OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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February 18th, 2015

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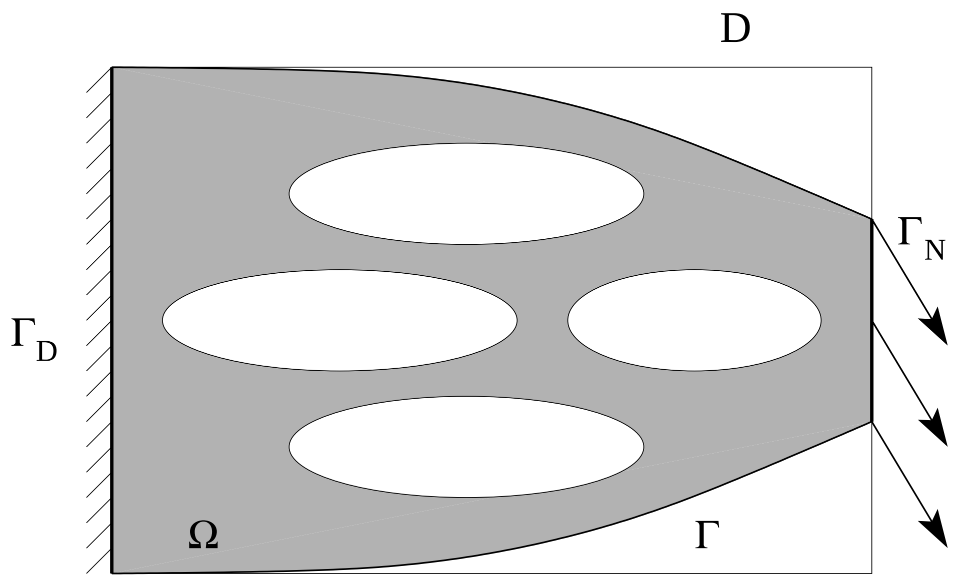
CHAPTER VII (the end)

TOPOLOGY OPTIMIZATION

BY THE HOMOGENIZATION METHOD

7.5 Shape optimization in the elasticity setting



Bounded working domain $D \in \mathbb{R}^N$ $(N = 2, 3)$.

Linear isotropic elastic material, with Hooke's law A

$$A = (\kappa - \frac{2\mu}{N}) I_2 \otimes I_2 + 2\mu I_4, \quad 0 < \kappa, \mu < +\infty$$

Homogenized formulation of shape optimization

We introduce composite structures characterized by a local volume fraction $\theta(x)$ of the phase A (taking any values in the range $[0,1]$) and an homogenized tensor $A^*(x)$, corresponding to its microstructure.

The set of admissible homogenized designs is

$$\mathcal{U}^*_{ad} = \left\{ (\theta, A^*) \in L^\infty \left(D; [0,1] \times \mathbb{R}^{N^4} \right), A^*(x) \in G_{\theta(x)} \text{ in } D \right\}.$$

The homogenized state equation is

$$\begin{cases} \sigma = A^* e(u) & \text{with } e(u) = \frac{1}{2} \left(\nabla u + (\nabla u)^t \right), \\ \operatorname{div}\sigma = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N). \end{cases}$$

The homogenized compliance is defined by

$$c(\theta, A^*) = \int_{\Gamma_N} g \cdot u \, ds.$$

The relaxed or homogenized optimization problem is

$$\min_{(\theta, A^*) \in \mathcal{U}^*_{ad}} \left\{ J(\theta, A^*) = c(\theta, A^*) + \ell \int_D \theta(x) \, dx \right\}.$$

Bad news: in the elasticity setting an explicit characterization of G_θ is still lacking !

Good news: for compliance one can replace G_θ by its explicit subset L_θ of laminated composites.

Furthermore, an optimal composite is a rank-N sequential laminate with lamination directions given locally by the eigendirections of the stress σ.

