1 Parametric optimization: 10 points

1. The variational formulation is: find $u \in H^1(\Omega)$ such that, for any test function $q \in H^1(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla q \, dx + \int_{\partial \Omega} kuq \, ds = \int_{\Omega} f q \, dx.$$ 

The Lagrangian is the sum of the objective function and of the variational formulation

$$\mathcal{L}(h, v, q) = \int_{\Omega} j(v(x)) \, dx + \int_{\Omega} (\nabla v \cdot \nabla q - f q) \, dx + \int_{\partial \Omega} kvq \, ds.$$ 

2. The partial derivative of the Lagrangian with respect to $v$ is

$$\left\langle \frac{\partial \mathcal{L}}{\partial v}, \psi \right\rangle = \int_{\Omega} j'(v) \psi \, dx + \int_{\Omega} \nabla \psi \cdot \nabla q \, dx + \int_{\partial \Omega} k \psi q \, ds.$$ 

Equating it to 0 and taking the value $v = u$ yields the variational formulation for the adjoint $p \in H^1(\Omega)$ where $\psi \in H^1(\Omega)$ is any test function. Disintegrating by parts yields the boundary value problem satisfied by $p$

$$\left\{ \begin{array}{ll}
-\Delta p = -j'(u) & \text{in } \Omega, \\
\frac{\partial p}{\partial n} + kp = 0 & \text{on } \partial \Omega.
\end{array} \right.$$ 

3. The formal derivative of $J(k)$ is given by the formula

$$\left\langle J'(k), \theta \right\rangle = \left\langle \frac{\partial \mathcal{L}}{\partial k}, (k, u, p), \theta \right\rangle.$$ 

Thus a simple computation (because the Lagrangian depends linearly on $k$!) yields

$$\int_{\Omega} J'(k) \, \theta \, dx = \int_{\partial \Omega} \theta up \, ds,$$

or equivalently

$$J'(k) = up \quad \text{on } \partial \Omega.$$
4. When \( j(v) = -fv \), we find \( p = u \) so \( J'(k) = u^2 \geq 0 \). Since the derivative is always non-negative, the optimality condition is satisfied for the minimal value of \( k \), namely \( k(x) = k_{min} \) on \( \partial \Omega \). Therefore, we expect this value to be the minimum of the objective function \( J(k) = -\int_{\Omega} f(x) u(x) \, dx \).

To make the proof rigorous, we rewrite \( J(k) \) as the minimum of the (primal) energy

\[
-\int_{\Omega} f(x) u(x) \, dx = \min_{v \in H^1(\Omega)} \int_{\Omega} |\nabla v|^2 \, dx + \int_{\partial \Omega} k v^2 \, ds - 2 \int_{\Omega} f v \, dx.
\]

The optimal design problem is thus equivalent to a double minimization

\[
\min_{(k,v) \in U_{ad} \times H^1(\Omega)} \int_{\Omega} |\nabla v|^2 \, dx + \int_{\partial \Omega} k v^2 \, ds - 2 \int_{\Omega} f v \, dx.
\]

For any fixed \( v \), the minimal value is clearly attained by \( k(x) = k_{min} \).

Thus \( k(x) = k_{min} \) is a global minimizer of the optimal design problem. (It may be not unique at those points \( x \in \partial \Omega \) where \( u(x) = 0 \).)

2 Geometric optimization: 7 points

1. By the chain rule lemma, the shape derivative of \( M_{\Omega}(f) \) is, for any vector field \( \theta \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2) \),

\[
M_{\Omega}(f)'(\theta) = \frac{1}{|\Omega|} \int_{\partial \Omega} f \theta \cdot n \, ds - \frac{1}{|\Omega|^2} \int_{\partial \Omega} \theta \cdot n \, ds \int_{\Omega} f(x) \, dx,
\]

which simplifies as

\[
M_{\Omega}(f)'(\theta) = \frac{1}{|\Omega|} \int_{\partial \Omega} (f - M_{\Omega}(f)) \theta \cdot n \, ds.
\]

Clearly, the derivative is zero (for any \( \theta \)) if and only if \( f = M_{\Omega}(f) \) on \( \partial \Omega \).

