# CONVERGENCE THEORY

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1. Maximum principle
2. Oscillating test function
3. Two-scale convergence
4. Application to homogenization
5. General theory ($H$-convergence)
6. Boundary layers

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# INTRODUCTION

We recall the model problem of diffusion

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega$ is a smooth bounded open set of $\mathbb{R}^N$, $f \in L^2(\Omega)$ and the coefficient tensor $A(y)$ is $Y$-periodic, uniformly coercive and bounded

$$\alpha|\xi|^2 \le \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \le \beta|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N, \text{ a.e. } y \in Y \quad (\beta \ge \alpha > 0).$$

**Lemma (a priori estimate).** If $\Omega$ is bounded, then there exists a constant $C > 0$ depending only on $\Omega$ such that

$$\|u_\epsilon\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Therefore, up to a subsequence, there exists a limit $u \in H^1(\Omega)$ such that the sequence $u_\epsilon$ of solutions of the model problem converges to $u$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, as $\epsilon$ goes to 0,

$$\lim_{\epsilon\to 0} \int_\Omega |u_\epsilon - u|^2 dx = 0, \quad \lim_{\epsilon\to 0} \int_\Omega \nabla(u_\epsilon - u) \cdot \phi\, dx = 0 \quad \forall\, \phi \in L^2(\Omega)^N.$$

Goal: find the equation satisfied by $u$.

## Proof of the a priori estimate

Variational formulation

$$\int_\Omega A\left(x, \frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi(x) dx = \int_\Omega f(x) \varphi(x) dx, \quad \forall\, \varphi \in H_0^1(\Omega).$$

Take $\varphi = u_\epsilon$ and use coercivity

$$\alpha \|\nabla u_\epsilon\|^2_{L^2(\Omega)} \leq \int_\Omega f(x) u_\epsilon(x) dx \leq \|f\|_{L^2(\Omega)} \|u_\epsilon\|_{L^2(\Omega)}$$

Poincaré inequality in $\Omega$

$$\|u_\epsilon\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla u_\epsilon\|_{L^2(\Omega)}$$

Thus

$$\|\nabla u_\epsilon\|_{L^2(\Omega)} \leq \frac{C(\Omega)\|f\|_{L^2(\Omega)}}{\alpha}$$

# -I- MAXIMUM PRINCIPLE

Restricted to scalar elliptic equations ! Need some smoothness of the coefficients.

Recall that, at least formally,

$$u_\epsilon(x) \approx u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right)$$

with

$$u_1(x,y) = \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) w_i(y) + \tilde{u}_1(x)$$

and

$$u_2(x,y) = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \chi_{ij}(y) + \sum_{i=1}^N \frac{\partial \tilde{u}_1}{\partial x_i}(x) w_i(y) + \tilde{u}_2(x)$$

Cell problem:

$$\begin{cases} -\mathrm{div_y} A(y) \left( e_i + \nabla_y w_i(y) \right) = 0 & \text{in } Y \\ y \to w_i(y) & Y\text{-periodic} \end{cases}$$

Second order cell problem:

$$\begin{cases} -\mathrm{div_y} A(y) \nabla_y u_2(x,y) = \mathrm{div_y} A(y) \nabla_x u_1 + \mathrm{div_x} A(y) \left( \nabla_y u_1 + \nabla_x u \right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad - \mathrm{div_x} (A^* \nabla_x u) & \text{in } Y \\ y \to u_2(x,y) & Y\text{-periodic} \end{cases}$$

which implies the form $u_2(x,y) = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \chi_{ij}(y)$ (up to addition of $\tilde{u}_1, \tilde{u}_2$).

Define the remainder

$$r_\epsilon(x) = u_\epsilon(x) - \left\{ u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) \right\}$$

