TRANSPORT IN POROUS MEDIA

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1. Introduction
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-I- INTRODUCTION

- Based on a joint work with A. Mikelic and A. Piatnitski.
- Infinite porous medium: (connected) fluid part Ω_f, solid part $\Omega_s = \mathbb{R}^n \setminus \Omega_f$.
- Saturated incompressible single phase flow in Ω_f and a single solute.
- The unknown is the concentration u in the fluid.

convection diffusion in the fluid:

$$\frac{\partial u}{\partial \tau} + b \cdot \nabla_y u - \mathrm{div}_y(D\nabla_y u) = 0 \quad \text{in} \ \ \Omega_f \times (0, \mathcal{T}),$$

Given incompressible and steady-state velocity:

$$\mathrm{div}\, b = 0 \text{ in } \Omega_f$$

no-flux boundary condition on the pore boundaries:

$$b \cdot n = 0 \quad \text{and} \quad -D\nabla_y u \cdot n = 0 \quad \text{on} \ \ \partial\Omega_f \times (0, \mathcal{T}),$$

Scaling

We want to upscale this model, so we define a large **macroscopic scale** ϵ^{-1} and we choose a **long time scale** of order ϵ^{-2} (parabolic or diffusion scaling)

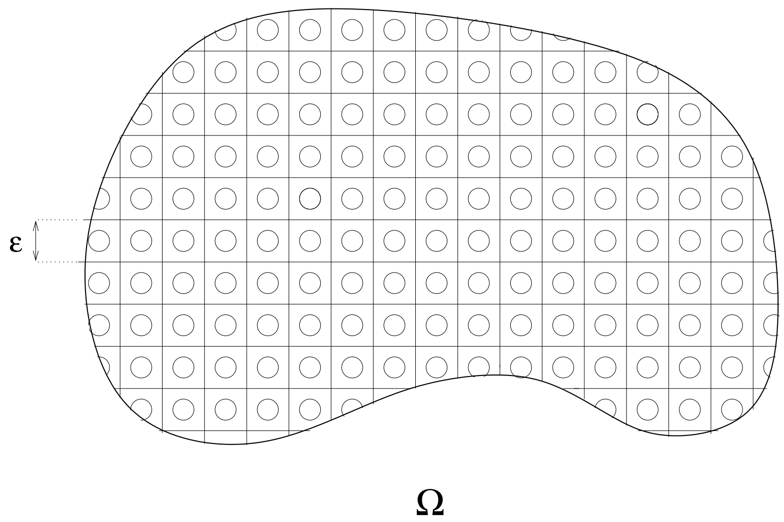
$$x = \epsilon y \quad \text{and} \quad t = \epsilon^2 \tau.$$

We define $u_\epsilon(t,x) = u(\tau, y)$ which is a solution of

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + \frac{1}{\epsilon} b_\epsilon \cdot \nabla_x u_\epsilon - \operatorname{div}_x(D_\epsilon \nabla_x u_\epsilon) = 0 \quad \text{in } \Omega_\epsilon \times (0,T) \\ u_\epsilon(x,0) = u^0(x), \quad x \in \Omega_\epsilon, \\ -D_\epsilon \nabla_x u_\epsilon \cdot n = 0 \quad \text{on } \partial\Omega_\epsilon \times (0,T) \end{cases}$$

with $T = \epsilon^2 \mathcal{T}$.

Periodicity assumption



- ✗ Periodic unit cell $Y = (0,1)^n = Y^* \cup \mathcal{O}$ with fluid part Y^*
- ✗ Periodic (infinite) porous media $x \in \Omega_\epsilon \Leftrightarrow y \in Y^*$
- ✗ Stationary incompressible periodic flow $b_\epsilon(x) = b\left(\frac{x}{\epsilon}\right)$ with $\mathrm{div}_y b = 0$ in Y^* and $b \cdot n = 0$ on $\partial\mathcal{O}$
- ✗ Periodic symmetric coercive diffusion $D_\epsilon(x) = D\left(\frac{x}{\epsilon}\right)$

Another approach to scaling: dimensional analysis

We write the same equations with dimensional constants denoted by $*$:

$$\frac{\partial c^*}{\partial t^*} + b^* \cdot \nabla_{x^*} c^* - \mathrm{div}_{x^*}(D^* \nabla_{x^*} c^*) = 0 \quad \text{in} \;\; \Omega_f \times (0, T^*),$$

$$-D^* \nabla_{x^*} c^* \cdot n = 0 \quad \text{on} \;\; \partial\Omega_f \times (0, T^*),$$

Dimensional analysis (ctd.)

We adimensionalize the equations as follows:

- ✗ Characteristic lengthscale L_R and timescale T_R.
- ✗ Period $\ell << L_R$: we introduce a small parameter $\epsilon = \frac{\ell}{L_R}$.
- ✗ Characteristic velocity b_R.
- ✗ Characteristic concentration c_R.
- ✗ Characteristic diffusivity D_R.

New adimensionalized variables and functions:

$$x = \frac{x^*}{L_R}, \; t = \frac{t^*}{T_R}, \; b_\epsilon(x,t) = \frac{b^*(x^*,t^*)}{b_R}, \; D = \frac{D^*}{D_R}, \; u_\epsilon = \frac{c^*}{c_R}$$

Dimensional analysis (ctd.)

Dimensionless equation

$$\frac{\partial u_\epsilon}{\partial t} + \frac{b_R T_R}{L_R} b_\epsilon \cdot \nabla_x u_\epsilon - \frac{D_R T_R}{L_R^2} \operatorname{div}_x (D \nabla_x u_\epsilon) = 0 \quad \text{in} \ \ \Omega_\epsilon \times (0,T)$$

and

$$-D \nabla_x u_\epsilon \cdot n = 0 \quad \text{on} \ \ \partial\Omega_\epsilon \times (0,T).$$

We choose a diffusion timescale, i.e., we assume $T_R = \frac{L_R^2}{D_R}$.