2. We rewrite the function \( J(\Omega) \) as

\[
J(\Omega) = \frac{1}{|\Omega|} \int_{\Omega} f^2(x) \, dx - (M_{\Omega}(f))^2.
\]

We deduce from the previous question that

\[
J'(\Omega)(\theta) = \frac{1}{|\Omega|} \int_{\partial \Omega} (f^2 - M_{\Omega}(f^2)) \theta \cdot n \, ds - 2M_{\Omega}(f) \frac{1}{|\Omega|} \int_{\partial \Omega} (f - M_{\Omega}(f)) \theta \cdot n \, ds.
\]

Recombining terms yields

\[
J'(\Omega)(\theta) = \frac{1}{|\Omega|} \int_{\partial \Omega} ((f - M_{\Omega}(f))^2 + (M_{\Omega}(f))^2 - M_{\Omega}(f^2)) \theta \cdot n \, ds.
\]
By Cauchy-Schwartz inequality we have $(\mathcal{M}_\Omega(f))^2 \leq \mathcal{M}_\Omega(f^2)$ and the inequality is strict if $f$ is not constant on $\Omega$. Therefore, if $f$ is not constant and $f = \mathcal{M}_\Omega(f)$ on $\partial \Omega$, we deduce that $J'(\Omega)(\theta) < 0$ if the domain increases, namely when $\theta \cdot n > 0$ on $\partial \Omega$. Thus, if $\Omega$ is such that $J((\text{Id} + \theta)\Omega) \leq \epsilon$ for a small enough $\theta$ satisfying $\theta \cdot n > 0$ on $\partial \Omega$, we still have $J((\text{Id} + \theta)\Omega) \leq \epsilon$ while the volume increases, $|((\text{Id} + \theta)\Omega)| > |\Omega|$. In other words, $(\text{Id} + \theta)\Omega$ is a better admissible design.

3. If the constraint is inactive, i.e., $J(\Omega) < \epsilon$, for a maximizer $\Omega$ with finite volume, then we can slightly increase its volume while keeping the constraint satisfied, therefore contradicting the assumption that $\Omega$ was a maximizer. Thus, for a finite-volume maximizer, the constraint must be active, i.e., $J(\Omega) = \epsilon$. In such a case, there exists a non-negative Lagrange multiplier $\lambda \geq 0$ such that, for any $\theta \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$,

$$\frac{\lambda}{|\Omega|} \int_{\partial \Omega} \left( (f - \mathcal{M}_\Omega(f))^2 + (\mathcal{M}_\Omega(f))^2 - \mathcal{M}_\Omega(f^2) \right) \theta \cdot n \, ds + \int_{\partial \Omega} \theta \cdot n \, ds = 0.$$ 

In other words, the optimality condition is

$$\frac{\lambda}{|\Omega|} \left( (f - \mathcal{M}_\Omega(f))^2 + (\mathcal{M}_\Omega(f))^2 - \mathcal{M}_\Omega(f^2) \right) + 1 = 0 \quad \text{on } \partial \Omega.$$

3 Homogenization: 3 points

In space dimension $N = 2$, for an isotropic homogenized tensor $A^* = a^* \text{Id}$, the Hashin-Shtrikman upper bound reduces to

$$\frac{2}{\beta - a^*} \leq \frac{1}{\beta - \lambda^-} + \frac{1}{\beta - \lambda^+}$$

with $\lambda^- = \left( \frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1}$ and $\lambda^+ = \theta \alpha + (1 - \theta) \beta$. Taking $\alpha = 0$ yields

$$\frac{2}{\beta - a^*} \leq \frac{1}{\beta} + \frac{1}{\theta \beta}.$$ 

A simple calculation gives the result $a^* \leq \frac{1-\theta}{1+\theta} \beta$. 

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