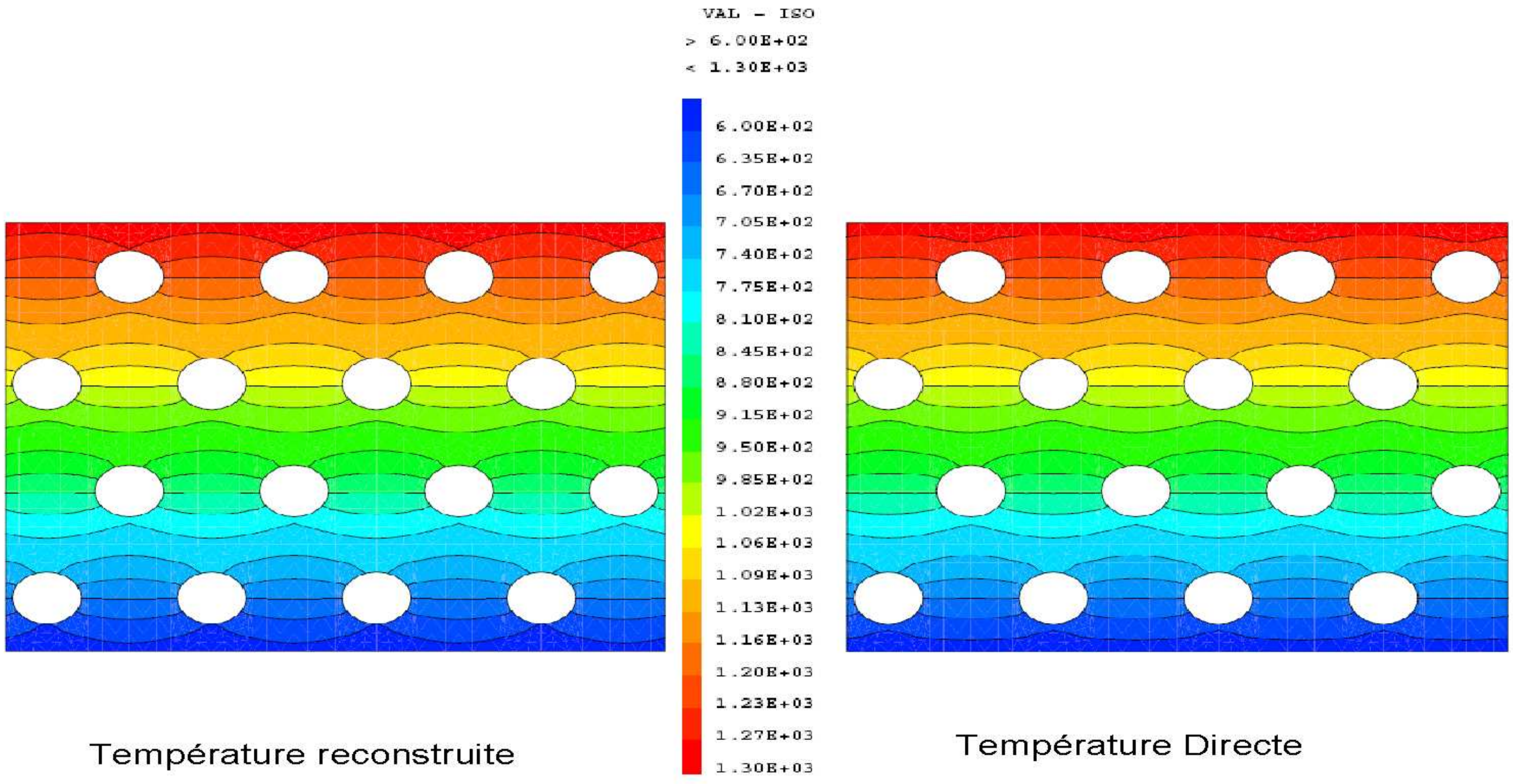
which satisfy

$$\left\{ \begin{array}{ll} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla r_\epsilon\right) = -\epsilon\left[\mathrm{div_x} A(\nabla_x u_1 + \nabla_y u_2) + \mathrm{div_y} A(\nabla_x u_2)\right]\left(x, \frac{x}{\epsilon}\right) & \\ \qquad\qquad - \epsilon^2\left[\mathrm{div_x} A(\nabla_x u_2)\right]\left(x, \frac{x}{\epsilon}\right) & \text{in } \Omega \\ r_\epsilon = -\epsilon u_1\left(x, \frac{x}{\epsilon}\right) - \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) & \text{on } \partial\Omega \end{array} \right.$$

**Theorem.** Assume $A(y) \in W^{1,\infty}(Y)$ and $u \in W^{4,\infty}(\Omega)$. Then

$$\|u_\epsilon(x) - u(x)\|_{L^\infty(\Omega)} \le C\epsilon$$

**Proof.** Apply the maximum principle to $r_\epsilon$.




Comparison of the exact and reconstructed solution $u + \epsilon u_1$ (K. El Ganaoui)

Correctors really matter !

# -II- Oscillating test function method (L. Tartar)

The goal is to mathematically prove the following

**Theorem.** The sequence $u_\epsilon(x)$ of solutions of the model problem converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, as $\epsilon$ goes to 0, to a limit $u(x)$ which is the unique solution of the homogenized problem.

Naive idea: pass to the limit in the variational formulation

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon(x)\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \quad \forall\,\varphi\in H_0^1(\Omega).$$

impossible because $A\left(\frac{x}{\epsilon}\right)$ and $\nabla u_\epsilon$ converge only weakly (the limit of the product is not the product of the limits).

## Difference between strong and weak convergence

$u_\epsilon$ converges strongly to $u$ in $L^2(\Omega)$ $\Leftrightarrow$ $\displaystyle\lim_{\epsilon\to 0} \int_\Omega |u_\epsilon(x) - u(x)|^2 dx = 0$

$v_\epsilon$ converges weakly to $v$ in $L^2(\Omega)$ $\Leftrightarrow$ $\displaystyle\lim_{\epsilon\to 0} \int_\Omega (v_\epsilon - v)\phi\, dx = 0 \quad \forall\, \phi \in L^2(\Omega)$

**Lemma.** Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function. Then $G(u_\epsilon)$ converges strongly to $G(u)$ in $L^2(\Omega)$.