Péclet number: $\mathbf{Pe} = \frac{L_R b_R}{D_R}$. We assume $\mathbf{Pe} = \epsilon^{-1}$.

$$\Rightarrow \quad \frac{\partial u_\epsilon}{\partial t} + \mathbf{Pe}\, b_\epsilon \cdot \nabla_x u_\epsilon - \operatorname{div}_x (D \nabla_x u_\epsilon) = 0 \quad \text{in} \ \ \Omega_\epsilon \times (0,T)$$

Microscopic model

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + \dfrac{1}{\epsilon} b_\epsilon \cdot \nabla_x u_\epsilon - \mathrm{div}_x (D_\epsilon \nabla_x u_\epsilon) = 0 \quad \text{in} \;\; \Omega_\epsilon \times (0,T) \\ u_\epsilon(x,0) = u^0(x), \quad x \in \Omega_\epsilon, \\ D_\epsilon \nabla_x u_\epsilon \cdot n = 0 \;\; \text{on} \;\; \partial\Omega_\epsilon \times (0,T) \end{cases}$$

Assumptions:

- ✗ Stationary incompressible periodic flow $\mathrm{div}_y b = 0$ in Y^*, $b \cdot n = 0$ on $\partial\mathcal{O}$
- ✗ Periodic symmetric coercive diffusion D
- ✗ Goal of homogenization: find the effective diffusion tensor.

-II- MAIN RESULT

Theorem. The solution u_ϵ satisfies

$$u_\epsilon(t,x) \approx u\left(t, x - \frac{b^*}{\epsilon} t\right)$$

with the effective drift

$$b^* = \frac{1}{|Y^*|} \int_{Y^*} b(y)\, dy$$

and u the solution of the homogenized problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(A^* \nabla u\right) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(t=0,x) = u^0(x) & \text{in } \mathbb{R}^n \end{cases}$$

Precise convergence

$$u_\epsilon(t,x) = u\left(t, x - \frac{b^*}{\epsilon}t\right) + r_\epsilon(t,x)$$

with

$$\lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^n} \left| r_\epsilon(t,x) \right|^2 dt\, dx = 0,$$

Homogenized diffusion tensor

$$A^*_{ij} = \int_{Y^*} D(e_i + \nabla_y \chi_i) \cdot (e_j + \nabla_y \chi_j) dy$$

where $\chi_i(y)$, $1 \leq i \leq n$, are solutions of the cell problems

$$\begin{cases} b(y) \cdot \nabla_y \chi_i - \mathrm{div_y}\left(D(y)\left(\nabla_y \chi_i + e_i\right)\right) = (b^* - b(y)) \cdot e_i & \text{in } Y^* \\ -D(y)\left(\nabla_y \chi_i + e_i\right) \cdot n = 0 & \text{on } \partial\mathcal{O} \\ y \to \chi_i(y) \ Y\text{-periodic} \end{cases}$$

Remark that the value of b^* is exactly the compatibility condition for the existence of χ_i.

Equivalent homogenized equation

Define $\tilde{u}_\epsilon(t,x) = u\left(t, x - \frac{b^*}{\epsilon}t\right)$. Then, it is solution of

$$\begin{cases} \dfrac{\partial \tilde{u}_\epsilon}{\partial t} + \dfrac{1}{\epsilon} b^* \cdot \nabla \tilde{u}_\epsilon - \operatorname{div}\left(A^* \nabla \tilde{u}_\epsilon\right) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ \tilde{u}_\epsilon(t=0,x) = u^0(x) & \text{in } \mathbb{R}^n \end{cases}$$

-III- TWO-SCALE ANSATZ WITH DRIFT

To motivate our result, let us start with a formal process.

Standard two-scale asymptotic expansions should be modified to introduce an **unknown large drift** $b^* \in \mathbb{R}^n$

$$u_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i u_i \left(t, x - \frac{b^* t}{\epsilon}, \frac{x}{\epsilon} \right),$$

with $u_i(t,x,y)$ a function of the macroscopic variable x and of the periodic microscopic variable $y \in Y = (0,1)^n$.

We plug these ansatz in the system of equations and use the usual chain rule derivation

$$\nabla\left(u_i\left(t,x-\frac{b^*t}{\epsilon},\frac{x}{\epsilon}\right)\right)=\left(\epsilon^{-1}\nabla_y u_i+\nabla_x u_i\right)\left(t,x-\frac{b^*t}{\epsilon},\frac{x}{\epsilon}\right),$$

plus a **new** contribution

$$\frac{\partial}{\partial t}\left(u_i\left(t,x-\frac{b^*t}{\epsilon},\frac{x}{\epsilon}\right)\right)=\left(\frac{\partial u_i}{\partial t}-\underbrace{\epsilon^{-1}b^*\cdot\nabla_x u_i}_{new\,term}\right)\left(t,x,\frac{x}{\epsilon}\right)$$

Fredholm alternative in the unit cell

Lemma. The boundary value problem

$$\begin{cases} b(y)\cdot\nabla_y\chi - \mathrm{div}_y\left(D(y)\nabla_y\chi\right) = f & \text{in } Y^* \\ -D(y)\nabla_y\chi\cdot n + g = 0 & \text{on } \partial\mathcal{O} \\ y \to \chi(y) \ Y\text{-periodic} & \end{cases}$$

admits a unique solution $\chi \in H^1(Y^*)/\mathbb{R}$ (up to an additive constant), **if and only if**

$$\int_{Y^*} f(y)\,dy + \int_{\partial\mathcal{O}} g(y)\,ds = 0.$$

Variational formulation of the cell problem

$$\int_{Y^*} b(y) \cdot \nabla_y \chi(y) \phi(y)\, dy + \int_{Y^*} D \nabla_y \chi(y) \cdot \nabla_y \phi(y)\, dy =$$

$$\int_{Y^*} f(y) \phi(y)\, dy + \int_{\partial \mathcal{O}} g(y) \phi(y)\, ds$$

for any test function $\phi \in H^1(Y^*)$.

