INTRODUCTION TO PERIODIC HOMOGENIZATION THEORY

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☞ First lecture: Two-scale asymptotic expansions.

☞ Second lecture: Two-scale convergence.

☞ Third lecture: Further generalizations.

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Content of the first lecture

1. Definition of periodic homogenization
2. Two-scale asymptotic expansions
3. Darcy's law in porous media
4. Linear Boltzman equation

-I- DEFINITION OF HOMOGENIZATION

☞ Rigorous version of averaging in p.d.e.'s

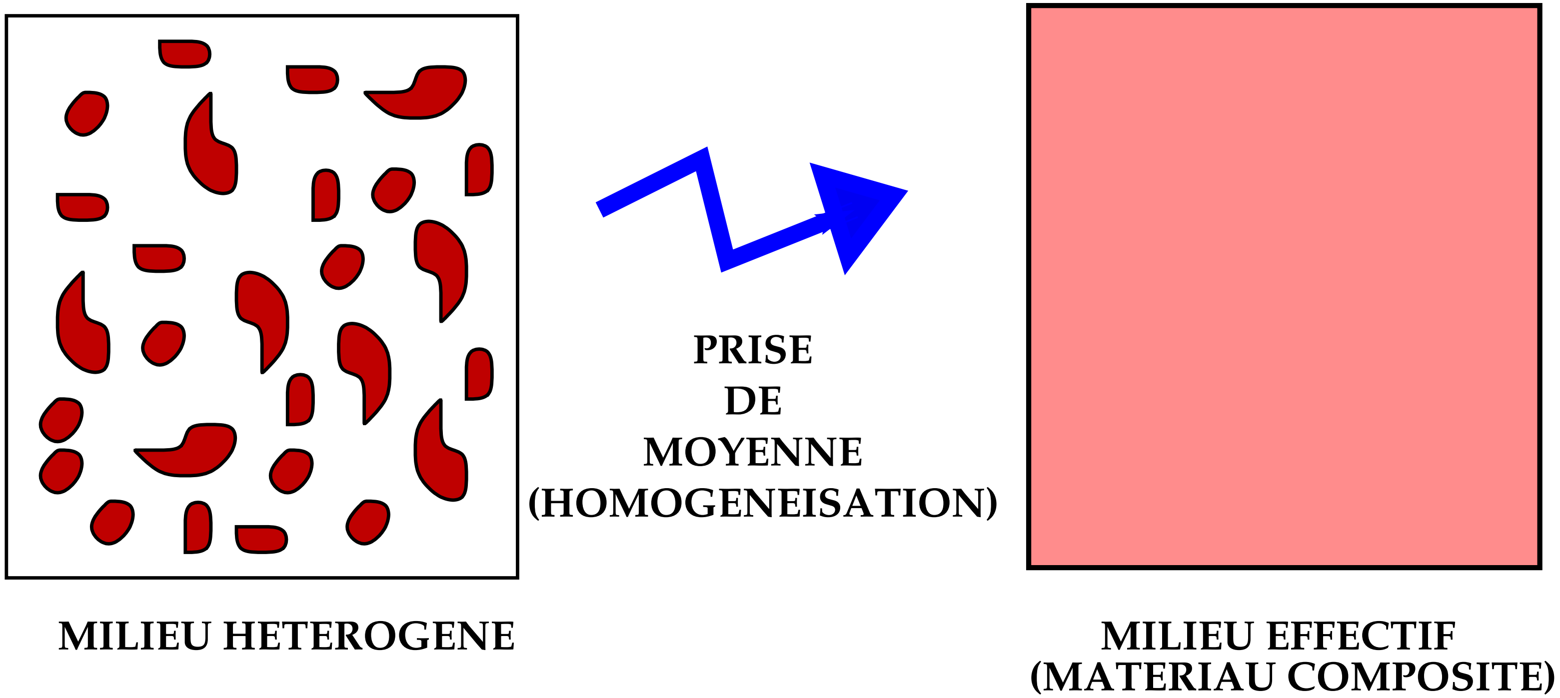
☞ Process of asymptotic analysis

☞ Extract effective or homogenized parameters for heterogeneous media

☞ Derive simpler macroscopic models from complicated microscopic models

☞ Different methods :

- two-scale asymptotic expansions for periodic media
- H- or G-convergence for general media
- stochastic homogenization
- variational methods (Γ-convergence)



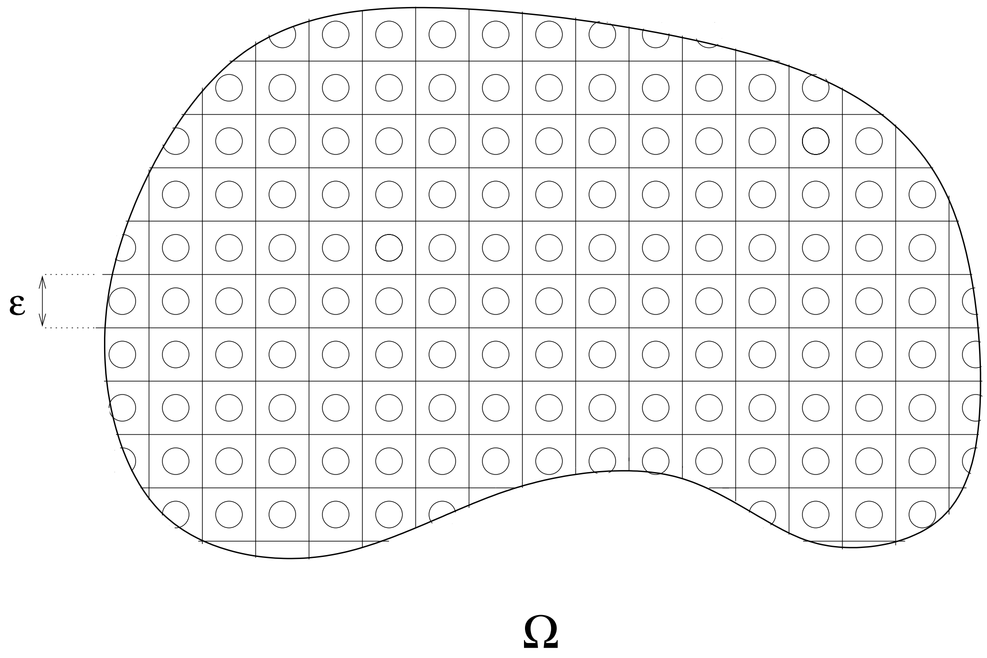
Motivation: composite materials, porous media, nuclear reactor physics, photonic crystals...