7.5.4 Homogenized formulation of shape optimization

$$\min_{\substack{div\sigma=0 \text{ in } D\\ \sigma n=g \text{ on } \Gamma_N\\ \sigma n=0 \text{ on } \partial D\setminus\Gamma_N\cup\Gamma_D}} \int_D \min_{\substack{0\leq\theta\leq 1\\ A^*\in G_\theta}} \left(A^{*-1}\sigma\cdot\sigma+\ell\theta\right)dx.$$

Optimality condition. If (θ, A^*, σ) is a minimizer, then A^* is a rank-N sequential laminate aligned with σ and with explicit proportions

$$A^{*-1} = A^{-1} + \frac{1-\theta}{\theta}\left(\sum_{i=1}^{N} m_i f_A^c(e_i)\right)^{-1},$$

and θ is given in 2-D (similar formula in 3-D)

$$\theta_{opt} = \min\left(1, \sqrt{\frac{\kappa+\mu}{4\mu\kappa\ell}}\left(|\sigma_1|+|\sigma_2|\right)\right),$$

where σ is the solution of the homogenized equation.

Existence theory

Original shape optimization problem

$$\inf_{\Omega\subset D} J(\Omega) = \int_{\Gamma_N} g\cdot u\, ds + \ell \int_\Omega dx. \tag{1}$$

Homogenized (or relaxed) formulation of the problem

$$\min_{\substack{A^*\in G_\theta \\ 0\leq\theta\leq 1}} J(\theta, A^*) = \int_{\Gamma_N} g\cdot u\, ds + \ell \int_D \theta\, dx. \tag{2}$$

Theorem 7.30. The homogenized formulation (2) is the relaxation of the original problem (1) in the sense where

1. there exists, at least, one optimal composite shape (θ, A^*) minimizing (2),
2. any minimizing sequence of classical shapes Ω for (1) converges, in the sense of homogenization, to a minimizer (θ, A^*) of (2),
3. the minimal values of the original and homogenized objective functions coincide.

7.5.5 Numerical algorithm

Double "alternating" minimization in σ and in (θ, A^*).

- intialization of the shape (θ_0, A_0^*)
- iterations $n \geq 1$ until convergence
  - given a shape $(\theta_{n-1}, A_{n-1}^*)$, we compute the stress σ_n by solving a linear elasticity problem (by a finite element method)
  - given a stress field σ_n, we update the new design parameters (θ_n, A_n^*) with the explicit optimality formula in terms of σ_n.

Remarks.