Coercive bilinear form on the orthogonal subspace to its kernel $\mathbb{R}$.

Cascade of equations

Equation of order ϵ^{-2}:

$$\left\{ \begin{array}{l} b(y) \cdot \nabla_y u_0 - \mathrm{div_y}\left(D(y) \nabla_y u_0 \right) = 0 \text{ in } Y^* \\ -D(y) \nabla_y u_0 \cdot n = 0 \text{ on } \partial \mathcal{O} \\ y \rightarrow u_0(t,x,y) \ Y\text{-periodic} \end{array} \right.$$

We deduce

$$\boxed{u_0(t,x,y) \equiv u(t,x)}$$

Equation of order ϵ^{-1}:

$$\left\{ \begin{array}{l} -b^* \cdot \nabla_x u_0 + b(y) \cdot (\nabla_x u_0 + \nabla_y u_1) - \mathrm{div_y} \left(D(y) \left(\nabla_x u_0 + \nabla_y u_1 \right) \right) = 0 \text{ in } Y^* \\ -D \left(\nabla_x u_0 + \nabla_y u_1 \right) \cdot n = 0 \text{ on } \partial \mathcal{O} \\ y \to u_1(t,x,y) \ Y\text{-periodic} \end{array} \right.$$

We deduce

$$\boxed{u_1(t,x,y) = \sum_{i=1}^{n} \frac{\partial u_0}{\partial x_i}(t,x) \chi_i(y)}$$

Cell problem

$$\begin{cases} b(y) \cdot \nabla_y \chi_i - \mathrm{div_y} \left(D(y) \left(\nabla_y \chi_i + e_i \right) \right) = \left(b^* - b(y) \right) \cdot e_i \text{ in } Y^* \\ -D(y) \left(\nabla_y \chi_i + e_i \right) \cdot n = 0 \text{ on } \partial \mathcal{O} \\ y \to \chi_i(y) \; Y\text{-periodic} \end{cases}$$

The compatibility condition (Fredholm alternative) for the existence of χ_i is

$$b^* = \frac{1}{|Y^*|} \int_{Y^*} b(y)\, dy$$

Equation of order ϵ^0:

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} - b^* \cdot \nabla_x u_1 + b \cdot (\nabla_x u_1 + \nabla_y u_2) - \mathrm{div_y}\left(D\left(\nabla_x u_1 + \nabla_y u_2\right)\right) \\ \qquad\qquad -\mathrm{div_x}\left(D(\nabla_y u_1 + \nabla_x u)\right) = 0 \quad \text{in } Y^* \\ -D\left(\nabla_x u_1 + \nabla_y u_2\right) \cdot n = 0 \text{ on } \partial\mathcal{O} \\ y \to u_2(t,x,y) \ Y\text{-periodic} \end{array} \right.$$

Compatibility condition for the existence of u_2 $\Rightarrow$ homogenized problem. (Recall that u_1 is linear in $\nabla_x u$.)

$$\left\{ \begin{array}{ll} \dfrac{\partial u}{\partial t} - \mathrm{div}\left(A^* \nabla u\right) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(t=0,x) = u^0(x) & \text{in } \mathbb{R}^n, \end{array} \right.$$

-IV- RIGOROUS PROOF

The proof is made of 3 steps

1. A priori estimates.
2. Passing to the limit by two-scale convergence with drift.
3. Strong convergence.

A priori estimates

Assume $u_0 \in L^2(\mathbb{R}^n)$. For any final time $T > 0$, there exists a constant $C > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega_\epsilon))} + \|\nabla u_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon)} \leq C$$

Proof. Multiply the fluid equation by u_ϵ and integrate by parts to get the usual parabolic estimates.

"Usual" two-scale convergence

Proposition.

Let w_ϵ be a bounded sequence in $L^2(\mathbb{R}^n)$. Up to a subsequence, there exist a limit $w(x,y) \in L^2(\mathbb{R}^n \times \mathbf{T}^n)$ such that w_ϵ two-scale converges to w in the sense that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} w_\epsilon(x) \phi\left(x, \frac{x}{\epsilon}\right) dx = \int_{\mathbb{R}^n} \int_{\mathbf{T}^n} w(x,y) \phi(x,y) \, dx \, dy$$

for all functions $\phi(x,y) \in L^2\left(\mathbb{R}^n; C(\mathbf{T}^n)\right)$.

Two-scale convergence with drift

Proposition (Marusic-Paloka, Piatnitski). Let $\mathcal{V} \in \mathbb{R}^N$ be a given drift velocity. Let w_ϵ be a bounded sequence in $L^2((0,T) \times \mathbb{R}^n)$. Up to a subsequence, there exist a limit $w_0(t,x,y) \in L^2((0,T) \times \mathbb{R}^n \times \mathbf{T}^n)$ such that w_ϵ two-scale converges with drift weakly to w_0 in the sense that

$$\lim_{\epsilon \to 0} \int_0^T \int_{\mathbb{R}^n} w_\epsilon(t,x) \phi\left(t, x + \frac{\mathcal{V}}{\epsilon} t, \frac{x}{\epsilon}\right) dt\, dx = \int_0^T \int_{\mathbb{R}^n} \int_{\mathbf{T}^n} w_0(t,x,y) \phi(t,x,y)\, dt\, dx\, dy$$

for all functions $\phi(t,x,y) \in L^2\left((0,T) \times \mathbb{R}^n; C(\mathbf{T}^n)\right)$.