PERIODIC HOMOGENIZATION

Periodic domain $\Omega \in \mathbb{R}^N$ with period ϵ. Rescaled unit cell $Y = (0,1)^N$.

$$x \in \Omega, \quad y = \frac{x}{\epsilon} \in Y$$

Example: Composite material with a periodic structure



MODEL PROBLEM

Conductivity or diffusion equation

$$\begin{cases} -\text{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

with a coefficient tensor $A(y)$ which is Y-periodic, uniformly coercive and bounded

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N, \forall\, y \in Y \quad (\beta \geq \alpha > 0).$$

HOMOGENIZATION AND ASYMPTOTIC ANALYSIS

- Direct solution too costly if ϵ is small
- Averaging: replace $A(y)$ by effective homogeneous coefficients
- Asymptotic analysis: limit as $\epsilon \rightarrow 0$
  yields a rigorous definition of the homogenized parameters
- Error estimates: compare exact and homogenized solutions
- Similar to Representative Volume Element method
- Huge literature

Representative Volume Element method

Mesoscale $\epsilon << h << 1$. A Representative Volume Element is a cube of size h. We average all quantities in this cube:

☞ u is the average of the field u_ϵ

☞ ξ is the average of the gradient ∇u_ϵ

☞ σ is the average of the flux $A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon$

☞ e is the average of the energy density $A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla u_\epsilon$

Definition of the homogenized tensor A^*:

$$\sigma = A^*\xi, \quad e = A^*\xi \cdot \xi, \quad \xi = \nabla u.$$

Questions: is it possible to find such a tensor A^* ? Does it depend on ϵ, h, f, u, the boundary conditions ? How to compute it ?

Asymptotic analysis

Rather than considering a single heterogeneous medium with a fixed lengthscale ϵ_0, the problem is embedded in a sequence of similar problems parametrized by a lengthscale ϵ.

Homogenization amounts to perform an asymptotic analysis when $\epsilon \to 0$

$$\lim_{\epsilon \to 0} u_\epsilon = u.$$

The limit u is the solution of an homogenized problem, the conductivity tensor of which is called the effective or homogenized conductivity.

This yields a coherent definition of homogenized properties which can be rigorously justified by quantifying the resulting error estimate.

-II- TWO-SCALE ASYMPTOTIC EXPANSIONS

Ansatz for the solution

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right),$$

with $u_i(x, y)$ function of both variables x and y, periodic in y

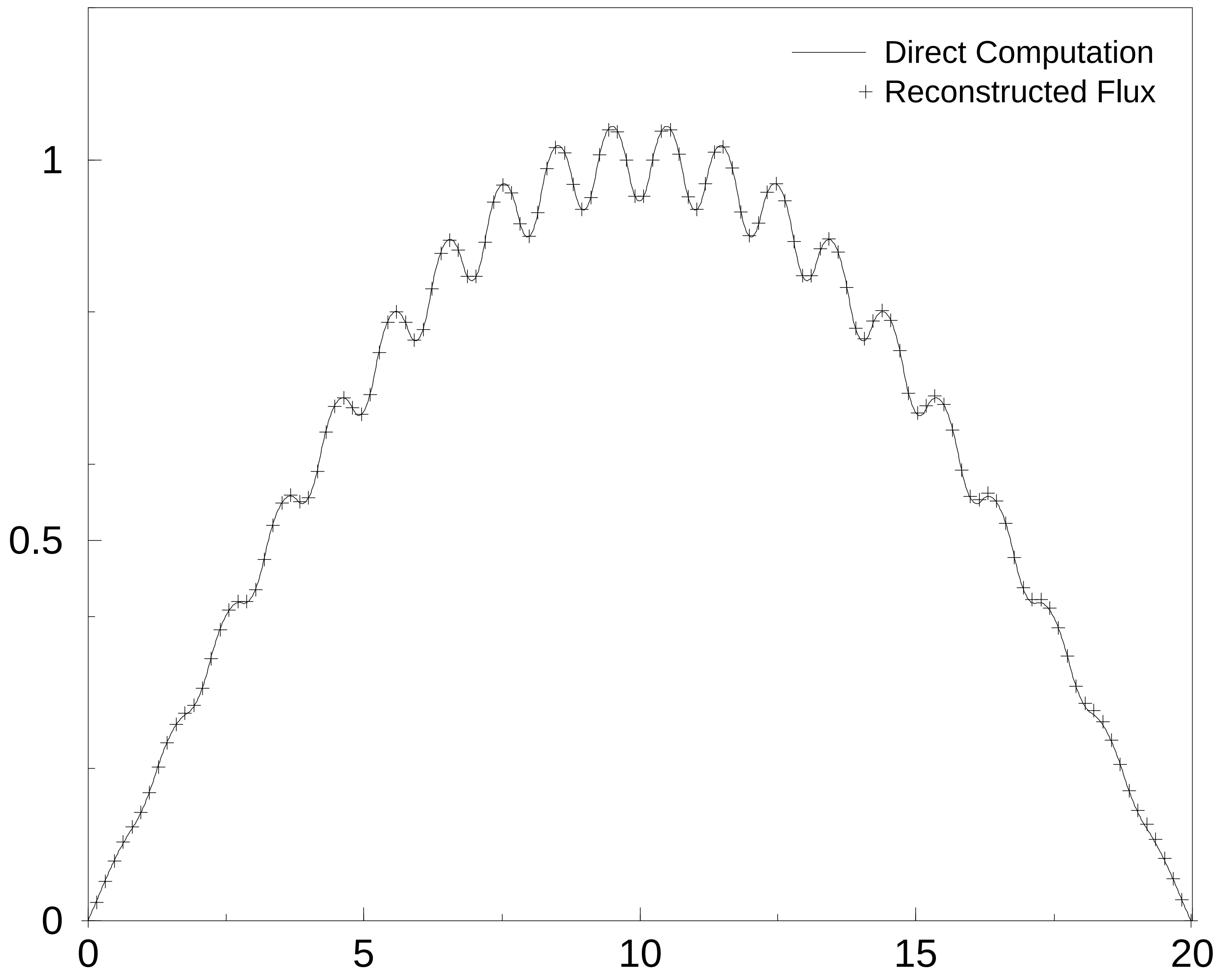
This is a **postulate !** Boundary layer terms are missing...