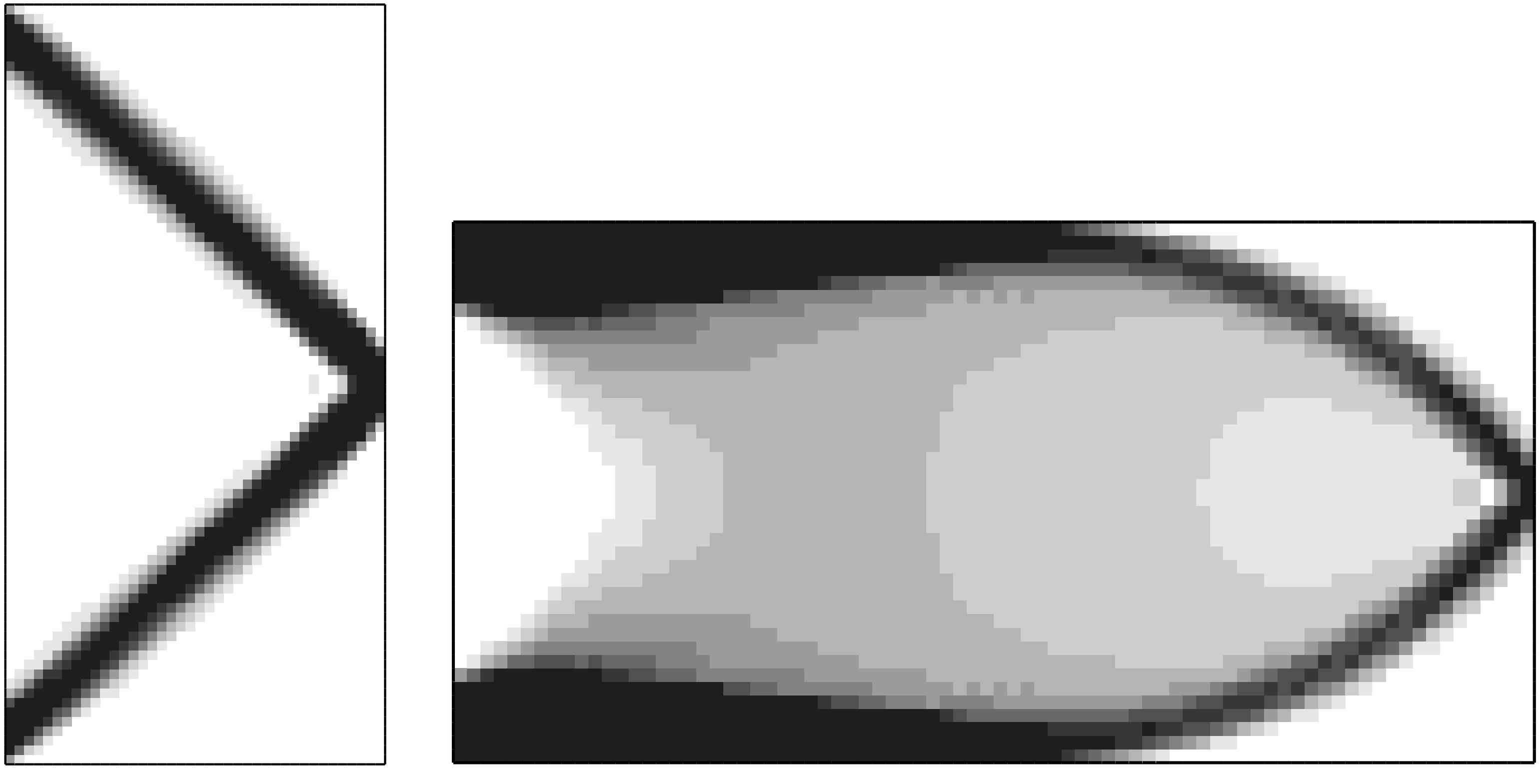
☞ For compliance, the problem is self-adjoint.

☞ Micro-macro method (local microstructure / global density).

Remarks

- ☞ The objective function always decreases.
- ☞ Algorithm of the type "optimality criteria".
- ☞ Algorithme of "shape capturing" on a fixed mesh of Ω.
- ☞ We replace void by a weak "ersatz" material, or we impose $\theta \geq 10^{-3}$ to get an invertible rigidity matrix.
- ☞ A few tens of iterations are sufficient to converge.

