the first eigenvector is the only one that can be chosen positive, $w(y) > 0$ in Y. Physically, the first eigencouple models the local equilibrium between the diffusion and reaction terms.

Theorem 2.2.1 *Let u_ϵ be the unique solution of (2.8). Define a new unknown*

$$v_\epsilon(\epsilon^2 t, x) = \frac{u_\epsilon(t,x)}{w\left(\frac{x}{\epsilon}\right)} e^{\lambda t}. \tag{2.10}$$

Then, for any time $T > 0$, $v_\epsilon(\tau, x)$ converges weakly in $L^2\left((0,T); H^1(\Omega)\right)$ to $u(\tau, x)$ which is the unique solution of the homogenized problem

$$\begin{cases} \overline{c}\dfrac{\partial u}{\partial \tau} - \operatorname{div}\left(\overline{A}\nabla u\right) = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = \overline{u}_0 & \text{in } \Omega, \end{cases} \tag{2.11}$$

with

$$\overline{c} = \int_Y c(y) w(y)^2 dy,$$

and $\overline{A}$ the homogenized diffusion tensor defined by its entries

$$\overline{A}_{ij} = \int_Y w^2 A\left(e_i + \nabla_y \chi_i\right) \cdot \left(e_j + \nabla_y \chi_j\right) dy,$$

where $\chi_i(y)$ are the unique solutions in $H^1_\#(Y)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\operatorname{div_y}\left(w^2 A\left(e_i + \nabla_y \chi_i(y)\right)\right) = 0 & \text{in } Y \\ y \to \chi_i(y) & Y\text{-periodic}, \end{cases}$$

with $(e_i)_{1\le i\le N}$ the canonical basis of $\mathbb{R}^N$.

In other words, Theorem 2.2.1 gives the following asymptotic expansion for the solution of (2.8)

$$u_\epsilon(t,x) \approx e^{-\lambda t} w\left(\frac{x}{\epsilon}\right) u\left(\epsilon^2 t, x\right).$$

Proof. Applying the change of unknown (2.10) and using the cell spectral equation (2.9), we find a simplified equation for v_ϵ

$$\begin{cases} (cw^2)\left(\dfrac{x}{\epsilon}\right)\dfrac{\partial v_\epsilon}{\partial \tau} - \operatorname{div}\left((w^2 A)\left(\dfrac{x}{\epsilon}\right)\nabla v_\epsilon\right) = 0 & \text{in } \Omega \times (0,T) \\ v_\epsilon = 0 & \text{on } \partial\Omega \times (0,T) \\ v_\epsilon(0) = v_0^\epsilon & \text{in } \Omega \end{cases} \tag{2.12}$$

with $v_0^\epsilon = a(x)/w(x/\epsilon)$. Remark that all scalings have disappear from (2.12) and we can therefore apply the standard homogenization theory to (2.12) (see subsection 1.2.4 in the first lecture). This easily yields (2.11). $\square$

Remark 2.2.2 *It is possible to consider a source term in the reaction-diffusion equation (2.8) of the form*

$$\epsilon^2 f(\epsilon^2 t, x) e^{-\lambda t}.$$

It yields a source term in the homogenized equation (2.11) of the type

$$\left(\int_Y w(y)\, dy \right) f(\tau, x).$$

2.2.2 Asymptotic expansions

In order to understand how we guessed the clever change of unknowns (2.10) in the previous subsection, we apply now the method of two-scale asymptotic expansions to the reaction-diffusion equation (2.8) (without any a priori knowledge of the result). Since there are two time scales, the fast one t and the slow one $\tau = \epsilon^2 t$, we start from the following *two-scale asymptotic expansion* (or *ansatz*)

$$u_\epsilon(t, x) = \sum_{i=0}^{+\infty} \epsilon^i u_i \left(t, \epsilon^2 t, x, \frac{x}{\epsilon} \right), \tag{2.13}$$

where each term $u_i(t, \tau, x, y)$ is a Y-periodic function. Remark however that there is no periodicity with respect to the time variables. This series is plugged into equation (2.8) and we get the usual cascade of equations. The ϵ^0 equation is

$$\begin{cases} c(y)\frac{\partial u_0}{\partial t} - \mathrm{div_y}\left(A(y)\nabla_y u_0\right) = \sigma(y) u_0 & \text{in } Y, \\ y \to u_0(t, \tau, x, y) & Y - \text{periodic.} \end{cases} \tag{2.14}$$