Derivation rule

$$\nabla\left(u_i\left(x, \frac{x}{\epsilon}\right)\right) = \left(\epsilon^{-1}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right)$$

$$\nabla u_\epsilon(x) = \epsilon^{-1}\nabla_y u_0\left(x, \frac{x}{\epsilon}\right) + \sum_{i=0}^{+\infty} \epsilon^i \left(\nabla_y u_{i+1} + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right)$$

Typical behavior of the function $x \to u_i\left(x, \frac{x}{\epsilon}\right)$



CASCADE OF EQUATIONS

$$-\epsilon^{-2}\left[\mathrm{div_y} A\nabla_y u_0\right]\left(x,\frac{x}{\epsilon}\right)$$

$$-\epsilon^{-1}\left[\mathrm{div_y} A(\nabla_x u_0+\nabla_y u_1)+\mathrm{div_x} A\nabla_y u_0\right]\left(x,\frac{x}{\epsilon}\right)$$

$$-\epsilon^{0}\left[\mathrm{div_x} A(\nabla_x u_0+\nabla_y u_1)+\mathrm{div_y} A(\nabla_x u_1+\nabla_y u_2)\right]\left(x,\frac{x}{\epsilon}\right)$$

$$-\sum_{i=1}^{+\infty}\epsilon^{i}\left[\mathrm{div_x} A(\nabla_x u_i+\nabla_y u_{i+1})+\mathrm{div_y} A(\nabla_x u_{i+1}+\nabla_y u_{i+2})\right]\left(x,\frac{x}{\epsilon}\right)$$

$$= f(x).$$

☞ We identify each power of ϵ.

☞ Notice that $\phi\left(x,\frac{x}{\epsilon},v\right)=0\ \forall x,\epsilon \quad\Leftrightarrow\quad \phi(x,y,v)\equiv 0\ \forall x,y.$

☞ Only the three first terms of the series really matter.

ϵ^{-2} equation

$$-\mathrm{div_y}\left(A(y)\nabla_y u_0(x,y)\right) = 0 \quad \text{in} \quad Y$$

where x is just a parameter.

Its unique solution does not depend on y

$$u_0(x,y) \equiv u(x)$$

Technical lemma on cell problems

Definition.

$$L^2_\#(Y) = \left\{ \phi(y) \ Y\text{-periodic, such that } \int_Y \phi(y)^2 dy < +\infty \right\}$$

$$H^1_\#(Y) = \{ \phi \in L^2_\#(Y) \text{ such that } \nabla \phi \in L^2_\#(Y)^N \}$$

Lemma. Let $f(y) \in L^2_\#(Y)$ be a periodic function. There exists a solution in $H^1_\#(Y)$ (unique up to an additive constant) of

$$\begin{cases} -\operatorname{div}\left(A(y)\nabla w(y)\right) = f & \text{in } Y \\ y \to w(y) & Y\text{-periodic,} \end{cases}$$

if and only if $\int_Y f(y)dy = 0$ (this is called the Fredholm alternative).

ϵ^{-1} equation

$$-\mathrm{div_y} A(y)\nabla_y u_1(x,y) = \mathrm{div_y} A(y)\nabla_x u(x) \quad \text{in} \quad Y$$

which is an equation for u_1. Introducing the cell problem

$$\begin{cases} -\mathrm{div_y} A(y)\left(e_i + \nabla_y w_i(y)\right) = 0 & \text{in } Y \\ y \to w_i(y) & Y\text{-periodic,} \end{cases}$$

by linearity we compute

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y).$$

ϵ^0 equation

$$-\mathrm{div_y} A(y)\nabla_y u_2(x,y) = \mathrm{div_y} A(y)\nabla_x u_1 + \mathrm{div_x} A(y)\left(\nabla_y u_1 + \nabla_x u\right) + f(x)$$

which is an equation for u_2. Its compatibility condition (Fredholm alternative) is

$$\int_Y \left(\mathrm{div_y} A(y)\nabla_x u_1 + \mathrm{div_x} A(y)\left(\nabla_y u_1 + \nabla_x u\right) + f(x)\right) dy = 0.$$

Replacing u_1 by its value yields the homogenized equation

$$\begin{cases} -\mathrm{div_x} A^* \nabla_x u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the constant homogenized tensor

$$A^*_{ij} = \int_Y \left[(A(y)\nabla_y w_i) \cdot e_j + A_{ij}(y) \right] dy = \int_Y A(y)\left(e_i + \nabla_y w_i\right) \cdot \left(e_j + \nabla w_j\right) dy.$$

COMMENTS

- Explicit formula for the effective parameters (no longer true for non-periodic problems).
- A^* does not depend on ϵ, f, u or the boundary conditions (still true in the non-periodic case).
- A^* is positive definite (not necessarily isotropic even if $A(y)$ was so).
- One can check that

$$\lim_{\epsilon\to 0} u_\epsilon = u, \quad \lim_{\epsilon\to 0} \nabla u_\epsilon = \nabla u, \quad \lim_{\epsilon\to 0} A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon = A^*\nabla u,$$

$$\lim_{\epsilon\to 0} A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\cdot\nabla u_\epsilon = A^*\nabla u\cdot\nabla u.$$

- Same results for evolution problems.
- Very general method, but heuristic and not rigorous.

