MASTER M2 NUMERICAL ANALYSIS AND P.D.E.s
UNIVERSITE PARIS 6 - ECOLE POLYTECHNIQUE
Course of G. Allaire, "Homogenization"
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The evaluation procedure will pay attention to the quality of the dissertation and most particularly to the clarity and readability in the proposed argumentation. As usual the subscript # denotes spaces of periodic functions. Throughout the problem C denotes a positive constant which does not depend on ϵ.

The goal of this problem is to study a model of mixed diffusive and radiative thermal transfer in a porous medium. Let Ω be a smooth bounded open set of $\mathbb{R}^N$ which represents a porous medium. The domain Ω is tiled by a square periodic tiling of size ϵ. The cubes of this tiling $(Y_p^\epsilon)_{1\le p\le n(\epsilon)}$, with $n(\epsilon) \approx |\Omega|\epsilon^{-N}$, are all equal, up to a translation, to $[0,\epsilon]^N$. Thus, after translation each cube is homothetic of ratio ϵ to the unit cell $Y = [0,1]^N$ which is decomposed in a solid part Y^* and a smooth, simply connected hole $T \Subset Y$, compactly included in Y, with boundary $\Gamma = \partial T$, with $Y = Y^* \cup T$. Using the same notation in each cube, $Y_p^\epsilon = Y_p^{*,\epsilon} \cup T_p^\epsilon$, we define the solid part Ω_ϵ of the porous medium Ω_ϵ by

$$\Omega_\epsilon = \Omega \setminus \left(\cup_{p=1}^{n(\epsilon)} T_p^\epsilon \right).$$

The holes T_p^ϵ being disconnected, the domain Ω_ϵ is connex. The interface Γ_ϵ between the holes and the solid part is defined by

$$\Gamma_\epsilon = \partial\Omega_\epsilon \setminus \partial\Omega = \cup_{p=1}^{n(\epsilon)} \partial T_p^\epsilon.$$

For simplicity we assume that no hole cut the outer boundary $\partial\Omega$. In the solid part Ω_ϵ, heat propagates by diffusion, while in the holes it propagates by radiation without absorption.

The diffusion tensor in the solid is $A\left(\frac{x}{\epsilon}\right)$ where $A(y) \in L^\infty_\#(Y_f)^{N\times N}$ is a periodic coercive symmetric matrix satisfying, for $0 < \alpha \le \beta$,

$$\alpha|\xi|^2 \le A(y)\xi\cdot\xi \le \beta|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^N, y \in Y_f.$$

The temperature in the porous medium is denoted by $u_\epsilon(x)$. We consider a (very) simplified model of radiative transfer such that any point x on the boundary ∂T_p^ϵ of one hole emits thermal radiations proportionally to its temperature and receives thermal radiations from all other points of the same hole's, equal to the average of the temperature on ∂T_p^ϵ. On each hole's boundary we write the continuity of the normal heat flux

$$-\epsilon A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot n = G_\epsilon(u_\epsilon) \text{ on } \partial T_p^\epsilon$$

where G_ϵ is an integral operator, defining the radiative transfer, given by

$$G_\epsilon(u_\epsilon)(x) = \sigma\left(u_\epsilon(x) - \frac{1}{|\partial T_p^\epsilon|}\int_{\partial T_p^\epsilon} u_\epsilon(x')\,ds(x') \right)$$

with a positive constant $\sigma > 0$, $ds(x')$ the surface measure (depending on the variable x'). The model is thus:

$$\begin{cases} -\mathrm{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega_\epsilon, \\ -A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon \cdot n = \dfrac{1}{\epsilon}G_\epsilon(u_\epsilon) & \text{on } \Gamma_\epsilon, \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where n is the exterior unit mormal to Ω_ϵ and $f(x) \in L^2(\Omega)$ is a source term. The scaling in the second line of (1) is natural since it ensures a perfect balance between diffusion and radiation, as we shall see.

Part I

In this part we study, at fixed ϵ, the properties of model (1).

