1 Parametric optimization: 10 points

1. Writing \( v = \langle u'(h), k \rangle \) and differentiating problem (1) yields
\[
\begin{cases}
-\text{div} (h \nabla v) = \text{div} (k \nabla u) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

2. Writing \( w = \langle v'(h), \tilde{k} \rangle \) and differentiating the previous problem leads to
\[
\begin{cases}
-\text{div} (h \nabla w) = \text{div} (\tilde{k} \nabla v) + \text{div} (k \nabla \tilde{v}) & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \tilde{v} = \langle u'(h), \tilde{k} \rangle \). Since \( v \) depends linearly on \( k \) and \( \tilde{v} \) depends linearly on \( \tilde{k} \) through the same linear operator, the right hand side of the above equation is symmetric in \((k, \tilde{k})\), and so is \( w \).

3. The first order derivative is
\[
\langle J'(h), k \rangle = \int_{\Omega} f v \, dx
\]
and the second order derivative is
\[
\langle J''(h), (k, \tilde{k}) \rangle = \int_{\Omega} f w \, dx.
\]

4. To eliminate \( w \) in the formula for \( J'' \) we first multiply the equation for \( u \) by \( w \) and integrate by parts
\[
\int_{\Omega} h \nabla u \cdot \nabla w \, dx = \int_{\Omega} f w \, dx.
\]
Second, we multiply the equation for \( w \) by \( u \) and integrate by parts
\[
\int_{\Omega} h \nabla w \cdot \nabla u \, dx = - \int_{\Omega} k \nabla \tilde{v} \cdot \nabla u \, dx - \int_{\Omega} \tilde{k} \nabla \tilde{v} \cdot \nabla u \, dx.
\]
Then, by comparison we get
\[
\langle J''(h), (k, \tilde{k}) \rangle = \int_{\Omega} h \nabla u \cdot \nabla w \, dx = - \int_{\Omega} k \nabla \tilde{v} \cdot \nabla u \, dx - \int_{\Omega} \tilde{k} \nabla \tilde{v} \cdot \nabla u \, dx.
\]
This last formula is symmetric in \((k, \tilde{k})\) by the same reasons as in question 2.
5. We multiply the equation for $v$ by $v$ to get

$$
\int_{\Omega} h \nabla v \cdot \nabla v \, dx = - \int_{\Omega} k \nabla v \cdot \nabla u \, dx.
$$

When $\tilde{k} = k$, we deduce from the above that

$$
\langle J''(h), (k, k) \rangle = -2 \int_{\Omega} k \nabla v \cdot \nabla u \, dx = 2 \int_{\Omega} h \nabla v \cdot \nabla v \, dx \geq 0.
$$

It implies that the objective function $h \to J(h)$ is convex. In particular, it implies that any local minimizer is actually a global minimizer.

2 Geometric optimization: 10 points

1. Since the boundary conditions are fixed (only the subdomain $\Omega$ is varying), we can use the standard variational formulation of problem (3) to build the Lagrangian. Introduce the space

$$
V = \{ \phi \in H^1(D) \text{ such that } \phi = 0 \text{ on } \Gamma_D \}.
$$

Thus, for any functions $v, q \in V$ and any subset $\Omega \subset D$, we define the Lagrangian

$$
\mathcal{L}(\Omega, v, q) = \frac{1}{2} \int_{\Gamma_N} |v - u_0|^2 \, ds + \int_{D} (\nabla v \cdot \nabla q + \chi_{\Omega} v q) \, dx - \int_{\Gamma_N} g q \, ds.
$$

2. To get the adjoint problem we differentiate the Lagrangian with respect to $v$ and set this partial derivative equal to 0. For any $\phi \in V$ we have

$$
\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \rangle = \int_{\Gamma_N} (v - u_0) \phi \, ds + \int_{D} (\nabla \phi \cdot \nabla q + \chi_{\Omega} \phi q) \, dx.
$$

By integration by parts, since $\phi = 0$ on $\Gamma_D$, we deduce

$$
\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \rangle = \int_{\Gamma_N} (v - u_0 + \frac{\partial q}{\partial n}) \phi \, ds + \int_{D} (-\Delta q + \chi_{\Omega} q) \phi \, dx.
$$

We first take a test function $\phi$ with compact support in $\Omega$, so we deduce that the optimal value of $q$, the adjoint $p$, satisfies

$$
-\Delta p + \chi_{\Omega} p = 0 \quad \text{in } D.
$$

Then we take $\phi \neq 0$ on $\Gamma_N$ so that

$$
\frac{\partial p}{\partial n} = -(u - u_0) \quad \text{on } \Gamma_N.
$$
Eventually, since $p \in V$ we have $p = 0$ on $\Gamma_D$. The adjoint problem is thus
\[
\begin{cases}
-\Delta p + \chi_{\Omega} p = 0 & \text{in } D, \\
p = 0 & \text{on } \Gamma_D \\
\frac{\partial p}{\partial n} = -(u - u_0) & \text{on } \Gamma_N.
\end{cases}
\]

3. Formally we know that the shape derivative is given by
\[J'(\Omega)(\theta) = \frac{\partial L}{\partial \Omega}(\Omega, u, p)(\theta).
\]
The only term which depends on $\Omega$ in $L(\Omega, v, q)$ is
\[\int_D \chi_{\Omega} v q \, dx = \int_{\Omega} v q \, dx.
\]
We compute its derivative
\[\frac{\partial L}{\partial \Omega}(\Omega, v, q)(\theta) = \int_{\partial \Omega} v q \theta \cdot n \, ds.
\]
Therefore
\[J'(\Omega)(\theta) = \int_{\partial \Omega} u p \theta \cdot n \, ds.
\]

4. We multiply the equation for $u$ by $u^- = \min(u, 0)$ to get
\[\int_D (\nabla u \cdot \nabla u^- + \chi_{\Omega} uu^-) \, dx = \int_{\Gamma_N} gu^- \, ds.
\]
Since $u^- = 0$ if $u > 0$, we deduce
\[\int_D (|\nabla u^-|^2 + \chi_{\Omega}|u^-|^2) \, dx = \int_{\Gamma_N} gu^- \, ds.
\]
The left hand side is non-negative while the right hand side is non-positive because $g \geq 0$ and $u^- \leq 0$. Therefore all terms are zero which implies that $u^- = 0$ everywhere in $D$. In other words, we have proved that $u \geq 0$ in $D$.

5. If $g \geq 0$, then $u \geq 0$. If $u \geq u_0$, then a similar argument as in the previous question shows that $p \leq 0$. Therefore, if $\theta \cdot n \geq 0$, we deduce
\[J'(\Omega)(\theta) = \int_{\partial \Omega} u p \theta \cdot n \, ds \leq 0.
\]
In other words the objective function decreases if the subdomain $\Omega$ is enlarged.

Of course, if $u \leq u_0$, then $p \geq 0$ and $J'(\Omega)(\theta) \leq 0$ for $\theta \cdot n \leq 0$ which means that the objective function decreases if the subdomain $\Omega$ is reduced.