This is false for weak convergence ! Counter-example: $v_\epsilon(x) = \sin\left(\frac{x}{\epsilon}\right)$.

## Oscillating test function method (ctd.)

The key idea is to replace $\varphi$ by

$$\varphi_\epsilon(x) = \varphi(x) + \epsilon \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right)$$

when $A$ is symmetric. Then, one can pass to the limit in the variational formulation using the cell equation satisfied by $w_i(y)$. Remark that the difference between the gradients of $\varphi_\epsilon$ and $\varphi$ is not small

$$\nabla \varphi_\epsilon(x) = \nabla \varphi(x) + \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x)(\nabla_y w_i)\left(\frac{x}{\epsilon}\right) + \mathcal{O}(\epsilon).$$

This idea can be used in multiscale numerical computations: replace the usual piecewise affine finite element basis by oscillating functions.

# -III- TWO-SCALE CONVERGENCE METHOD

This is a simpler method to prove the convergence theorem.

**Definition.** A sequence of functions $u_\epsilon$ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^2(\Omega \times Y)$ if, for any $Y$-periodic smooth function $\varphi(x,y)$, it satisfies

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \varphi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y u_0(x,y)\varphi(x,y) dx dy.$$

**Theorem 1.** From each bounded sequence $u_\epsilon$ in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to $u_0$.

**Theorem 2.** Let $u_\epsilon$ be a bounded sequence in $H^1(\Omega)$. Then, up to a subsequence, $u_\epsilon$ two-scale converges to a limit $u_0(x,y) \equiv u(x) \in H^1(\Omega)$, and $\nabla u_\epsilon$ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$ with $u_1 \in L^2(\Omega; H^1_\#(Y))$.

**Lemma 3.** For a bounded open set $\Omega$, let $B = C(\bar{\Omega}; C_\#(Y))$ be the space of continuous functions $\varphi(x,y)$ on $\bar{\Omega} \times Y$ which are $Y$-periodic in $y$. Then, $B$ is a separable Banach space (i.e. it contains a dense countable family), is dense in $L^2(\Omega \times Y)$, and there exists $C > 0$ such that

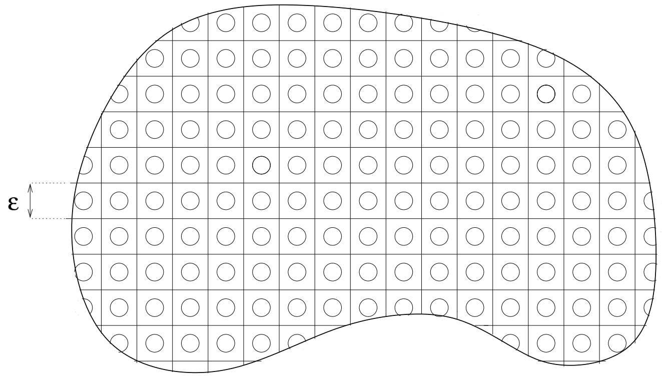
$$\int_\Omega |\varphi\left(x, \frac{x}{\epsilon}\right)|^2 dx \leq C \|\varphi\|_B^2,$$

and

$$\lim_{\epsilon \to 0} \int_\Omega |\varphi\left(x, \frac{x}{\epsilon}\right)|^2 dx = \int_\Omega \int_Y |\varphi(x,y)|^2 dx dy,$$

for any $\varphi(x,y) \in B$.

**Remark.** The same works with $B = L^2(\Omega; C_\#(Y))$ and $\Omega$ not necessarily bounded.




$\Omega$

**Proof of Lemma 3.** We mesh $\Omega$ with cubes of the type $(0,\epsilon)^N$

$$\Omega = \cup_{1\le i\le n(\epsilon)} Y_i^\epsilon \quad \text{with } Y_i^\epsilon = x_i^\epsilon + (0,\epsilon)^N.$$

$$\int_\Omega \left|\varphi\left(x,\frac{x}{\epsilon}\right)\right|^2 dx = \sum_{i=1}^{n(\epsilon)} \int_{Y_i^\epsilon} \left|\varphi\left(x,\frac{x}{\epsilon}\right)\right|^2 dx = \sum_{i=1}^{n(\epsilon)} \int_{Y_i^\epsilon} \left|\varphi\left(x_i^\epsilon,\frac{x}{\epsilon}\right)\right|^2 dx + o(1)$$

$$= \sum_{i=1}^{n(\epsilon)} \epsilon^N \int_Y \left|\varphi\left(x_i^\epsilon, y\right)\right|^2 dy + o(1) = \int_\Omega\int_Y \left|\varphi\left(x,y\right)\right|^2 dx\,dy + o(1)$$

**Proof of Theorem 1.** By Schwarz inequality, we have

$$\left| \int_\Omega u_\epsilon(x) \varphi\left(x, \frac{x}{\epsilon}\right) dx \right| \leq C \left| \int_\Omega \varphi\left(x, \frac{x}{\epsilon}\right) dx \right|^{\frac{1}{2}} \leq C \|\varphi\|_B.$$