Example: optimal cantilever



Penalization

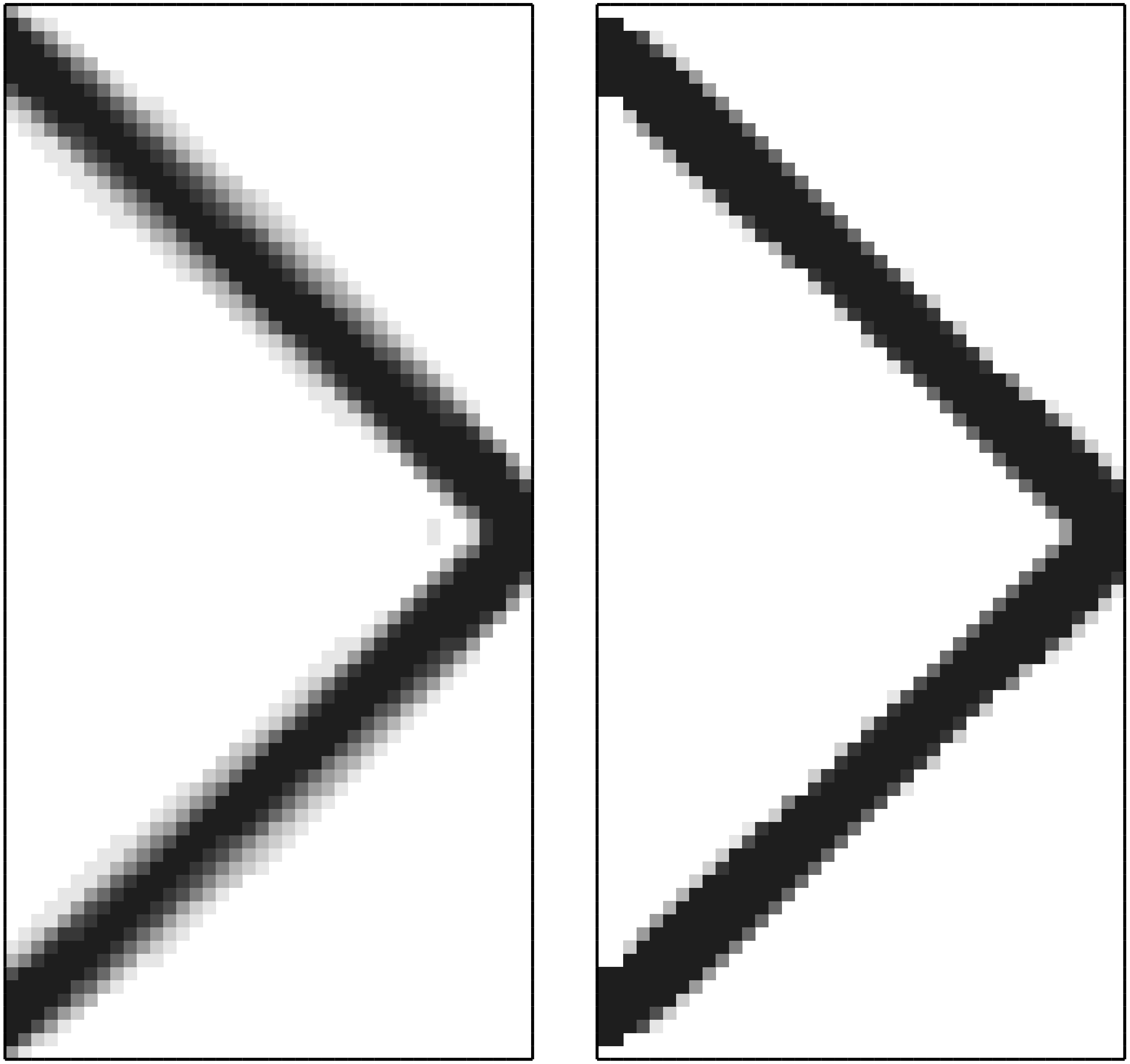
The previous algorithm compute **composite** shapes instead of **classical** shapes.