Variational characterization of the homogenized coefficients

Equivalent formula for A^* with $\xi \in \mathbb{R}^N$

$$A^*\xi \cdot \xi = \int_Y A(y)\,(\xi + \nabla_y w_\xi) \cdot (\xi + \nabla_y w_\xi)\,dy,$$

where w_ξ is the solution of

$$\begin{cases} -\mathrm{div_y} A(y)\,(\xi + \nabla_y w_\xi(y)) = 0 & \text{in } Y, \\ y \to w_\xi(y) & Y\text{-periodic.} \end{cases}$$

If the tensor $A(y)$ is symmetric, this is the Euler-Lagrange equation of the following variational principle

$$A^*\xi \cdot \xi = \min_{w(y)\in H^1_\#(Y)} \int_Y A(y)\,(\xi + \nabla_y w) \cdot (\xi + \nabla_y w)\,dy.$$

Bounds on the homogenized coefficients

Taking $w(y) = 0$ in the variational principle yields the so-called arithmetic mean upper bound

$$A^* \xi \cdot \xi \leq \left(\int_Y A(y) dy \right) \xi \cdot \xi.$$

Replacing any gradient $\nabla_y w(y)$ (which has zero-average over Y) by any zero-average vector field yields the so-called harmonic mean lower bound

$$A^* \xi \cdot \xi \geq \left(\int_Y A^{-1}(y) dy \right)^{-1} \xi \cdot \xi = \min_{\substack{\zeta(y) \in L^2_\#(Y)^N \\ \int_Y \zeta(y) dy = 0}} \int_Y A(y) \left(\xi + \zeta(y)\right) \cdot \left(\xi + \zeta(y)\right) dy.$$

In general, these bounds are strict inequalities.

-III- DARCY'S LAW IN POROUS MEDIA

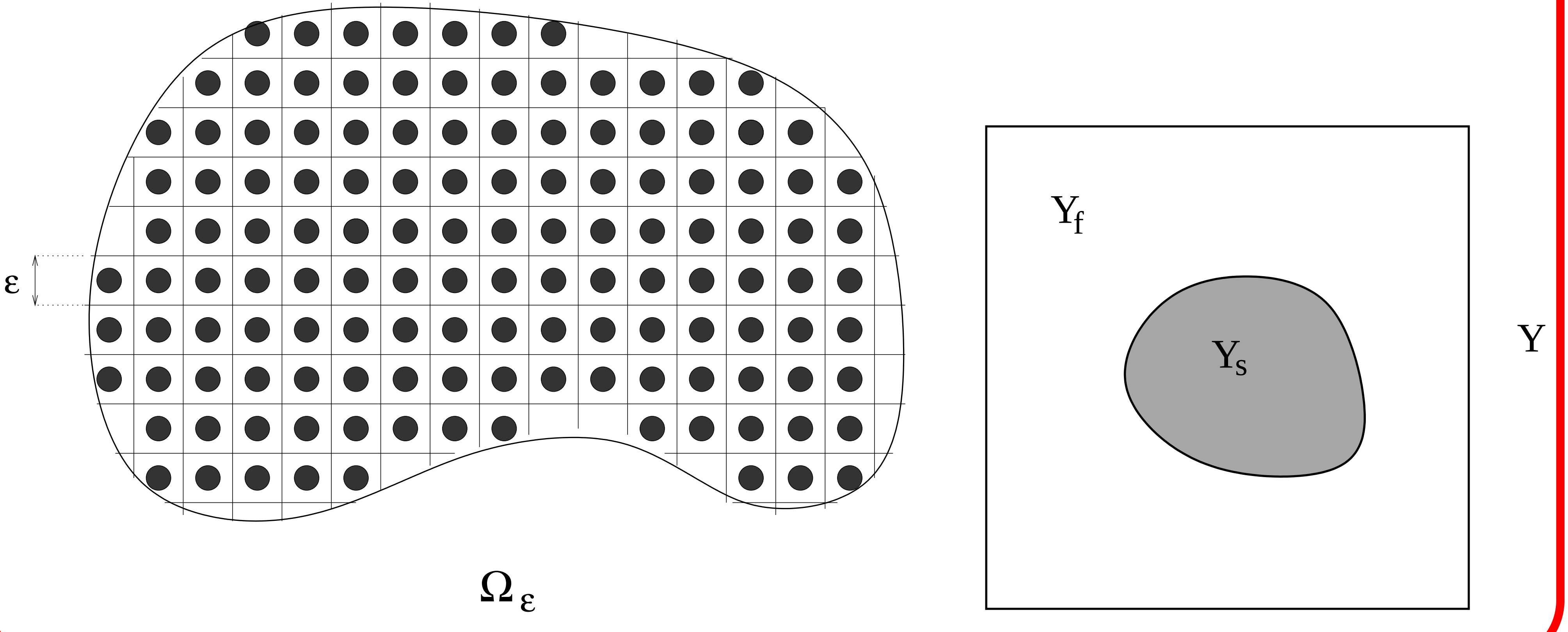
The goal of this section (and the next one) is to show that homogenization is a modelling tool for deriving new macroscopic models.

As an example we consider a viscous fluid flowing in a porous media and show that it obeays Darcy's law.

PERIODIC POROUS MEDIUM

Periodic domain with period ϵ: Ω_ϵ is the fluid part of the porous medium.
Rescaled unit cell $Y = (0,1)^N = Y_f \cup Y_s$ (fluid and solid parts, respectively).

$$x \in \Omega_\epsilon \quad \Leftrightarrow \quad y = \frac{x}{\epsilon} \in Y_f$$



MICROSCOPIC MODEL

Stokes equations (incompressible viscous fluid)

$$\begin{cases} \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

which admits a unique solution

$$u_\epsilon \in H_0^1(\Omega_\epsilon)^N, \; p_\epsilon \in L^2(\Omega_\epsilon)/\mathbb{R},$$

the pressure being uniquely defined up to an additive constant. (The space of the solution is changing with ϵ.)

MACROSCOPIC MODEL

Darcy's law (slow filtration of a fluid in a porous medium)

$$\begin{cases} u(x) = \frac{1}{\mu} A\left(f(x) - \nabla p(x)\right) & \text{in } \Omega \\ \mathrm{div} u(x) = 0 & \text{in } \Omega \\ u(x) \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits a unique solution $(u, p) \in L^2(\Omega)^N \times H^1(\Omega)/\mathbb{R}$. The velocity can be eliminated from Darcy's law (second-order elliptic equation for the pressure).

A is called the permeability tensor.