Thus, the l.h.s. is a continuous linear form on $B$ which can be identified to a duality product $\langle \mu_\epsilon, \varphi \rangle_{B',B}$ for some bounded sequence of measures $\mu_\epsilon$. Since $B$ is separable, by the Banach-Alaoglu theorem, one can extract a subsequence and there exists a limit $\mu_0$ such $\mu_\epsilon$ converges to $\mu_0$ in the weak * topology of $B'$ (the dual of $B$). On the other hand, Lemma 3 allows us to pass to the limit in the middle term above. It yields

$$|\langle \mu_0, \varphi \rangle_{B',B}| \leq C \left| \int_\Omega \int_Y |\varphi(x,y)|^2 dx dy \right|^{\frac{1}{2}} .$$

Therefore $\mu_0$ is actually a continuous linear form on $L^2(\Omega \times Y)$, by density of $B$ in this space. Thus, there exists $u_0(x,y) \in L^2(\Omega \times Y)$ such that

$$\langle \mu_0, \varphi \rangle_{B',B} = \int_\Omega \int_Y u_0(x,y) \varphi(x,y) dx dy.$$

**Proof of Theorem 2.** Since $u_\epsilon$ and $\nabla u_\epsilon$ are bounded in $L^2(\Omega)$, up to a subsequence, they two-scale converge to limits $u_0(x,y) \in L^2(\Omega \times Y)$ and $\xi_0(x,y) \in L^2(\Omega \times Y)^N$. Thus

$$\lim_{\epsilon \to 0} \int_\Omega \nabla u_\epsilon(x) \cdot \vec{\psi}\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y \xi_0(x,y) \cdot \vec{\psi}(x,y) dx dy \quad \forall \vec{\psi} \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right).$$

Integrating by parts the left hand side gives

$$\epsilon \int_\Omega \nabla u_\epsilon(x) \cdot \vec{\psi}\left(x, \frac{x}{\epsilon}\right) dx = -\int_\Omega u_\epsilon(x) \left(\mathrm{div_y} \vec{\psi}\left(x, \frac{x}{\epsilon}\right) + \epsilon \mathrm{div_x} \vec{\psi}\left(x, \frac{x}{\epsilon}\right)\right) dx.$$

Passing to the limit yields

$$0 = -\int_\Omega \int_Y u_0(x,y) \mathrm{div_y} \vec{\psi}(x,y) dx dy \quad \Rightarrow \quad u_0(x,y) \equiv u(x) \in L^2(\Omega).$$

Next, we choose $\vec{\psi}$ such that $\mathrm{div_y} \vec{\psi}(x,y) = 0$. We obtain

$$\int_\Omega \nabla u_\epsilon(x) \cdot \vec{\psi}\left(x, \frac{x}{\epsilon}\right) dx = -\int_\Omega u_\epsilon(x) \mathrm{div_x} \vec{\psi}\left(x, \frac{x}{\epsilon}\right) dx.$$

Passing to the two-scale limit

$$\int_\Omega \int_Y \xi_0(x,y)\cdot\vec{\psi}(x,y)dxdy = -\int_\Omega \int_Y u(x)\mathrm{div_x}\vec{\psi}(x,y)dxdy.$$

If $\vec{\psi}$ does not depend on $y$, it proves that $u(x)\in H^1(\Omega)$. Furthermore,

$$\int_\Omega \int_Y (\xi_0(x,y)-\nabla u(x))\cdot\vec{\psi}(x,y)dxdy = 0 \quad \forall\vec{\psi} \text{ with } \mathrm{div_y}\vec{\psi}=0.$$

The orthogonal of divergence-free functions are exactly the gradients. Thus, there exists a unique function $u_1(x,y)$ in $L^2(\Omega;H^1_\#(Y)/\mathbb{R})$ such that

$$\xi_0(x,y)=\nabla u(x)+\nabla_y u_1(x,y).$$

**Theorem 4.** Let $u_\epsilon \in L^2(\Omega)$ two-scale converge to $u_0(x,y) \in L^2(\Omega \times Y)$.

1. Then, $u_\epsilon$ converges weakly in $L^2(\Omega)$ to $u(x) = \int_Y u_0(x,y)dy$, and we have

$$\lim_{\epsilon \longrightarrow 0} \|u_\epsilon\|^2_{L^2(\Omega)} \geq \|u_0\|^2_{L^2(\Omega\times Y)} \geq \|u\|^2_{L^2(\Omega)}.$$

2. Assume further that $u_0(x,y)$ is smooth and that

$$\lim_{\epsilon \longrightarrow 0} \|u_\epsilon\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega\times Y)}.$$

   Then, we have

$$\left\| u_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right) \right\|^2_{L^2(\Omega)} \to 0.$$

**Remark.** In the last case we say that $u_\epsilon$ two-scale converges **strongly** to $u_0$.

**Proof of Theorem 4.** Take a test function depending only on $x$

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(x)\varphi\,(x)\,dx = \int_\Omega\int_Y u_0(x,y)\varphi(x)dxdy = \int_\Omega u(x)\,\varphi(x)\,dx.$$