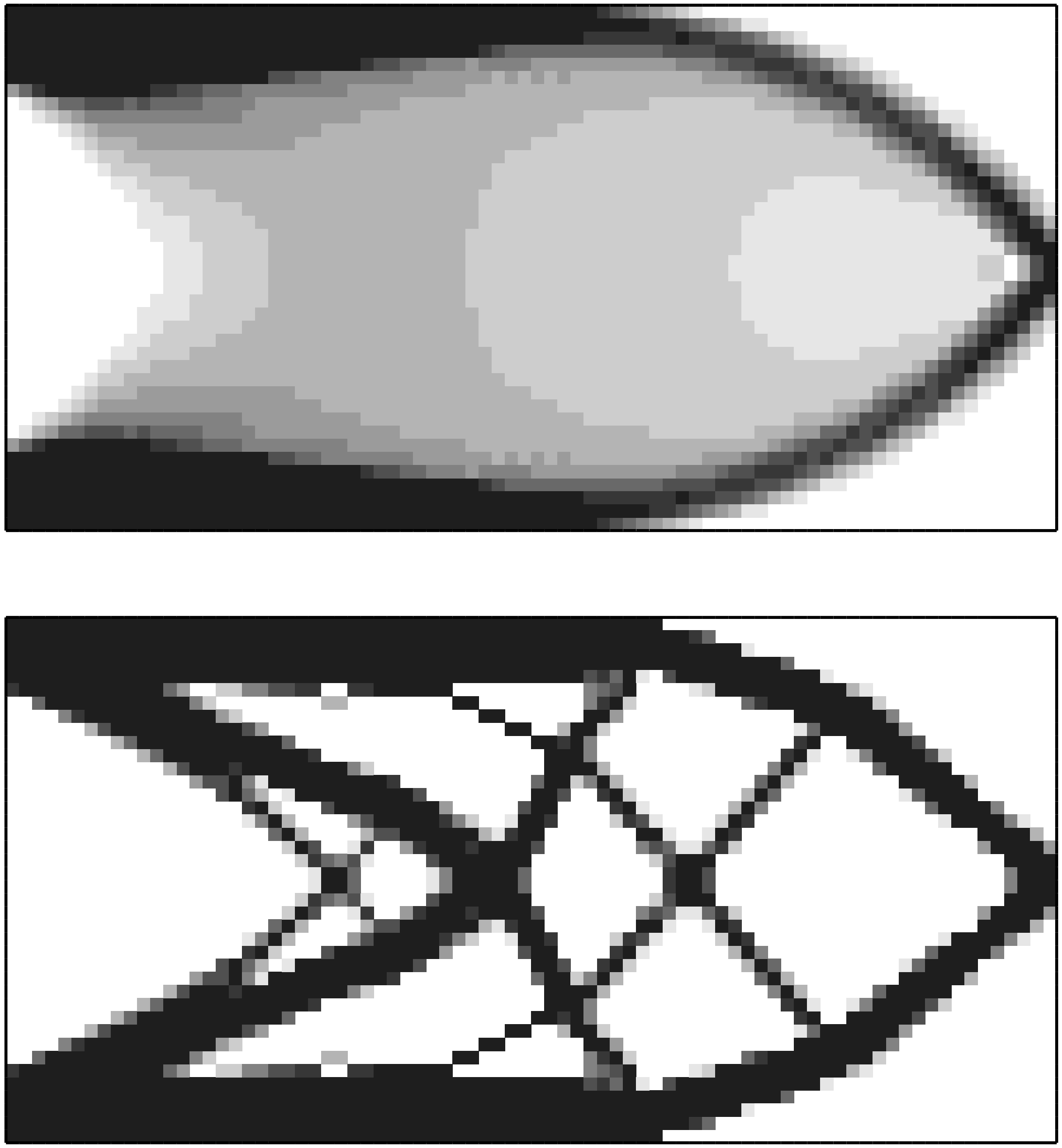
We thus use a penalization technique to force the density in taking values close to 0 or 1.

Algorithm: after convergence to a composite shape, we perform a few more iterations with a penalized density