CONVERGENCE RESULT

Theorem. An extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ to the whole of Ω of the solution (u_ϵ, p_ϵ) of Stokes equations converges to the unique solution (u, p) of the homogenized Darcy's law. The permeability tensor is defined by

$$A_{ij} = \int_{Y_f} \nabla w_i(y) \cdot \nabla w_j(y) dy$$

where $w_i(y)$ is the unique solution of the cell Stokes problem

$$\left\{ \begin{array}{ll} \nabla q_i - \Delta w_i = e_i & \text{in } Y_f \\ \text{div} w_i = 0 & \text{in } Y_f \\ w_i = 0 & \text{in } Y_s \\ y \to q_i, w_i & Y\text{-periodic.} \end{array} \right.$$

Precise convergence

$$p_\epsilon \;\rightarrow\; p \quad \text{strongly in } L^2(\Omega)$$

$$\left(\tilde{u}_\epsilon(x) - \sum_{i=1}^{N} w_i(\frac{x}{\epsilon})u_i(x)\right) \rightarrow 0 \quad \text{strongly in } L^2(\Omega)^N$$

where $(w_i)_{1\leq i\leq N}$ are the cell velocities and $(u_i)_{1\leq i\leq N}$ the components of u.

Two-scale asymptotic expansions

Ansatz

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right), \quad p_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i p_i\left(x, \frac{x}{\epsilon}\right),$$

where each term $u_i(x, y)$ or $p_i(x, y)$ is a function of both variables x and y, Y-periodic in y.

The cascade of equations is

$$\left\{ \begin{array}{l} \epsilon^{-1} \nabla_y p_0 \left(x, \frac{x}{\epsilon}\right) + \epsilon^0 \left[\nabla_x p_0 + \nabla_y p_1 - \mu \Delta_{yy} u_0\right] \left(x, \frac{x}{\epsilon}\right) + \mathcal{O}(\epsilon) = f(x) \\ \\ \epsilon^{-1} \mathrm{div_y} u_0 \left(x, \frac{x}{\epsilon}\right) + \epsilon^0 \left[\mathrm{div_x} u_0 + \mathrm{div_y} u_1\right] \left(x, \frac{x}{\epsilon}\right) + \mathcal{O}(\epsilon) = 0. \end{array} \right.$$

ϵ^{-1} equation for the pressure

$$\nabla_y p_0(x,y) = 0 \quad \text{in} \quad Y,$$

from which we deduce that

$$p_0(x,y) \equiv p(x).$$

ϵ^{-1} equation for the incompressibility condition and the ϵ^0 equation from the momentum equation

$$\begin{cases} \nabla_y p_1 - \mu \Delta_{yy} u_0 = f(x) - \nabla_x p(x) \quad \text{in} \quad Y_f \\ \\ \mathrm{div_y} u_0 = 0 \quad \text{in} \quad Y_f \end{cases}$$

which is a Stokes equation for the velocity u_0 and pressure p_1 in the periodic unit cell Y. By linearity we find

$$u_0(x,y) = \frac{1}{\mu} \sum_{i=1}^{N} w_i(y) \left(f_i - \frac{\partial p}{\partial x_i} \right)(x), \quad p_1(x,y) = \sum_{i=1}^{N} q_i(y) \left(f_i - \frac{\partial p}{\partial x_i} \right)(x),$$

where w_i is the cell velocity and q_i is the cell pressure, solutions of the cell Stokes problem.

ϵ^0 equation for the incompressibility condition

$$\mathrm{div}_{\mathrm{x}} u_0(x,y) + \mathrm{div}_{\mathrm{y}} u_1(x,y) = 0 \quad \text{in} \quad Y_f$$

We average this equation in the unit cell Y and apply Stokes theorem

$$\int_{Y_f} \mathrm{div}_{\mathrm{y}} u_1(x,y)\,dy = \int_{\partial Y} u_1 \cdot n\,ds + \int_{\partial Y_s} u_1 \cdot n\,ds = 0$$

because of the periodicity and the no-slip condition on the solid part Y_s. With $u(x) \equiv \int_Y u_0(x,y)\,dy$ this implies that

$$\mathrm{div}_{\mathrm{x}} u(x) = \int_Y \mathrm{div}_{\mathrm{x}} \left[\sum_{i=1}^{N} w_i(y) \left(f_i - \frac{\partial p}{\partial x_i} \right)(x) \right] dy = 0,$$

which simplifies to

$$-\mathrm{div}_{\mathrm{x}} A \left(\nabla_x p(x) - f(x) \right) = 0 \qquad \text{in } \Omega,$$

which is a second-order elliptic equation for the pressure p.

Darcy's law with memory

Microscopic problem: unsteady Stokes equations

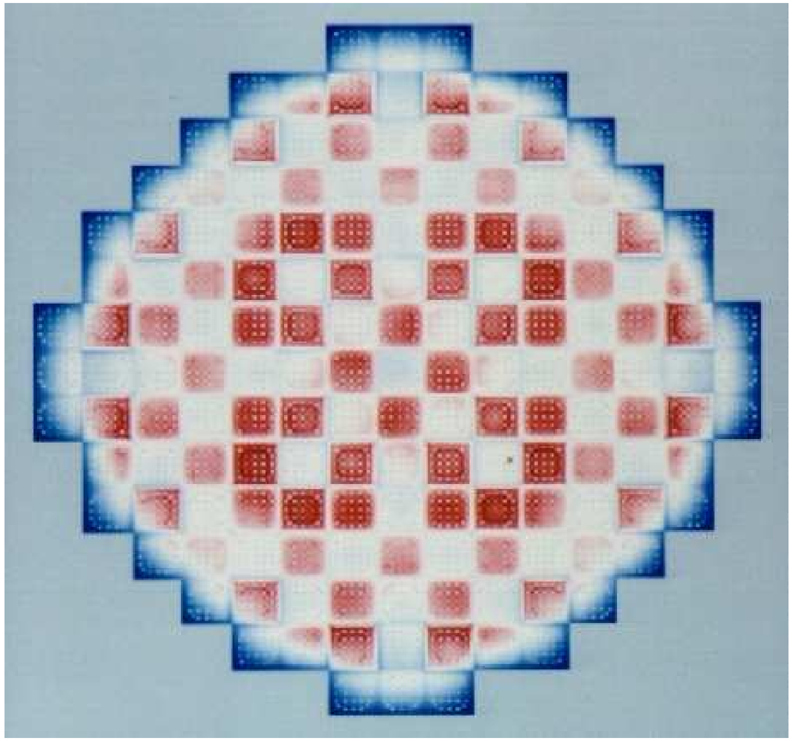
$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } (0,T) \times \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } (0,T) \times \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } (0,T) \times \partial\Omega_\epsilon \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \Omega_\epsilon \text{ at time } t = 0. \end{cases}$$

Macroscopic problem: Darcy's law with memory

$$\begin{cases} u(t,x) = v(t,x) + \frac{1}{\mu} \int_0^t A(t-s) \left(f - \nabla p \right)(s,x) ds & \text{in } (0,T) \times \Omega \\ \text{div} u(t,x) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) \cdot n = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

with an unsteady Stokes cell problem.