Thus, $u_\epsilon$ converges weakly to $u$ in $L^2(\Omega)$. Then, developing the inequality

$$\int_\Omega \left| u_\epsilon(x) - \varphi\left(x, \frac{x}{\epsilon}\right)\right|^2 dx \geq 0$$

$$\int_\Omega |u_\epsilon(x)|^2 dx - 2\int_\Omega u_\epsilon(x)\varphi\left(x,\frac{x}{\epsilon}\right)dx + \int_\Omega |\varphi\left(x,\frac{x}{\epsilon}\right)|^2 dx \geq 0$$

$$\liminf_{\epsilon\to 0}\int_\Omega |u_\epsilon(x)|^2 dx - 2\int_\Omega\int_Y u_0(x,y)\varphi(x,y)dx\,dy + \int_\Omega\int_Y |\varphi\,(x,y)\,|^2 dx\,dy \geq 0$$

Take $\varphi = u_0$ to get

$$\lim_{\epsilon\to 0}\|u_\epsilon\|^2_{L^2(\Omega)} \geq \|u_0\|^2_{L^2(\Omega\times Y)}.$$

**Proof of Theorem 4 (continued).** If we assume

$$\lim_{\epsilon\to 0} \|u_\epsilon\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega\times Y)},$$

the same computation yields

$$\liminf_{\epsilon\to 0} \int_\Omega \left| u_\epsilon(x) - \varphi\left(x, \frac{x}{\epsilon}\right)\right|^2 dx = \int_\Omega \int_Y |u_0(x,y) - \varphi(x,y)|^2 dx\, dy$$

If $u_0$ is smooth enough to be a test function $\varphi$ (Carathéodory function), it gives the desired result

$$\left\| u_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right)\right\|^2_{L^2(\Omega)} \to 0.$$

**Theorem 5.**

1. Let $u_\epsilon$ be a bounded sequence in $L^2(\Omega)$ such that $\epsilon\nabla u_\epsilon$ is also bounded in $L^2(\Omega)^N$. Then, there exists a two-scale limit $u_0(x,y) \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, $u_\epsilon$ two-scale converges to $u_0(x,y)$, and $\epsilon\nabla u_\epsilon$ to $\nabla_y u_0(x,y)$.

2. Let $u_\epsilon$ be a bounded sequence in $L^2(\Omega)^N$ such that $\mathrm{div} u_\epsilon$ is also bounded in $L^2(\Omega)$. Then, there exists a two-scale limit $u_0(x,y) \in L^2(\Omega \times Y)^N$ with $\mathrm{div_y} u_0 = 0$ and $\mathrm{div_x} u_0 \in L^2(\Omega \times Y)$ such that, up to a subsequence, $u_\epsilon$ two-scale converges to $u_0(x,y)$, and $\mathrm{div} u_\epsilon$ to $\mathrm{div_x} u_0(x,y)$.

**Proof of Theorem 5 (first part).** We have

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(x)\varphi\left(x,\frac{x}{\epsilon}\right)dx = \int_\Omega\int_Y u_0(x,y)\varphi(x,y)dxdy$$

$$\lim_{\epsilon\to 0}\int_\Omega \epsilon\nabla u_\epsilon(x)\cdot\vec{\psi}\left(x,\frac{x}{\epsilon}\right)dx = \int_\Omega\int_Y \xi_0(x,y)\cdot\vec{\psi}(x,y)dxdy$$

By integration by parts

$$\int_\Omega \epsilon\nabla u_\epsilon(x)\cdot\vec{\psi}\left(x,\frac{x}{\epsilon}\right)dx = -\int_\Omega u_\epsilon(x)\left(\mathrm{div_y}\vec{\psi}+\epsilon\mathrm{div_x}\vec{\psi}\right)\left(x,\frac{x}{\epsilon}\right)dx$$

Passing to the two-scale limit

$$\int_\Omega\int_Y \xi_0(x,y)\cdot\vec{\psi}(x,y)dxdy = -\int_\Omega\int_Y u_0(x,y)\mathrm{div_y}\vec{\psi}(x,y)dxdy$$

which implies that $\xi_0(x,y)=\nabla_y u_0(x,y)$.

# -IV- APPLICATION TO HOMOGENIZATION

Conductivity or diffusion equation

$$\begin{cases} -\mathrm{div}\left(A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

with a coefficient tensor $A(x,y)$ which is $Y$-periodic, uniformly coercive and bounded

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(x,y)\xi_i\xi_j \leq \beta|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N, \forall\, y \in Y, \forall\, x \in \Omega \quad (\beta \geq \alpha > 0).$$

A priori estimate. If $\Omega$ is bounded, then

$$\|u_\epsilon\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

## Two-scale convergence method

**First step.** We deduce from the a priori estimates the precise form of the two-scale limit of the sequence $u_\epsilon$.

By application of Theorem 2, there exist two functions, $u(x) \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$, such that, up to a subsequence, $u_\epsilon$ two-scale converges to $u(x)$, and $\nabla u_\epsilon$ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$.

