1 Parametric optimization: 14 points

1. Writing \( v = \langle u'(h), k \rangle \), \( \Lambda = \langle \lambda'(h), k \rangle \) and differentiating problem (1) yields

\[
\begin{cases}
- \text{div} (h \nabla v) - \text{div} (k \nabla u) = \lambda \rho h v + \lambda \rho k u + \Lambda \rho h u & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Differentiating the normalization condition \( \int_{\Omega} \rho hu^2 dx = 1 \) we obtain

\[
\int_{\Omega} \rho u(2hv + ku) dx = 0.
\]

2. Multiplying the above problem for \( v \) by \( u \) and integrating by parts leads to

\[
\int_{\Omega} (h \nabla v \cdot \nabla u + k \nabla u \cdot \nabla u) dx = \int_{\Omega} (\lambda \rho h v u + \lambda \rho k u^2 + \Lambda \rho h u^2) dx.
\]

From the equation for \( u \) we know that

\[
\int_{\Omega} h \nabla v \cdot \nabla u dx = \int_{\Omega} \lambda \rho h v u dx.
\]

Therefore, from the normalization \( \int_{\Omega} \rho hu^2 dx = 1 \), we deduce

\[
\Lambda = \int_{\Omega} k (|u|^2 - \lambda \rho u^2) dx.
\]

In other words, we found \( \lambda'(h) = |\nabla u|^2 - \lambda \rho u^2 \) which is a function in \( L^1(\Omega) \).

3. From the previous question and since, by assumption, \( \nabla u \neq 0 \) on \( \partial \Omega \) and \( u = 0 \) on \( \partial \Omega \), we find \( \lambda'(h) > 0 \) on \( \partial \Omega \) (and thus by continuity on a neighborhood of \( \partial \Omega \)). Therefore, for an optimal thickness \( h \), the lower constraint \( h \geq h_{\text{min}} \) must be saturated in this region, i.e., \( h = h_{\text{min}} \) near \( \partial \Omega \). On the contrary, at the point where \( u \) is maximum, we have \( \nabla u = 0 \) and \( u > 0 \), thus \( \lambda'(h) < 0 \). It implies that, in the vicinity of the maximum of \( u \), we must have \( h = h_{\text{max}} \).
4. The objective function is

\[ J(h) = \int_{\Omega} j\left( \frac{u(h)}{\|u(h)\|} \right) \, dx. \]

It is clear that, since \( j \) is even, i.e., \( j(-w) = j(w) \), we have for any \( t \neq 0 \)

\[ j\left( \frac{tu(h)}{\|tu(h)\|} \right) = j\left( \frac{u(h)}{\|u(h)\|} \right), \]

so that the objective function is independent from the normalization of the eigenfunction.

For \( h \in \mathcal{U}_{ad}, \hat{\lambda} \in \mathbb{R}, \hat{u} \in H^1_0(\Omega) \) and \( \hat{p} \in H^1_0(\Omega) \) we define the Lagrangian

\[ \mathcal{L}(h, \hat{\lambda}, \hat{u}, \hat{p}) = \int_{\Omega} j\left( \frac{\hat{u}}{\|\hat{u}\|} \right) \, dx + \int_{\Omega} \left( h\nabla \hat{u} \cdot \nabla \hat{p} - \hat{\lambda} \rho h \hat{u} \hat{p} \right) \, dx. \]

5. For \( \hat{u} \in L^2(\Omega) \) we define

\[ F(\hat{u}) = \int_{\Omega} j\left( \frac{\hat{u}}{\|\hat{u}\|} \right) \, dx. \]

By the chain rule lemma, since the derivative of \( \|\hat{u}\| \) in the direction of \( \phi \) is \( \langle \hat{u}, \phi \rangle / \|\hat{u}\| \) with the notation \( \langle \hat{u}, \phi \rangle = \int_{\Omega} \hat{u} \phi \, dx \), we obtain

\[ \langle F'(\hat{u}), \phi \rangle = \int_{\Omega} F'(\hat{u}) \phi \, dx = \int_{\Omega} j'\left( \frac{\hat{u}}{\|\hat{u}\|} \right) \left( \frac{\phi}{\|\hat{u}\|} - \frac{\langle \hat{u}, \phi \rangle}{\|\hat{u}\|^3} \hat{u} \right) \, dx. \]

Clearly we find \( \langle F'(\hat{u}), \hat{u} \rangle = 0 \). Equivalently,

\[ \int_{\Omega} F'(\hat{u}) \phi \, dx = \int_{\Omega} j'\left( \frac{\hat{u}}{\|\hat{u}\|} \right) \phi \|\hat{u}\| \, dx - \left( \int_{\Omega} j'\left( \frac{\hat{u}}{\|\hat{u}\|} \right) \frac{\hat{u}}{\|\hat{u}\|^3} \, dx \right) \int_{\Omega} \hat{u} \phi \, dx \]

which implies

\[ F'(\hat{u}) = \frac{1}{\|\hat{u}\|^2} j'\left( \frac{\hat{u}}{\|\hat{u}\|} \right) - \left( \int_{\Omega} j'\left( \frac{\hat{u}}{\|\hat{u}\|} \right) \frac{\hat{u}}{\|\hat{u}\|^3} \, dx \right) \hat{u}. \]

6. The variational formulation of the adjoint equation is, by definition,

\[ \langle \frac{\mathcal{L}}{\partial \hat{u}}(h, \lambda, u, p), \phi \rangle = 0 \quad \forall \phi \in H^1_0(\Omega). \]

