

Mini-project for MAP411

Damping for the elastic wave equation

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Abstract

Résumé: Le but de ce projet est une investigation des techniques d'amortissement pour l'équation d'onde. Cette étude est intéressante tant du point de vue de la physique et que de l'analyse numérique. En effet l'amortissement apparaît fréquemment dans les systèmes avec friction. Du point de vue mathématique, l'amortissement donne une meilleure régularité de l'équation élastique qui est importante par exemple pour l'analyse théorique des interactions fluide-structure.

Abstract

Abstract: The goal of this project is to investigate damping techniques for the elastic wave equation. These are interesting from physical and numerical/mathematical point of view. The reason is that damping (e.g., friction) is often present in a system. Mathematically, damping leads to better regularity of the solid PDE equation, which is important for theoretical analysis of fluid-structure interaction for instance.

1 Problem statement

Let $\Omega = (0, 1)$ to be an open and bounded set. Furthermore, let $\partial\Omega$ be the boundary. The time interval is given by $I := (0, T]$ for some fixed end time value $T > 0$.

We consider the initial/boundary-value problem:

Formulation 1.1. Let $f : \Omega \times I \rightarrow \mathbb{R}$ and $g, h : \Omega \rightarrow \mathbb{R}$ be given. Furthermore, let $\gamma_w, \gamma_s \geq 0$. We seek the unknown function $u : \bar{\Omega} \times I \rightarrow \mathbb{R}$ such that

$$\partial_t^2 u + Lu + \gamma_w \partial_t u + \gamma_s \partial_t Lu = f \quad \text{in } \Omega \times I, \quad (1)$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T], \quad (2)$$

$$u(0) = g \quad \text{in } \Omega \times \{t = 0\}, \quad (3)$$

$$\partial_t u(0) = h \quad \text{in } \Omega \times \{t = 0\}. \quad (4)$$

Furthermore, the linear second-order differential operator is defined by:

$$Lu := - \sum_{i,j=1}^n \partial_{x_j} (a_{ij}(x, t) \partial_{x_i} u) = -\nabla \cdot (a \nabla u),$$

for a given (possibly spatially and time-dependent) coefficient function a_{ij} . In 1D (one dimension) it holds:

$$Lu := -\frac{d}{dx}(a(x)u'(x)).$$

Question 1. Give an explication/idea (mathematically and/or physically) why the first term with γ_w is called weak damping and the second one with γ_s is called strong damping.

Question 2. Derive a mixed form of Formulation 1.1 in which the velocity $v := \partial_t u$ appears explicitly.

Hint: Replace whenever possible $\partial_t u$ by v and create a 2nd equation: $\partial_t u - v = 0$.

In the following, we now derive a variational (weak) form by multiplying with suitable test functions from suitable function spaces.

Question 3. Derive a weak formulation using the correct function spaces of Formulation 1.1 and the second form derived in Question 2. What do you observe with respect to the damping terms when we assume not only Dirichlet boundaries but also Neumann boundaries, i.e., $\partial\Omega = \Gamma_D \cup \Gamma_N$? Hint. The last question is meant w.r.t. to partial integration of the second order operator.

Hint to Question 3: For the Laplace problem, the weak formulation is obtained as:

$$-\Delta u = f \quad \Rightarrow \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx - \int_{\Gamma_N} \partial_n u \varphi \, ds = \int_{\Omega} f \varphi \, dx.$$

2 Theory

In this section, we look a little bit on the theory of the wave equation. The main characteristic is that in contrast to elliptic and parabolic problems, no maximum principle holds true and that energy conservation holds true.

Question 4 (Optional - since not a topic in MAP411). Recapitulate well-posedness and justify existence and uniqueness of Formulation 1.1 for the case $\gamma_w = \gamma_s = 0$. What are the regularity requirements for u and $\partial_t u$?

Question 5 (Optional/Difficult!). We add now damping terms $\gamma_w > 0$ and $\gamma_s > 0$. How do the regularity requirements change for u and $\partial_t u$?

Question 6. An important aspect of the pure undamped wave equation ($\gamma_w = \gamma_s = 0$) is energy conservation. Thus we take the weak form and take as test function $\partial_t u$. Please derive the energy conservation law. We add now solid damping and perform the same derivation. Give an interpretation of the resulting law. Does energy conservation still hold? Give also a physical explanation (for instance in form of an example).