In view of these limits, $u_\epsilon$ is expected to behave as $u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right)$.

# Two-scale convergence method

**Second step.** We multiply the p.d.e. by a test function similar to the limit of $u_\epsilon$, namely $\varphi(x) + \epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right)$, where $\varphi(x) \in \mathcal{D}(\Omega)$ and $\varphi_1(x, y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. This yields

$$\int_\Omega A\left(x, \frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot \left(\nabla\varphi(x) + \nabla_y\varphi_1\left(x, \frac{x}{\epsilon}\right) + \epsilon\nabla_x\varphi_1\left(x, \frac{x}{\epsilon}\right)\right) dx$$

$$= \int_\Omega f(x) \left(\varphi(x) + \epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right)\right) dx.$$

Regarding $A^t\left(x, \frac{x}{\epsilon}\right)\left(\nabla\varphi(x) + \nabla_y\varphi_1\left(x, \frac{x}{\epsilon}\right)\right)$ as a test function for the two-scale convergence, we pass to the two-scale limit

$$\int_\Omega\int_Y A(x, y)\left(\nabla u(x) + \nabla_y u_1(x, y)\right)\cdot\left(\nabla\varphi(x) + \nabla_y\varphi_1(x, y)\right) dx dy = \int_\Omega f(x)\varphi(x) dx.$$

## Two-scale convergence method

**Third step.** We read off a variational formulation for $(u, u_1)$. Take $(\varphi, \varphi_1)$ in the Hilbert space $H_0^1(\Omega) \times L^2\left(\Omega; H_\#^1(Y)/\mathbb{R}\right)$ endowed with the norm

$$\sqrt{(\|\nabla u(x)\|^2_{L^2(\Omega)} + \|\nabla_y u_1(x,y)\|^2_{L^2(\Omega\times Y)})}$$

The assumptions of the Lax-Milgram lemma are easily checked. The main point is the coercivity of the bilinear form defined by the left hand side

$$\int_\Omega \int_Y A(x,y)\left(\nabla\varphi(x) + \nabla_y \varphi_1(x,y)\right) \cdot \left(\nabla\varphi(x) + \nabla_y \varphi_1(x,y)\right) dx dy \geq$$

$$\alpha \int_\Omega \int_Y |\nabla\varphi(x) + \nabla_y \varphi_1(x,y)|^2 dx dy = \alpha \int_\Omega |\nabla\varphi(x)|^2 dx + \alpha \int_\Omega \int_Y |\nabla_y \varphi_1(x,y)|^2 dx dy.$$

$$\int_\Omega \int_Y A(x,y) \left(\nabla u(x) + \nabla_y u_1(x,y)\right)\cdot\left(\nabla \varphi(x) + \nabla_y \varphi_1(x,y)\right) dxdy = \int_\Omega f(x)\varphi(x)dx.$$

By application of the Lax-Milgram lemma, there exists a unique solution $(u,u_1) \in H_0^1(\Omega) \times L^2\left(\Omega; H^1_\#(Y)/\mathbb{R}\right)$. Consequently, the entire sequences $u_\epsilon$ and $\nabla u_\epsilon$ converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x,y)$.

An easy integration by parts shows that the associated p.d.e.'s are the so-called "two-scale homogenized problem",

$$\begin{cases} -\mathrm{div_y}\left(A(x,y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)\right) = 0 & \text{in } \Omega \times Y \\ -\mathrm{div_x}\left(\int_Y A(x,y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right) dy\right) = f(x) & \text{in } \Omega \\ y \to u_1(x,y) & Y\text{-periodic} \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

## Two-scale convergence method

**Fourth (and optional) step.** Eliminate the $y$ variable and the $u_1$ unknown

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(x,y),$$

where $w_i(x,y)$ are the unique solutions in $H^1_\#(Y)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\mathrm{div_y}\left(A(x,y)\left(\vec{e}_i + \nabla_y w_i(x,y)\right)\right) = 0 & \text{in } Y \\ y \to w_i(x,y) & Y\text{-periodic}, \end{cases}$$

at each point $x \in \Omega$, and

$$\begin{cases} -\mathrm{div_x}\left(A^*(x)\nabla u(x)\right) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $A^*_{ij}(x) = \int_Y A(x,y)\left(\vec{e}_i + \nabla_y w_i(x,y)\right) \cdot \left(\vec{e}_j + \nabla_y w_j(x,y)\right) dy.$

## Conclusion

We have just proved.

**Theorem.** The sequence of solution $u_\epsilon$ converges weakly to $u$ in $H_0^1(\Omega)$ and the sequence $\nabla u_\epsilon$ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$ where $u$ is the solution of the homogenized problem and $u_1(x, y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(x, y)$.