-IV- Linear Boltzman equation



Motivation: neutron distribution in a nuclear reactor.

Phase space $\Omega \times V$: space variable $x \in \Omega \subset \mathbb{R}^N$, velocity variable $v \in V$ (typically $V = \mathbf{S}_{N-1}$).

Unknown = density of neutrons $u_\epsilon(x, v)$.

Modèle

Section efficace variable: $\sigma(y)$ fonction Y-périodique, avec $Y = (0,1)^N$.

$$\sigma(y + e_i) = \sigma(y) \quad \forall e_i \; i\text{-ème vecteur de la base canonique.}$$

On remplace y par $\frac{x}{\epsilon}$:

$$x \to \sigma\left(\frac{x}{\epsilon}\right) \text{ périodique de période } \epsilon \text{ dans toutes les directions.}$$

Même définition pour $\tilde{\sigma}(x, \frac{x}{\epsilon})$. On considère

$$\begin{cases} \epsilon^{-1} v \cdot \nabla u_\epsilon + \epsilon^{-2} \sigma(\frac{x}{\epsilon}) \left(u_\epsilon - \int_V u_\epsilon \, dv \right) + \tilde{\sigma}(x, \frac{x}{\epsilon}) u_\epsilon = S(x, \frac{x}{\epsilon}, v) & \text{dans } \Omega \times V \\ u_\epsilon(x,v) = 0 & \text{sur } \Gamma^- \end{cases}$$

Nous faisons l'hypothèse de sous-criticité

$$\tilde{\sigma}(x,y) \geq 0 \text{ pour } (x,y) \in \Omega \times Y.$$

Remarques

- La mise à l'échelle choisie (scaling) provient d'une hypothèse de libre parcours moyen des particules de l'ordre de grandeur de la période. Elle permet d'obtenir une limite de diffusion.
- Domaine convexe borné régulier Ω.
- Bord rentrant $\Gamma^- = \{x \in \partial\Omega, v \in V, v \cdot n(x) < 0\}$.
- Pour simplifier on suppose que $V = \mathbf{S}_{N-1}$, la sphère unité, et que la mesure dv est telle que
$$\int_V dv = 1\,.$$
- Un calcul direct de u_ϵ peut être très cher (car il faut un maillage de taille $h < \epsilon$), donc on cherche les valeurs moyennes de u_ϵ.

Anstaz (série formelle)

On suppose que la solution est sous la forme

$$u_\epsilon(x, v) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}, v\right),$$

où chaque terme $u_i(x, y, v)$ est une fonction de trois variables $x \in \Omega$, $y \in Y = (0, 1)^N$ et $v \in V$, qui est périodique en y de période Y.

C'est un postulat !

On peut justifier les 2 premiers termes seulement...

(Il manque des termes de couches limites.)

Règle de dérivation

On injecte cette série dans l'équation et on utilise la règle

$$\nabla\left(u_i\left(x,\frac{x}{\epsilon},v\right)\right)=\left(\epsilon^{-1}\nabla_y u_i+\nabla_x u_i\right)\left(x,\frac{x}{\epsilon},v\right).$$

On a donc

$$\nabla u_\epsilon(x,v)=\epsilon^{-1}\nabla_y u_0\left(x,\frac{x}{\epsilon},v\right)+\sum_{i=0}^{+\infty}\epsilon^i\left(\nabla_y u_{i+1}+\nabla_x u_i\right)\left(x,\frac{x}{\epsilon},v\right).$$

L'équation devient une série en ϵ

$$\epsilon^{-2}\left[v\cdot\nabla_y u_0+\sigma(y)\left(u_0-\int_V u_0\,dv\right)\right]\left(x,\frac{x}{\epsilon},v\right)$$

$$+\epsilon^{-1}\left[v\cdot\nabla_y u_1+v\cdot\nabla_x u_0+\sigma(y)\left(u_1-\int_V u_1\,dv\right)\right]\left(x,\frac{x}{\epsilon},v\right)$$

$$+\sum_{i=0}^{+\infty}\epsilon^i\left[v\cdot\nabla_y u_{i+2}+v\cdot\nabla_x u_{i+1}+\sigma(y)\left(u_{i+2}-\int_V u_{i+2}\,dv\right)\right.$$
$$\left.+\tilde\sigma(x,y)u_i\right]\left(x,\frac{x}{\epsilon},v\right)$$

$$=S\left(x,\tfrac{x}{\epsilon},v\right).$$

☞ On identifie chaque puissance de ϵ.

☞ On remarque que $\phi\left(x,\frac{x}{\epsilon},v\right)=0\ \forall x,\epsilon \quad\Leftrightarrow\quad \phi(x,y,v)\equiv 0\ \forall x,y.$

☞ Seuls les 3 premiers termes de la série seront importants.

On commence par un lemme technique.