3 Discretization

For discretization we need to discretize in space and time. For spatial discretization we adopt finite elements. For temporal discretization we adopt finite differences.

A rather general form of time-stepping can be achieved by using a One-Step- θ scheme in which a parameter $\theta \in [0, 1]$ is introduced. As an example we consider the heat equation:

$$\begin{aligned} \partial_t u - \Delta u &= f \\ \Rightarrow \frac{u^n - u^{n-1}}{k} - \theta \Delta u^n - (1 - \theta) \Delta u^{n-1} &= \theta f^n + (1 - \theta) f^{n-1}, \end{aligned}$$

where $k = t^n - t^{n-1}$ is the time step size and $u^n := u(t^n)$ the current time step solution and $u^{n-1} := u(t^{n-1})$ the previous time step solution.

For $\theta < 0.5$ we obtain explicit schemes, for instance for $\theta = 0$ the (forward) Euler scheme. For $\theta \geq 0.5$ we obtain implicit time-stepping schemes. For $\theta = 0.5$ we obtain the (2nd order) Crank-Nicolson scheme, and $\theta = 1$ yields the backward Euler scheme.

Question 7 (Optional - since ODE numerics is not done in MAP411). Recapitulate (with or without rigorous mathematical proofs) the main properties of the above schemes: A-stability, global stability, strong A-stability, and the convergence order (first or second). Hint: Definition and examples to A-stability can be found in

http://www.cmap.polytechnique.fr/~wick/map_502_winter_2016_eng1.html

Question 8. Based on the mixed weak form from Question 2, derive a One-Step- θ scheme based on the weak mixed formulation.

Question 9. With regard to energy conservation, derive an energy law for temporally discretized (but spatially still continuous) system. How does the system depends on time-stepping?

Question 10. Take the previous system and discretize now in space with the finite element method as shown in the lecture MAP411.

4 Simulations

As we have seen in the previous sections, damping in the wave equation can be achieved in two different ways: physically and numerically. It is important however, that numerical schemes represent as well as possible the properties of the original system. Thus, the undamped wave equation, should not be artificially damped through numerical discretization. To get a feeling of such findings, we perform several numerical tests in this final section. The main goal is to get a better understanding between physical and numerical damping.

Question 11. Implement the temporally and spatially discretized mixed formulation in 1D (one space dimension).

For the numerical simulations in 1D, the boundary $\partial\Omega$ consists actually of two points: $u(0)$ and $u(1)$. On $u(0)$ we prescribe a time-dependent non-homogeneous Dirichlet condition:

$$u(0) = g(t) = \sin(t).$$

On $u(1)$ we either fix the solution $u(1) = 0$ or we leave it free (homogeneous Neumann conditions). The right hand side is chosen as $f = 0$. The coefficient vector is given by $a = a_{ij} = 1$. The two initial conditions are $u(0) = v(0) = 0$.

Question 12. 1. Run simulations using the Crank-Nicolson scheme for the undamped wave equation.

2. Run simulations using backward Euler for the undamped wave equation.

3. Run simulations using Crank-Nicolson for the damped wave equation.

4. Run simulations using backward Euler for the damped wave equation.

Question 13. Compare the results to all four numerical simulations and interpret them.

5 Pour aller plus loin ...

In some applications (for instance in well-posedness analysis of fluid-structure interaction), it is a problem that the velocity v has not enough regularity. On the other hand, damping helps to increase the regularity as we have studied before. Let us try to develop a scheme that maybe helps to address this difficulty.

Question 14 (Difficult!). Develop and analyze the following scheme: Run wave undamped equation with Crank-Nicolson and add ‘from time to time’ damping. What do you observe?

Another important question is concerned with the reflection of waves on (artificial) boundaries. Let us finally create some imagination what could be done here:

Question 15 (Difficult!). We have a string and oscillate the string up and down on the left boundary. Due to this oscillation, the string will move in wave form. And these waves are transported from left to the right. At some time, these waves reach the right boundary. It is of importance which kind of boundary conditions are prescribed on this right boundary since Dirichlet conditions and Neumann conditions both will reflect these ‘outgoing’ waves back to the left. Question: What could be done to avoid reflection of outgoing waves?