**Remark.** $u_\epsilon$ converges strongly to $u$ in $L^2(\Omega)$ but its gradient does not converge strongly !

## Corrector and strong two-scale convergence

**Proposition.** Assume that $A(x,y) \in C(\overline{\Omega}; L^\infty_\#(Y))$. Then $u_1(x,y)$ is indeed a corrector in the sense that

$$\lim_{\epsilon \to 0} \int_\Omega \left| \nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right) \right|^2 dx = 0$$

Key ingredient: strong two-scale convergence. If $v_\epsilon \in L^2(\Omega)$ two-scale converge to a smooth $v_0(x,y)$ and

$$\lim_{\epsilon \to 0} \|v_\epsilon\|^2_{L^2(\Omega)} = \|v_0\|^2_{L^2(\Omega \times Y)},$$

then

$$\lim_{\epsilon \to 0} \left\| v_\epsilon(x) - v_0\left(x, \frac{x}{\epsilon}\right) \right\|^2_{L^2(\Omega)} = 0.$$

## Proof

We develop

$$\int_\Omega A\left(x,\frac{x}{\epsilon}\right)\left(\nabla u_\epsilon - \nabla u - \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right)\cdot\left(\nabla u_\epsilon - \nabla u - \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right)dx =$$

$$\int_\Omega A\left(x,\frac{x}{\epsilon}\right)\left(\nabla u(x) + \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right)\cdot\left(\nabla u(x) + \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right)dx$$

$$-2\int_\Omega A^{sym}\left(x,\frac{x}{\epsilon}\right)\nabla u_\epsilon\cdot\left(\nabla u(x) + \nabla_y u_1\left(x,\frac{x}{\epsilon}\right)\right)dx$$

$$+\int_\Omega A\left(x,\frac{x}{\epsilon}\right)\nabla u_\epsilon\cdot\nabla u_\epsilon dx$$

The last term is equal to $\int_\Omega f u_\epsilon dx \to \int_\Omega f u dx.$

# Proof (ctd.)

Passing to the limit

$$\lim_{\epsilon\to 0} \alpha \int_\Omega \left| \nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right) \right|^2 dx \leq$$

$$\lim_{\epsilon\to 0} \int_\Omega A\left(x, \frac{x}{\epsilon}\right)\left(\nabla u_\epsilon - \nabla u - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right)\right)\cdot\left(\nabla u_\epsilon - \nabla u - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right)\right) dx =$$

$$\int_\Omega f u dx$$

$$-2 \int_\Omega \int_Y A^{sym}(x,y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)\cdot\left(\nabla u(x) + \nabla_y u_1(x,y)\right) dxdy$$

$$+ \int_\Omega \int_Y A(x,y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)\cdot\left(\nabla u(x) + \nabla_y u_1(x,y)\right) dxdy$$

$$= 0$$

because of the variational formulation of the two-scale limit problem !

## Error estimate

**Proposition.** Assume that $A(x,y) \in C(\overline{\Omega}; L^\infty_\#(Y))$ and $u \in W^{2,\infty}(\Omega)$. Then

$$\left\| u_\epsilon(x) - u(x) - \epsilon u_1\left(x, \frac{x}{\epsilon}\right) \right\|_{H^1(\Omega)} \leq C\sqrt{\epsilon}$$

# -V- GENERAL THEORY (H-convergence)

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$, and let $0 < \alpha \leq \beta$. We introduce the set $\mathcal{M}(\alpha, \beta, \Omega)$ of all possible matrices $A(x)$ such that, a.e. in $\Omega$

$$\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \quad \text{and} \quad \beta|\xi|^2 \leq A^{-1}(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^N.$$

We consider a sequence $A_\epsilon(x)$ in $\mathcal{M}(\alpha, \beta, \Omega)$, indexed by $\epsilon$, going to 0, which is not associated to any specific lengthscale or statistical property of the elastic medium. In other words, no special assumptions (like periodicity or stationarity) are placed on the sequence $A_\epsilon$.

For $f(x) \in L^2(\Omega)$, there exists a unique solution $u_\epsilon \in H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}\,(A_\epsilon(x)\nabla u_\epsilon) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

**Definition.** The sequence $A_\epsilon(x) \in \mathcal{M}(\alpha, \beta, \Omega)$ is said to $H$-converge to a limit $A^*(x)$, as $\epsilon$ goes to 0, if, for any $f \in L^2(\Omega)$, the sequence $u_\epsilon$ sastisfies

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla u_\epsilon \rightharpoonup A^* \nabla u \text{ weakly in } L^2(\Omega)^N, \end{cases}$$

where $u$ is the solution of the homogenized equation associated to $A^*$

$$\begin{cases} -\mathrm{div}\,(A^*(x)\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark that, by definition, the homogenized tensor $A^*$ is independent of the source term $f$. We shall see that it is also independent of the boundary condition and of the domain.

**Theorem (compactness).** For any sequence $A_\epsilon$ in $\mathcal{M}(\alpha, \beta, \Omega)$, there exist a subsequence (still denoted by $\epsilon$) and a homogenized limit $A^*$, belonging to $\mathcal{M}(\alpha, \beta, \Omega)$, such that $A_\epsilon$ $H$-converges to $A^*$.