Alternative de Fredholm

Lemme. Soit $g \in L^2(Y \times V)$. Le problème aux limites

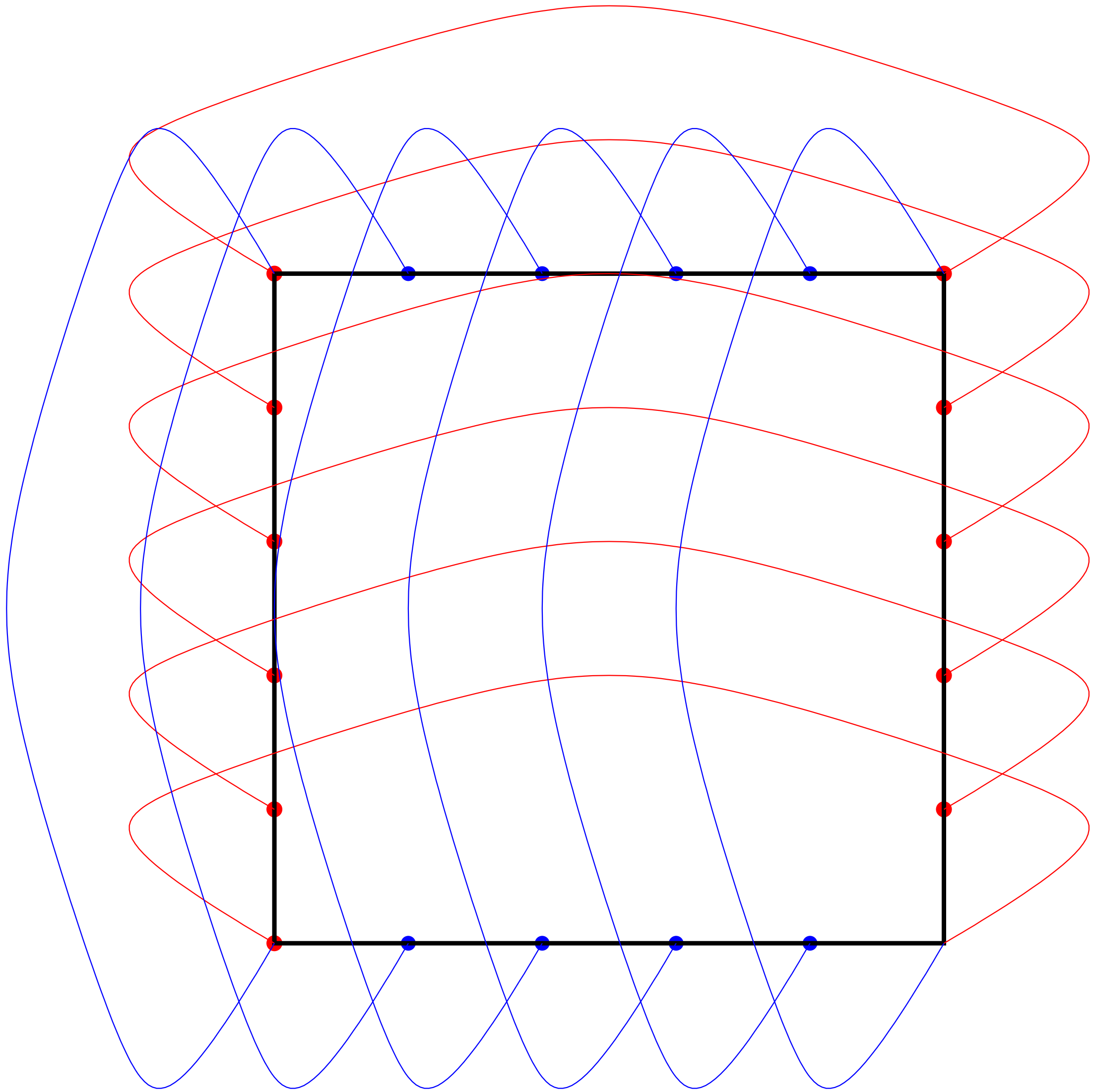
$$\begin{cases} v \cdot \nabla_y \phi + \sigma(y) \left(\phi - \int_V \phi \, dv \right) = g(y, v) & \text{dans } Y \times V \\ y \rightarrow \phi(y, v) \; Y\text{-périodique} \end{cases}$$

admet une unique solution $\phi \in L^2(Y \times V)/\mathbb{R}$ (à une constante additive près) si et seulement si

$$\int_V \int_Y g(y, v) \, dy \, dv = 0.$$

Preuve. Clairement la solution ϕ est définie à l'addition d'une constante près puisque $\int_V dv = 1$.

Condition aux limites de périodicité dans Y



Preuve (suite)

On se contente de vérifier la condition nécessaire d'existence d'une solution. On intègre l'équation sur Y et le terme de transport disparait car

$$\int_Y v \cdot \nabla_y \phi \, dy = \int_{\partial Y} v \cdot n \phi \, ds = 0$$

à cause des conditions aux limites de périodicité. On obtient donc

$$\int_Y \sigma \left(\phi - \int_V \phi \, dv \right) dy = \int_Y g \, dy$$

que l'on intègre par rapport à v

$$0 = \int_V \int_Y \sigma(y) \left(\phi - \int_V \phi \, dv \right) dy \, dv = \int_V \int_Y g \, dy \, dv$$

car

$$\int_V \left(\phi - \int_V \phi \, dv \right) dv = 0.$$

L'équation en ϵ^{-2} est

$$v \cdot \nabla_y u_0 + \sigma(y) \left(u_0 - \int_V u_0 \, dv \right) = 0,$$

qui s'interprète comme une équation dans la cellule unité $Y \times V$ avec des conditions aux limites de périodicité (x n'est qu'un paramètre).

Par Fredholm la solution u_0 est une fonctions constante par rapport à (y, v) mais qui peut néanmoins dépendre de x

$$u_0(x, y, v) \equiv u(x).$$

L'équation en ϵ^{-1} est

$$v \cdot \nabla_y u_1 + \sigma(y) \left(u_1 - \int_V u_1 \, dv \right) = -v \cdot \nabla_x u(x),$$

qui est une équation pour l'inconnue u_1 dans la cellule de périodicité $Y \times V$. Comme $V = \mathbf{S}_{N-1}$ est symétrique, on a

$$\int_V v \cdot \nabla_x u(x) \, dv = 0.$$

Par Fredholm il existe donc une unique solution, à une constante additive près, ce qui nous permet de calculer $u_1(x, y, v)$ en fonction du gradient $\nabla_x u(x)$.

Problèmes de cellule

Pour chaque vecteur $(e_i)_{1\leq i\leq N}$, on appelle **problème de cellule**

$$\left\{ \begin{array}{ll} v\cdot\nabla_y w_i + \sigma(y)\left(w_i - \int_V w_i\,dv\right) = -v\cdot e_i & \text{dans } Y\times V \\ y \to w_i(y,v) & Y\text{-périodique.} \end{array} \right.$$

Par linéarité, on calcule facilement

$$u_1(x,y,v) = \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) w_i(y,v).$$

(En fait u_1 est défini à l'addition d'une fonction de x près, mais cela n'importera pas dans la suite.)