1. Prove that the operator G_ϵ is linear continuous self-adjoint from $L^2(\Gamma_\epsilon)$ into $L^2(\Gamma_\epsilon)$ and that it is positive in the sense that

$$\int_{\Gamma_\epsilon} G_\epsilon(u)\, u\, ds \geq 0 \quad \forall\, u \in L^2(\Gamma_\epsilon).$$

2. Deduce the existence and uniqueness of a solution to (1) in the space $H^1(\Omega_\epsilon) \cap H^1_0(\Omega)$.
3. Show that the kernel of G_ϵ is made of functions in $L^2(\Gamma_\epsilon)$ which are constant on each component ∂T^ϵ_p.

Partie II

In this part the method of formal two-scale asymptotic expansions is applied in order to find the homogenized problem for (1). It is thus assumed that the solution u_ϵ can be written as a series

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i(x, \frac{x}{\epsilon})$$

with Y-periodic functions $y \to u_i(x, y)$.

1. Let $\phi(y) \in L^2(\Gamma)$ be extended by Y-periodicity and define $\phi^\epsilon(x) = \phi(\frac{x}{\epsilon}) \in L^2(\Gamma_\epsilon)$. Show that $G_\epsilon(\phi^\epsilon)(x) = [G(\phi)](y = \frac{x}{\epsilon})$ with the operator G defined on $L^2(\Gamma)$ by

$$G(\phi)(y) = \sigma\left(\phi(y) - \frac{1}{|\Gamma|}\int_\Gamma \phi(y')\, ds(y')\right).$$

Unfortunately, if $\phi(x,y)$ is a smooth, Y-periodic in y finction, for $\phi^\epsilon(x) = \phi(x, \frac{x}{\epsilon})$ we usually have

$$G_\epsilon(\phi^\epsilon)(x) \neq \Big[G(\phi(x,\cdot))\Big](y = \frac{x}{\epsilon}),$$

where the operator G is integral in y (but not in x). This difference is at the root of additional difficulties in the application of the method of two-scale asymptotic expansions...

2. For each cell Y_p^ϵ, x_ϵ^p denote its origin in such a way that the following change of variables holds true $x = x_\epsilon^p + \epsilon y$ with $x \in Y_p^\epsilon$ and $y \in Y$. Show that a smooth function $u_i(x,y)$ satisfies, for any $x \in Y_p^\epsilon$,

$$\begin{aligned} u_i(x, \frac{x}{\epsilon}) &= u_i(x_\epsilon^p, \frac{x}{\epsilon}) + (x - x_\epsilon^p)\cdot\nabla_x u_i(x_\epsilon^p, \frac{x}{\epsilon}) \\ &+ \frac{1}{2}(x - x_\epsilon^p)\otimes(x - x_\epsilon^p)\cdot\nabla_x\nabla_x u_i(x_\epsilon^p, \frac{x}{\epsilon}) + \mathcal{O}(\epsilon^3). \end{aligned} \tag{2}$$

3. Prove that, on each boundary ∂T_p^ϵ,

$$G_\epsilon(u_\epsilon)(x) = Q_0(x_\epsilon^p, \frac{x}{\epsilon}) + \epsilon Q_1(x_\epsilon^p, \frac{x}{\epsilon}) + \epsilon^2 Q_2(x_\epsilon^p, \frac{x}{\epsilon}) + \mathcal{O}(\epsilon^3)$$

with

$$Q_0(x_\epsilon^p, y) = G\Big(u_0(x_\epsilon^p, y)\Big),$$

$$Q_1(x_\epsilon^p, y) = G\Big(u_1(x_\epsilon^p, y) + y\cdot\nabla_x u_0(x_\epsilon^p, y)\Big),$$

$$Q_2(x_\epsilon^p, y) = G\Big(u_2(x_\epsilon^p, y) + y\cdot\nabla_x u_1(x_\epsilon^p, y) + \frac{1}{2} y\otimes y\cdot\nabla_x\nabla_x u_0(x_\epsilon^p, y)\Big),$$

where the operator G is integral in y only. Show also that, for any $x \in \partial T_p^\epsilon$,

$$Q_i(x_\epsilon^p, y) = Q_i(x,y) - (x - x_\epsilon^p)\cdot\nabla_x Q_i(x,y) + \mathcal{O}(\epsilon^2). \tag{3}$$

4. Plugging the ansatz in (1) and using both (2) and (3), deduce the equations and boundary conditions satisfied by u_0, u_1 and u_2. Hint: these equations must involve only the x and y variables ; the points x_ϵ^p should have been eliminated.