Since we are interested in the long time asymptotic of the problem, and because t is the fast time variable (compared to τ), we ignore the initial condition in (2.14) and we decide to retain only the behavior of u_0 for very large time t. (The initial condition will be applied to the Cauchy problem with respect to the slow time variable τ.) It is well-known that any solution of (2.14) has the same asymptotic profile, as time t goes to infinity, whatever the initial condition. This limit profile is precisely

$$C e^{-\lambda t} w(y),$$

where C is a constant depending on the initial condition, and (λ, w) is the first eigencouple of the underlying operator (2.9). Therefore, we deduce that, for large t,

$$u_0(t, \tau, x, y) = u(\tau, x) e^{-\lambda t} w(y).$$

The ϵ equation is

$$\begin{cases} c(y)\frac{\partial u_1}{\partial t} - \mathrm{div_y}\left(A(y)\nabla_y u_1\right) = \sigma(y) u_1 + g_1 & \text{in } Y, \\ y \to u_1(t, \tau, x, y) & Y - \text{periodic,} \end{cases} \tag{2.15}$$

with the source term

$$g_1 = \mathrm{div_x}\left(A(y)\nabla_y u_0\right) + \mathrm{div_y}\left(A(y)\nabla_x u_0\right).$$

If we want the series (2.13) to converge for large time t, its second term u_1 should not grow faster in t than its first term u_0. To avoid any resonance effect in (2.15), the right hand side g_1 must be orthogonal to the first eigenfunction $w(y)$, namely

$$\int_Y g_1(y)w(y)\,dy = 0. \tag{2.16}$$

This condition is always satisfied since

$$\int_Y g_1(y)w(y)\,dy = e^{-\lambda t}\int_Y \left(A(y)\nabla_y w(y)\cdot\nabla_x u(x)w(y) - A(y)w(y)\nabla_x u(x)\cdot\nabla_y w(y)\right)dy = 0$$

because A is a symmetric matrix. Thus, for large time t, the solution u_1 is

$$u_1(t,\tau,x,y) = e^{-\lambda t}\chi(\tau,x,y),$$

where χ solves

$$\begin{cases} -\lambda c(y)\chi - \mathrm{div_y}\left(A(y)\nabla_y\chi\right) = \sigma(y)\chi + g_1 & \text{in } Y, \\ y \to \chi(t,\tau,x,y) & Y-\text{periodic.}\end{cases}$$

The ϵ^2 equation is

$$\begin{cases} c(y)\frac{\partial u_2}{\partial t} - \mathrm{div_y}\left(A(y)\nabla_y u_2\right) = \sigma(y)u_2 + g_2 & \text{in } Y, \\ y \to u_2(t,\tau,x,y) & Y-\text{periodic,}\end{cases} \tag{2.17}$$

with the source term

$$g_2 = \mathrm{div_x}\left(A\nabla_y u_1\right) + \mathrm{div_y}\left(A\nabla_x u_1\right) - c\frac{\partial u_0}{\partial\tau} + \mathrm{div_x}\left(A\nabla_x u_0\right).$$

Once again, if we want the series (2.13) to converge, its third term u_2 should not grow faster in t than u_0. To avoid any resonance effect in (2.17), the right hand side g_2 must be orthogonal to the first eigenfunction $w(y)$, namely

$$\int_Y g_2(y)w(y)\,dy = 0. \tag{2.18}$$

After some algebra, the compatibility condition (2.18) yield a macroscopic equation

$$\overline{c}\frac{\partial u}{\partial\tau} - \mathrm{div_x}\left(\overline{A}\nabla_x u\right) = 0$$

to which we add the initial condition in order to recover the homogenized problem (2.11).

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