Finalement, **l'équation en ϵ^0** est

$$v \cdot \nabla_y u_2 + \sigma(y) \left(u_2 - \int_V u_2 \, dv \right) = -v \cdot \nabla_x u_1 - \tilde{\sigma}(x,y) u + S,$$

qui est une équation pour l'inconnue u_2 dans la cellule de périodicité $Y \times V$.

Par Fredholm il existe une solution si la condition de compatibilité suivante est vérifiée

$$\int_Y \int_V \left[-v \cdot \nabla_x u_1(x,y,v) - \tilde{\sigma}(x,y) u(x) + S(x,y,v) \right] dy \, dv = 0.$$

On remplace u_1 par son expression en fonction de $\nabla_x u$ et on obtient le problème homogénéisé pour u.

Puisque

$$u_1(x,y,v) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y,v),$$

on calcule

$$\int_Y \int_V -v \cdot \nabla_x u_1(x,y,v)\, dy\, dv =$$

$$-\sum_{i=1}^{N} \nabla_x \left(\frac{\partial u}{\partial x_i} \right)(x) \cdot \int_Y \int_V v\, w_i(y,v)\, dy\, dv =$$

$$\sum_{i,j=1}^{N} D^*_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

Seule compte la partie symétrique de D^*.

Formule de Kubo

Le tenseur homogénéisé D^* est défini par (formule de Kubo)

$$D^*_{ij} = \mathrm{Sym}\left(- \int_Y \int_V v_j w_i(y,v)\, dy\, dv \right).$$

(Remarquons que l'addition d'une constante à w_i ne change pas la valeur de D^*_{ij} car $\int_V v_j dv = 0$.)

On introduit les moyennes

$$\sigma^*(x) = \int_Y \tilde{\sigma}(x,y)\, dy \quad \text{et} \quad S^*(x) = \int_Y \int_V S(x,y,v)\, dy\, dv$$

On obtient **l'équation homogénéisée**

$$\begin{cases} -\mathrm{div_x}\Big(D^* \nabla_x u(x) \Big) + \sigma^*(x) u(x) = S^*(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

Lemme. Le tenseur D^* est défini positif.

Preuve. Montrons que $D^*\xi \cdot \xi > 0$ pour $\xi \neq 0 \in \mathbb{R}^N$. Soit

$$w_\xi(y,v) = \sum_{i=1}^{N} \xi_i w_i(y,v) \quad \text{solution de}$$

$$\begin{cases} v \cdot \nabla_y w_\xi + \sigma(y)\left(w_\xi - \int_V w_\xi \, dv\right) = -v \cdot \xi & \text{dans } Y \times V \\ y \to w_\xi(y,v) & Y\text{-périodique.} \end{cases}$$

On multiplie l'équation par w_ξ et on l'intègre sur Y

$$\int_Y v \cdot \nabla_y w_\xi \, w_\xi \, dy = \frac{1}{2} \int_{\partial Y} v \cdot n w_\xi^2 \, ds = 0$$

à cause des conditions aux limites de périodicité. On obtient donc

$$\int_Y \sigma \left(w_\xi - \int_V w_\xi \, dv \right) w_\xi \, dy = - \int_Y v \cdot \xi \, w_\xi \, dy$$

On intègre par rapport à v

$$\int_V \int_Y \sigma \left(w_\xi - \int_V w_\xi \, dv \right) w_\xi \, dy \, dv = - \int_V \int_Y v \cdot \xi \, w_\xi \, dy \, dv.$$

Comme la fonction $\left(w_\xi - \int_V w_\xi \, dv\right)$ est de moyenne nulle en v, on a

$$\int_V \int_Y \sigma \left(w_\xi - \int_V w_\xi \, dv \right) \left(\int_V w_\xi \, dv \right) dy \, dv = 0.$$

En combinant les deux on en déduit

$$0 \leq \int_V \int_Y \sigma \left(w_\xi - \int_V w_\xi \, dv \right)^2 dy \, dv = - \int_V \int_Y v \cdot \xi \, w_\xi \, dy \, dv = D^* \xi \cdot \xi$$

Montrons que cette inégalité est stricte. Si $D^* \xi \cdot \xi = 0$ pour un vecteur $\xi \neq 0$, alors on en déduit que $w_\xi \equiv \int_V w_\xi \, dv$ est indépendant de v et en reportant dans l'équation on obtient

$$v \cdot \nabla_y (w_\xi(y) + \xi \cdot y) = 0 \text{ dans } Y \times V.$$

Comme v est quelconque et w_ξ ne dépend pas de v, cela implique que $w_\xi(y) = -\xi \cdot y + C$ qui ne peut pas être périodique ! Contradiction.

Origine de la condition aux limites

Développement asymptotique sur le bord, au premier ordre ϵ^0 :

$$u_0(x, y, v) \equiv u(x) = 0 \text{ sur } \Gamma^- = \{x \in \partial\Omega, v \in V, v \cdot n(x) < 0\}.$$

Comme $u(x)$ ne dépend pas de v, on en déduit que cette fonction doit être nulle sur tout le bord $\partial\Omega$.

Remarquons qu'à l'ordre suivant ϵ^1 il n'est pas possible, en général, d'imposer que

$$u_1(x, y, v) \equiv \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y, v) = 0 \text{ sur } \Gamma^-$$

La série formelle est donc fausse: il faut la corriger par des "couches limites".

Conclusion

$$u_\epsilon(x,v) \approx u(x) + \epsilon \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(v, \frac{x}{\epsilon}\right)$$

- On remplace le problème exact par le problème homogénéisé.
- On doit calculer les solutions $w_i(y,v)$ des problèmes de cellule pour obtenir le tenseur homogénéisé constant D^*.
- D^* ne dépend ni de Ω, ni des sources S, ni des conditions aux limites.
- **Le tenseur D^* caractérise la microstructure.**
- On est passé du transport pour u_ϵ à de la diffusion pour u.