Lemma.

Let $\phi(t,x,y) \in L^2\left((0,T)\times \mathbf{T}^N; C_c(\mathbb{R}^N)\right)$. Then

$$\lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^N} \left|\phi\left(t, x+\frac{\mathcal{V}}{\epsilon}t, \left(\frac{x}{\epsilon}\right)\right)\right|^2 dt\, dx = \int_0^T \int_{\mathbb{R}^N} \int_{\mathbf{T}^N} |\phi(t,x,y)|^2 dt\, dx\, dy.$$

Proof. Change of variables $x' = x + \frac{\mathcal{V}}{\epsilon}t$

$$\int_0^T \int_{\mathbb{R}^N} \left|\phi\left(t, x+\frac{\mathcal{V}}{\epsilon}t, \left(\frac{x}{\epsilon}\right)\right)\right|^2 dt\, dx = \int_0^T \int_{\mathbb{R}^N} \left|\phi\left(t, x', \frac{x'}{\epsilon} - \frac{\mathcal{V}}{\epsilon^2}t\right)\right|^2 dt\, dx'$$

We mesh $\mathbb{R}^N$ with cubes of size ϵ, $\mathbb{R}^N = \cup_{i\in\mathbb{Z}} Y_i^\epsilon$ with $Y_i^\epsilon = x_i^\epsilon + (0,\epsilon)^N$

$$\int_{\mathbb{R}^N} \left|\phi\left(t, x', \frac{x'}{\epsilon} - \frac{\mathcal{V}}{\epsilon^2}t\right)\right|^2 dx = \sum_i \int_{Y_i^\epsilon} \left|\phi\left(t, x_i^\epsilon, \frac{x'}{\epsilon} - \frac{\mathcal{V}}{\epsilon^2}t\right)\right|^2 dx + o(1)$$

$$= \sum_{i\in\mathbb{Z}} \epsilon^N \int_{\mathbf{T}^N} |\phi(x_i^\epsilon, y)|^2 dy + o(1) = \int_\Omega \int_Y |\phi(x,y)|^2 dx\, dy + o(1)$$

Passing to the limit

We multiply the equation by an oscillating test function with drift $\mathcal{V} = -b^*$

$$\Psi_\epsilon = \phi\left(t, x + \frac{\mathcal{V}}{\epsilon}t\right) + \epsilon\,\phi_1\left(t, x + \frac{\mathcal{V}}{\epsilon}t, \frac{x}{\epsilon}\right)$$

and we use the two-scale convergence with drift to get the homogenized equation.

STRONG CONVERGENCE

We use the notion of strong two-scale convergence with drift.

Proposition. If $w_\epsilon(t,x)$ two-scale converges with drift weakly to $w_0(t,x,y)$ (assumed to be smooth enough) and

$$\lim_{\epsilon\to 0} \|w_\epsilon\|_{L^2((0,T)\times\mathbb{R}^n)} = \|w_0\|_{L^2((0,T)\times\mathbb{R}^n\times\mathbf{T}^n)},$$

then it converges strongly in the sense that

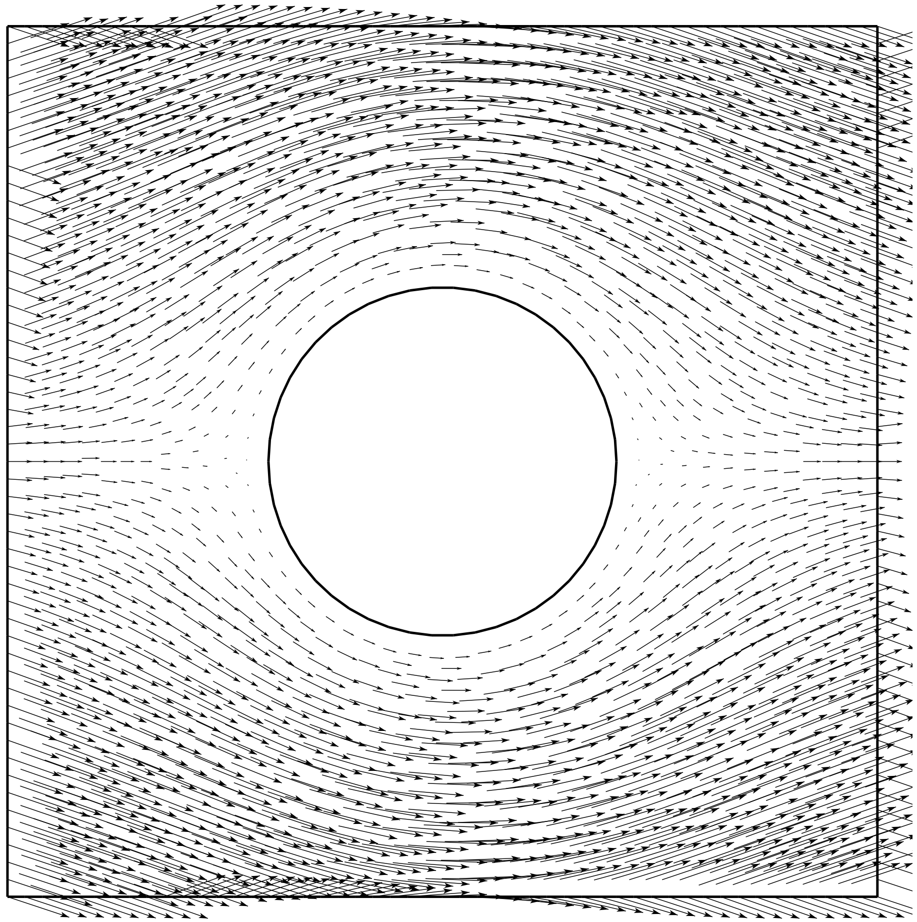
$$\lim_{\epsilon\to 0} \left\| w_\epsilon(t,x) - w_0\left(t, x - \frac{b^*}{\epsilon}t, \frac{x}{\epsilon}\right)\right\|_{L^2((0,T)\times\mathbb{R}^n)} = 0$$