5. Let $g(y) \in L^2_\#(Y^*)$ and $h(y) \in L^2(\Gamma)$. Prove that the following problem admits a unique solution in $H^1_\#(Y^*)/\mathbb{R}$

$$\begin{cases} -\mathrm{div}_y\,(A(y)\nabla_y w) = g & \text{in } Y^* \\ -A(y)\nabla_y w\cdot n = G(w) - h & \text{on } \Gamma \\ y \to w(y)\ Y\text{-periodic} \end{cases} \tag{4}$$

if and only if the data satisfy

$$\int_{Y^*} g(y)dy + \int_\Gamma h(y)ds = 0.$$

One should rely on the self-adjointness of G and on its explicit kernel.

6. Deduce that $u_0(x,y)$ does not depend on y, and that $u_1(x,y)$ can be written in terms of the gradient of u_0 and of solutions $w_k(y)$ of cell problems which must be given explicitly.

7. Write the necessary and sufficient condition of existence of $u_2(x,y)$. Deduce from it that the homogenized equation is a diffusion equation in Ω with a constant homogenized tensor A^* which must be given explicitly.

8. Using the variational formulation of the cell problem, prove that A^* is positive definite.

Partie III

In this part we use two-scale convergence to rigorously prove a homogenization theorem. We recall the following Poincaré inequality (uniform in ϵ): there exists a constant $C > 0$ such that

$$\|\phi\|_{L^2(\Omega_\epsilon)} \le C\|\nabla\phi\|_{L^2(\Omega_\epsilon)^N} \quad \forall\, \phi \in H^1(\Omega_\epsilon) \cap H^1_0(\Omega).$$

1. Prove that there exists $C > 0$ such that the solution u_ϵ of (1) satisfies

$$\|u_\epsilon\|_{L^2(\Omega_\epsilon)} + \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)^N} + \sqrt{\epsilon}\|u_\epsilon\|_{L^2(\Gamma_\epsilon)} \le C\|f\|_{L^2(\Omega)}. \tag{5}$$

2. Recall the results from the course on the structure of the two-scale limit of a sequence u_ϵ satisfying the a priori estimate (5).

3. Let $\phi_1(x,y)$ be a smooth Y-periodic test function with compact support in $x \in \Omega$. Prove that there exists at least one vector-valued function $\theta(x,y)$ (with the same compact support in x) which is a solution of

$$\begin{cases} -\mathrm{div}_y\theta(x,y) = 0 & \text{in } Y^*, \\ \theta(x,y)\cdot n = G(\phi_1)(x,y) & \text{on } \Gamma, \\ y \ \to\ \theta(x,y) & Y\text{-periodic.} \end{cases}$$

4. Multiplying (1) by a test function $\epsilon\phi_1(x,\frac{x}{\epsilon})$, find the cell problem. One shall first use the Taylor expansion (2) for ϕ_1 before applying to it the G_ϵ operator and then the previous question for $x = x^p_\epsilon$.

5. Multiplying (1) by a test function of the type $\phi(x) + \epsilon\phi_1(x,\frac{x}{\epsilon})$, with a smooth compactly supported in Ω function ϕ and with ϕ_1 explicitly given by

$$\phi_1(x,y) = \sum_{k=1}^{N} \frac{\partial\phi}{\partial x_k}(x)w_k(y),$$

   where the w_k's are the solutions of the cell problems, find the homogenized problem. One shall use again the Taylor expansion (2) for ϕ, ϕ_1 before applying the operator G_ϵ, as well as the cell equation defining $w_k(y)$.

   Prove that the homogenized problem has a unique solution and deduce the convergence of the entire sequence u_ϵ.