**Proposition (locality of $H$-convergence).** Let $A^\epsilon(x)$ and $B^\epsilon(x)$ be two sequences in $\mathcal{M}(\alpha, \beta, \Omega)$, which $H$-converge to $A^*(x)$ and $B^*(x)$, respectively. Let $\omega$ be an open subset compactly embedded in $\Omega$, i.e., $\overline{\omega} \subset \Omega$. If $A^\epsilon(x) = B^\epsilon(x)$ in $\omega$, then $A^*(x) = B^*(x)$ in $\omega$.

**Proposition (energy convergence).** Let $A^\epsilon(x)$ be a sequence in $\mathcal{M}(\alpha, \beta, \Omega)$ that $H$-converges to $A^*(x)$. For any $f \in L^2(\Omega)$, the sequence $u_\epsilon$ satisfies

$$A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \rightharpoonup A^* \nabla u \cdot \nabla u \text{ in } L^1(\Omega)$$

and

$$\int_\Omega A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon dx \to \int_\Omega A^* \nabla u \cdot \nabla u dx,$$

## $G$-convergence

When restricted to symmetric matrices, $H$ convergence simplifies and is called $G$-convergence.

**Proposition ($G$-convergence).** Let $A^\epsilon(x)$ be a sequence of symmetric matrices in $\mathcal{M}(\alpha, \beta, \Omega)$ that $H$-converges to $A^*(x)$. Then $A^*$ is symmetric too. Furthermore, $H$-convergence is equivalent to so-called $G$-convergence which is defined by: for any $f \in L^2(\Omega)$, $u_\epsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ where $u$ is the solution of the homogenized equation.

**Remark.** The convergence of the flux $A^\epsilon \nabla u_\epsilon$ is not required in $G$-convergence and is rather an automatic property (due to the symmetry).

**Bibliography.** $H$-convergence was introduced by Murat and Tartar. $G$-convergence was older and due to De Giorgi and Spagnolo.

# -VI- BOUNDARY LAYERS

$$u_\epsilon(x) \approx u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right)$$

but $u_1$ does not satisfy the Dirichlet boundary condition.

We correct it by introducing a boundary layer solution of

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon^{bl}\right) = 0 & \text{in } \Omega \\ u_\epsilon^{bl} = -u_1\left(x, \frac{x}{\epsilon}\right) & \text{on } \partial\Omega \end{cases}$$

which cannot be computed explicitly except in special cases. In particular

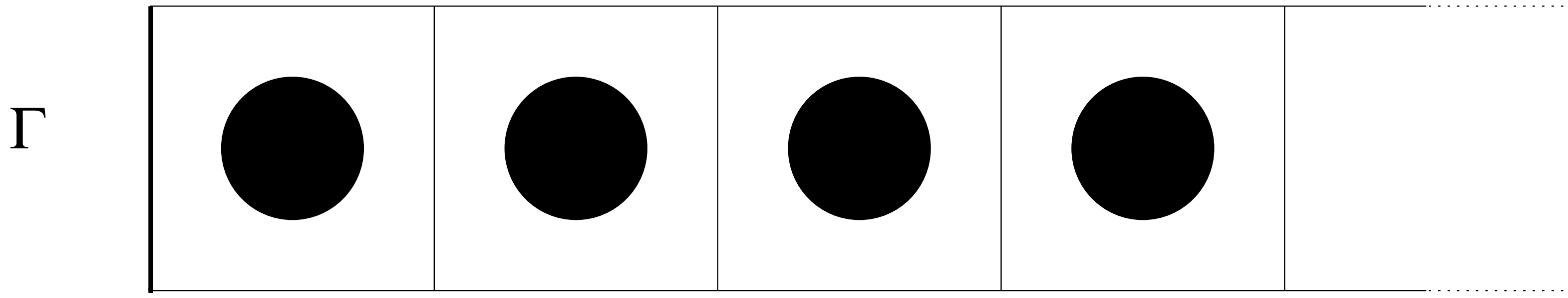
$$\|u_\epsilon^{bl}\|_{H^1(\Omega)} = \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right)$$

**Proposition.** Assume that $u \in W^{2,\infty}(\Omega)$. Then

$$\left\| u_\epsilon(x) - u(x) - \epsilon u_1\left(x, \frac{x}{\epsilon}\right) - \epsilon u_\epsilon^{bl}(x) \right\|_{H^1(\Omega)} \leq C\epsilon$$

Consider a rectangular domain $\Omega$. In such a case we can approximate the boundary layer as follows.

Introduce the semi-infinite band $G$ (in the direction $y_N$)




$$\left\{ \begin{array}{ll} -\mathrm{div_y}\left(A(y)\nabla_y w_i^{bl}(y)\right) = 0 & \text{in } G \\ w_i^{bl} = -w_i & \text{on } \Gamma \\ y' \to w_i^{bl}(y', y_N) & Y\text{-periodic} \end{array} \right.$$

Approximate the boundary layer (on just one side) by

$$u_\epsilon^{bl}(x) \approx \frac{\partial u}{\partial x_N}(x) w_N^{bl}\left(\frac{x}{\epsilon}\right)$$