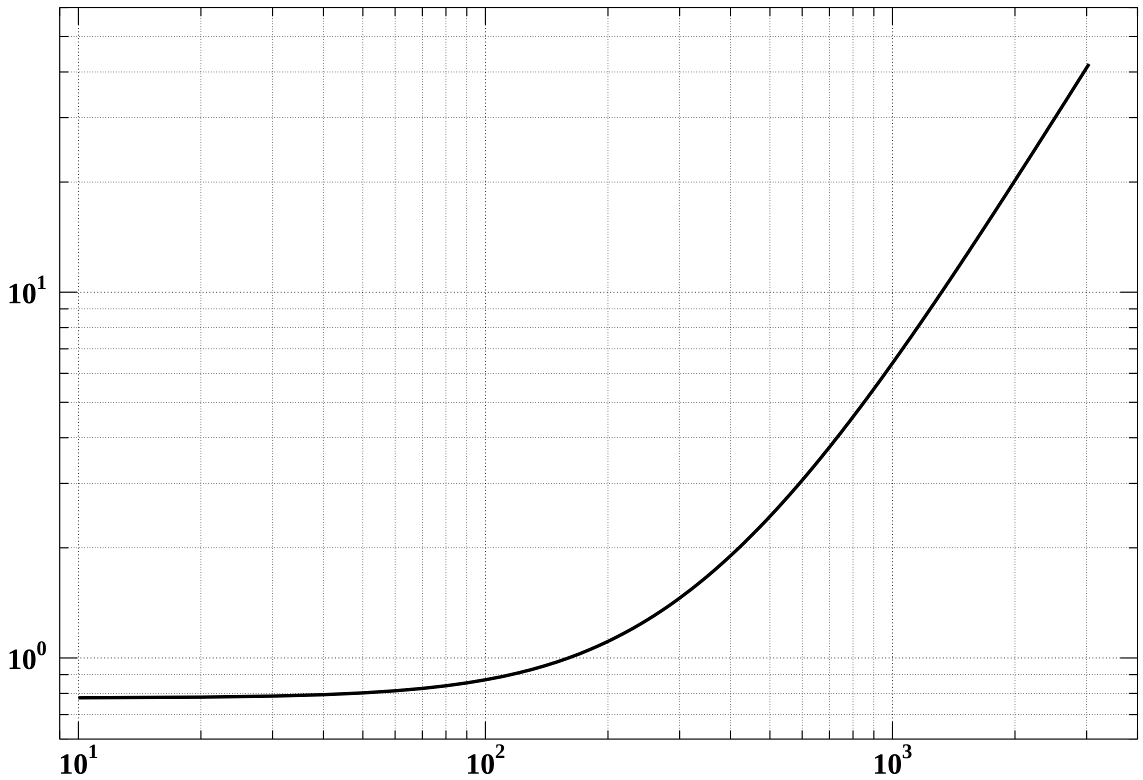
-V- NUMERICAL RESULTS

Numerical computations done with FreeFem++ in 2-d for circular obstacles.

The velocity $\mathbf{b}(y)$ is generated by solving the following filtration problem in the fluid part Y^* of the unit cell Y

$$\begin{cases} \nabla_y p - \Delta_y \mathbf{b} = \mathbf{e}_i & \text{in } \ Y^*; \\ \operatorname{div}_y \mathbf{b} = 0 & \text{in } \ Y^*; \\ \mathbf{b} = 0 & \text{on } \ \partial\mathcal{O}; \\ p, \mathbf{b} \quad \text{are } \ Y-\text{periodic.} & \end{cases}$$





Log-log plot of the longitudinal dispersion A^*_{11} as a function of the local Péclet's number (asymptotic slope ≈ 1.7).

-VI- THE CASE OF BOUNDED DOMAINS

Consider now a bounded domain Ω with a Dirichlet boundary condition on $\partial\Omega$.