We compute

\[ \langle \frac{\mathcal{L}}{\partial \hat{u}}(h, \lambda, u, p), \phi \rangle = \int_{\Omega} F'(u) \phi \, dx + \int_{\Omega} (h \nabla \phi \cdot \nabla p - \lambda \rho \phi \phi) \, dx. \]
7. By integration by parts we find that the adjoint \( p \) is a solution of
\[
\begin{align*}
-\text{div} \left( h \nabla p \right) - \lambda \rho h p &= -\frac{1}{\|u\|} j' \left( \frac{u}{\|u\|} \right) + \alpha u \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
with
\[\alpha = \left( \int_{\Omega} j' \left( \frac{u}{\|u\|} \right) \frac{u}{\|u\|^2} \, dx \right)\]
We check that the right hand side is orthogonal to \( u \) since
\[\int_{\Omega} \frac{u}{\|u\|} j' \left( \frac{u}{\|u\|} \right) \, dx = \alpha \int_{\Omega} u^2 \, dx = \alpha \|u\|^2.\]
Clearly, if \( p \) is a solution, then \( p + Cu \) is another possible solution. To determine the value of the constant \( C \) we use
\[\frac{L}{\partial \lambda}(h, \lambda, u, p) = -\int_{\Omega} \rho h u p \, dx = 0\]
which implies that \( C = -\int_{\Omega} \rho h u p \, dx \).

8. The derivative satisfies
\[\langle J'(h), k \rangle = \langle \frac{\partial L}{\partial h}(h, \lambda, u, p), k \rangle.\]
We compute
\[\langle \frac{\partial L}{\partial h}(h, \lambda, u, p), k \rangle = \int_{\Omega} (k \nabla u \cdot \nabla p - \lambda \rho k u p) \, dx,\]
which implies
\[J'(h) = \nabla u \cdot \nabla p - \lambda \rho u p.\]

2 Geometric optimization: 6 points

1. To define the Lagrangian we introduce two Lagrange multipliers \( q \in H^1(\mathbb{R}^N) \) and \( \mu \in H^1(\mathbb{R}^N) \), which, together with \( v \in H^1(\mathbb{R}^N) \), are the arguments of \( L \)
\[L(\Omega, v, q, \mu) = \int_{\Omega} j(v) \, dx + \int_{\Omega} (\Delta v + f)q \, dx + \int_{\partial \Omega} (v - g)\mu \, ds.\]

2. To get the adjoint problem we differentiate the Lagrangian with respect to \( v \) and set this partial derivative equal to 0. Before that we perform two successive integration by parts
\[L(\Omega, v, q, \mu) = \int_{\Omega} j(v) \, dx + \int_{\Omega} (fq - \nabla q \cdot \nabla v) \, dx + \int_{\partial \Omega} \left( \frac{\partial v}{\partial n} - (v - g)\mu \right) \, ds.\]
\[ \mathcal{L}(\Omega, v, q, \mu) = \int_{\Omega} j(v) \, dx + \int_{\Omega} (f q + \Delta q v) \, dx + \int_{\partial\Omega} \left( \frac{\partial v}{\partial n} q - \frac{\partial q}{\partial n} v + (v - g)\mu \right) \, ds \]

For any \( \phi \in H^1(\mathbb{R}^N) \) we have

\[ \langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q, \mu), \phi \rangle = \int_{\Omega} j'(v) \phi \, dx + \int_{\Omega} \Delta q \phi \, dx + \int_{\partial\Omega} \left( \frac{\partial \phi}{\partial n} q - \frac{\partial q}{\partial n} \phi + \phi \mu \right) \, ds \] (1)

We first take a test function \( \phi \) with compact support in \( \Omega \), so we deduce that the optimal value of \( q \), the adjoint \( p \), satisfies

\[- \Delta p = j'(u) \quad \text{in } \Omega.\]

Then we take \( \phi = 0 \) on \( \partial\Omega \) but with no restriction on the value of \( \frac{\partial \phi}{\partial n} \) on \( \partial\Omega \), so that

\[ p = 0 \quad \text{on } \partial\Omega. \]

This yields the adjoint problem

\[ \begin{cases} - \Delta p = j'(u) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \]

Eventually, varying the trace of \( \phi \) on \( \partial\Omega \) gives the optimal value of the Lagrange multiplier

\[ \lambda = \frac{\partial p}{\partial n} \text{ on } \partial\Omega. \]

3. Formally we know that the shape derivative is given by

\[ J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u, p)(\theta). \]

We compute the partial derivative

\[ \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, v, q, \mu)(\theta) = \int_{\partial\Omega} \left( j(v) + f q - \nabla q \cdot \nabla v \right) \theta \cdot n \, ds + \int_{\partial\Omega} \left( \frac{\partial h}{\partial n} + H h \right) \theta \cdot n \, ds, \]

with \( h = \frac{\partial \mu}{\partial n} q + (v - g)\mu \). Taking into account \( p = 0 \) and \( u = g \) on \( \partial\Omega \), we deduce

\[ J'(\Omega)(\theta) = \int_{\partial\Omega} \left( j(u) - \nabla p \cdot \nabla u \right) \theta \cdot n \, ds + \int_{\partial\Omega} \left( \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} + \mu \frac{\partial (u - g)}{\partial n} \right) \theta \cdot n \, ds. \]

Since \( \lambda = \frac{\partial p}{\partial n} \) on \( \partial\Omega \) and \( \nabla \epsilon p = 0 \) on \( \partial\Omega \), it leads to

\[ J'(\Omega)(\theta) = \int_{\partial\Omega} \left( j(u) + \frac{\partial p}{\partial n} (u - g) \right) \theta \cdot n \, ds. \]