To study the long time behavior we consider the eigenvalue problem

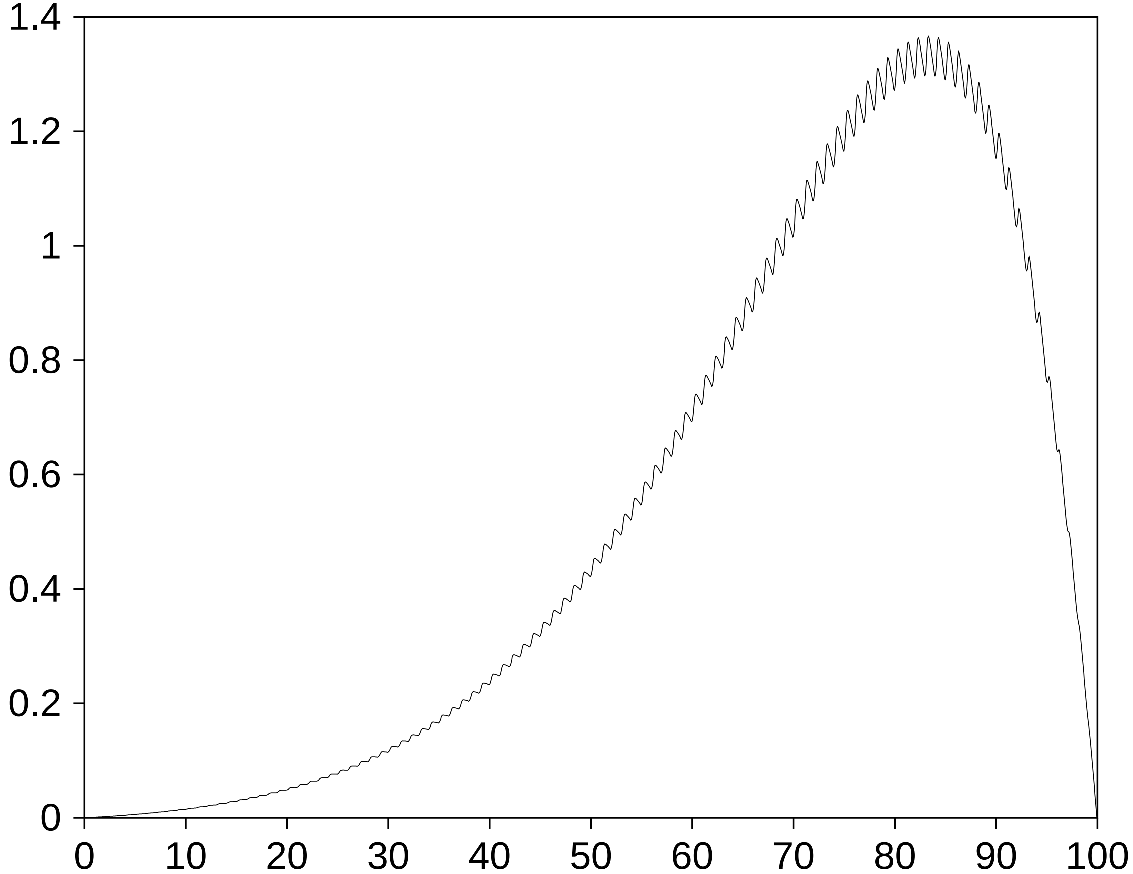
$$\begin{cases} \frac{1}{\epsilon} b\left(\frac{x}{\epsilon}\right) \cdot \nabla u_\epsilon - \mathrm{div}(D\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon) = \lambda_\epsilon u_\epsilon \quad \text{in} \ \ \Omega_\epsilon \\ u_\epsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_\epsilon, \\ D\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot n = 0 \ \text{ on } \ \partial\Omega_\epsilon \setminus \partial\Omega \end{cases}$$

where λ_ϵ and u_ϵ are the first eigenvalue and eigenfunction (which exists by the Krein-Rutman theorem).

Assumptions:

✗ Stationary incompressible periodic flow $\mathrm{div}_y b = 0$ in Y^*, $b \cdot n = 0$ on $\partial\mathcal{O}$

✗ Periodic symmetric coercive diffusion D



Typical expected behavior of the first eigenfunction:
because of the large drift it behaves like a boundary layer.

Following Bensoussan-Lions-Papanicolaou and Capdeboscq, for $\theta \in \mathbb{R}^N$, we introduce the spectral cell problem

$$\begin{cases} -\mathrm{div}_y \left(D(y) \nabla_y \psi_\theta \right) + b(y) \cdot \nabla_y \psi_\theta = \lambda(\theta) \psi_\theta & \text{in } Y^* \\ D(y) \nabla_y \psi_\theta \cdot n = 0 \quad \text{on } \partial\mathcal{O} \\ y \to \psi_\theta(y) e^{-\theta \cdot y} \quad Y\text{-periodic} \end{cases}$$

where $\lambda(\theta)$ and ψ_θ are the first eigenvalue and eigenfunction.

We also introduce the adjoint spectral cell problem

$$\begin{cases} -\mathrm{div}_y \left(D(y) \nabla_y \psi^*_\theta \right) - \mathrm{div}_{\mathrm{y}} \left(b(y) \psi^*_\theta \right) = \lambda(\theta) \psi^*_\theta & \text{in } Y^* \\ D(y) \nabla_y \psi^*_\theta \cdot n = 0 \quad \text{on } \partial\mathcal{O} \\ y \to \psi^*_\theta(y) e^{+\theta \cdot y} \quad Y\text{-periodic} \end{cases}$$

The first eigenfunctions ψ_θ and ψ^*_θ can be chosen positive and normalized by

$$\int_{\mathbf{T}^N} |\psi_\theta(y) e^{-\theta \cdot y}|^2 dy = 1 \quad \text{and} \quad \int_{\mathbf{T}^N} \psi_\theta(y) \psi^*_\theta(y) \, dy = 1$$

Lemma. The function $\theta \to \lambda(\theta)$ is strictly concave from $\mathbb{R}^N$ into $\mathbb{R}$ and admits a maximum λ_∞ which is obtained for a unique $\theta = \theta_\infty$.

Denoting $\psi_\infty = \psi_{\theta_\infty}$ and $\psi^*_\infty = \psi^*_{\theta_\infty}$, the vector field

$$\tilde{b}(y) = \psi^*_\infty \psi_\infty b(y) + \psi_\infty D^* \nabla_y \psi^*_\infty(y) - \psi^*_\infty D \nabla_y \psi_\infty(y)$$

satisfies

$$\mathrm{div_y} \tilde{b} = 0 \text{ in } \mathbf{T}^N, \quad \int_{\mathbf{T}^N} \tilde{b}(y)\, dy = 0.$$

Change of unknown

Define a new unknown function

$$\tilde{u}_\epsilon(x) = \frac{u_\epsilon(x)}{\psi_\infty\left(\frac{x}{\epsilon}\right)}$$

and multiply the equation by $\psi_\infty^*\left(\frac{x}{\epsilon}\right)$. We obtain

$$\left\{\begin{array}{l} \frac{1}{\epsilon}\tilde{b}\left(\frac{x}{\epsilon}\right)\cdot\nabla\tilde{u}_\epsilon - \operatorname{div}(\tilde{D}\left(\frac{x}{\epsilon}\right)\nabla\tilde{u}_\epsilon) = \mu_\epsilon(\psi_\infty\psi_\infty^*)\left(\frac{x}{\epsilon}\right)\tilde{u}_\epsilon \quad \text{in} \quad \Omega_\epsilon \\ \tilde{u}_\epsilon = 0 \quad \text{on } \partial\Omega\cap\partial\Omega_\epsilon, \\ \tilde{D}\left(\frac{x}{\epsilon}\right)\nabla\tilde{u}_\epsilon\cdot n = 0 \quad \text{on} \quad \partial\Omega_\epsilon\setminus\partial\Omega \end{array}\right.$$

with $\tilde{D}(y) = \psi_\infty(y)\psi_\infty^*(y)D(y)$ and

$$\mu_\epsilon = \lambda_\epsilon - \frac{\lambda(\theta_\infty)}{\epsilon^2}$$

Homogenization

Theorem 1. Since the new velocity $\tilde{b}$ is divergence-free and has zero average, classical periodic homogenization can be applied

$$\lim_{\epsilon \to 0} \mu_\epsilon = \mu, \quad \tilde{u}_\epsilon \rightharpoonup \tilde{u} \text{ weakly in } H_0^1(\Omega)$$

where $(\mu, \tilde{u})$ is the first eigencouple of

$$\begin{cases} -\text{div}(\tilde{D}^* \nabla \tilde{u}) = \mu \tilde{u} & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \partial\Omega \end{cases}$$

with $\tilde{D}^*$ the usual homogenized matrix for $\tilde{D}$.

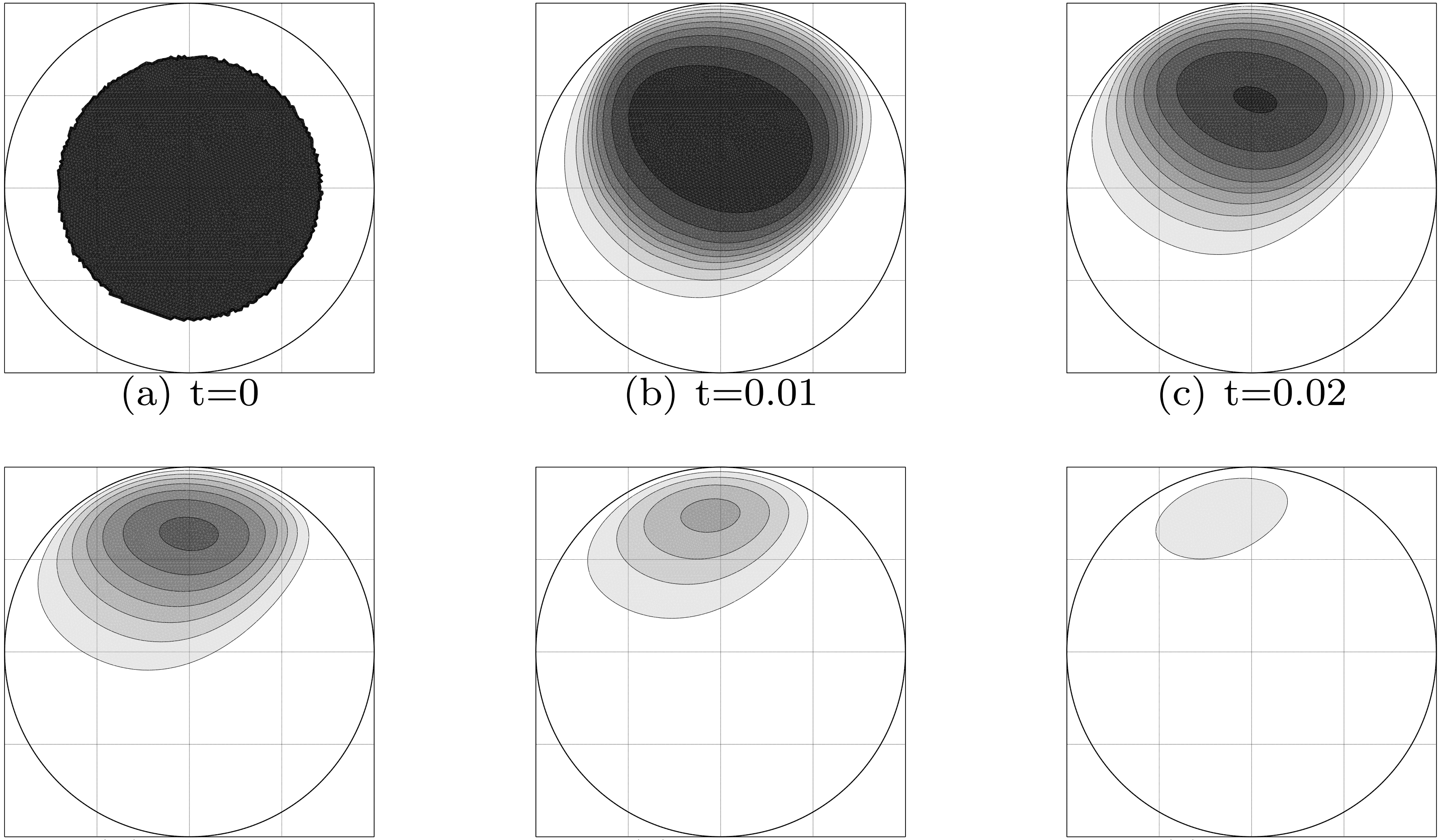
Conclusion

Theorem 2. The asymptotic behavior of the first eigencouple of the original problem is

$$\lambda_\epsilon = \frac{\lambda(\theta_\infty)}{\epsilon^2} + \mu + o(1), \qquad u_\epsilon(x) \approx e^{+\frac{\theta_\infty \cdot x}{\epsilon}} \phi_\infty\left(\frac{x}{\epsilon}\right) \tilde{u}(x)$$

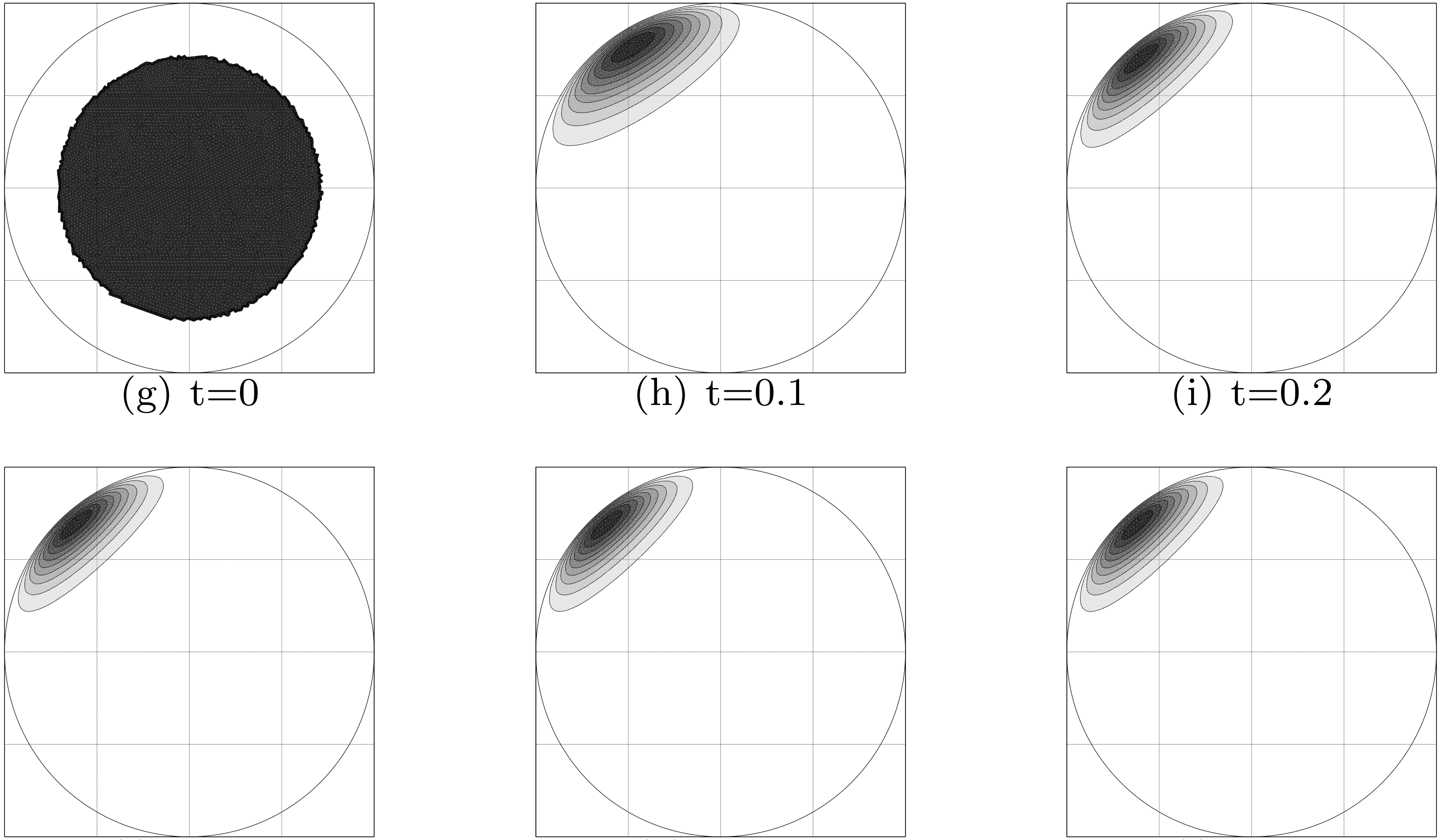
where $\phi_\infty(y) = \psi_{\theta_\infty}(y) e^{-\theta_\infty \cdot y}$ is a Y-periodic function.

Remark. The direction of concentration θ_∞ is different from the drift $b^* = \int_{Y^*} b(y)\, dy$!



(a) t=0 (b) t=0.01 (c) t=0.02 (d) t=0.03 (e) t=0.04 (f) t=0.05

Isovalues of u_ϵ for various t in the parabolic case
(computations by I. Pankratova)



(g) t=0 (h) t=0.1 (i) t=0.2

(j) t=0.3 (k) t=0.4 (l) t=0.5

Rescaled isovalues of u_ϵ for larger times t in the parabolic case
(computations by I. Pankratova)