$$\theta_{pen} = \frac{1 - \cos(\pi \theta_{opt})}{2}.$$

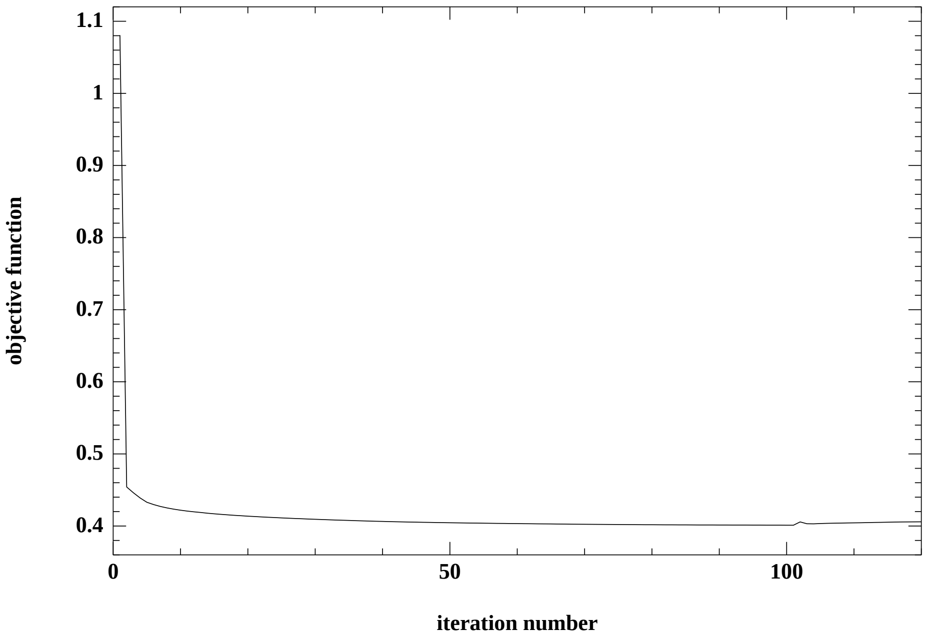
If $0 < \theta_{opt} < 1/2$, then $\theta_{pen} < \theta_{opt}$, while, if $1/2 < \theta_{opt} < 1$, then $\theta_{pen} > \theta_{opt}$.

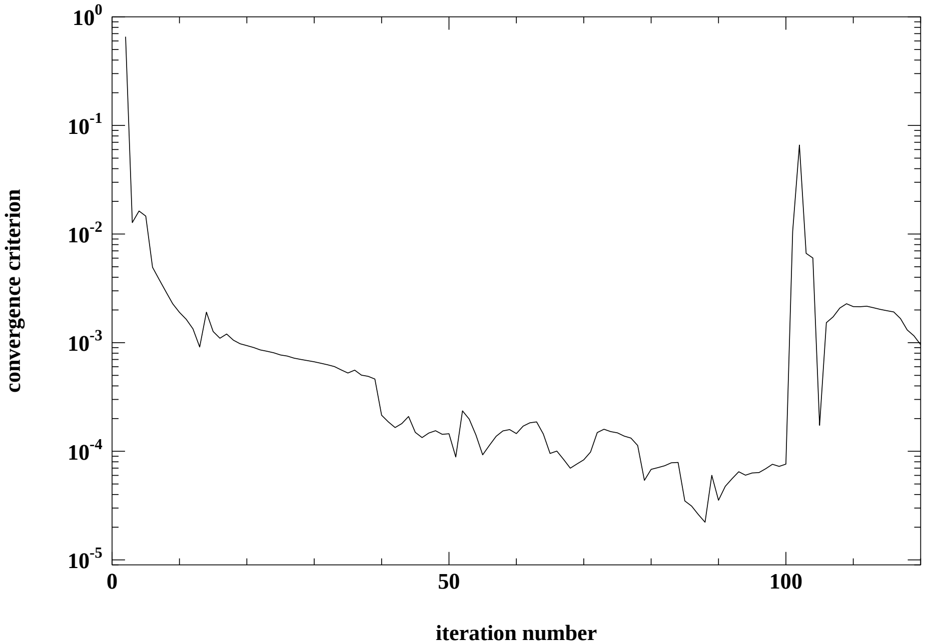




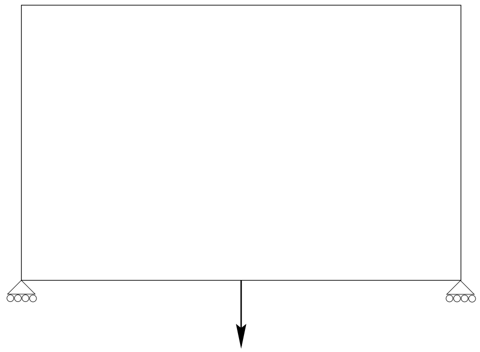
Convergence history:

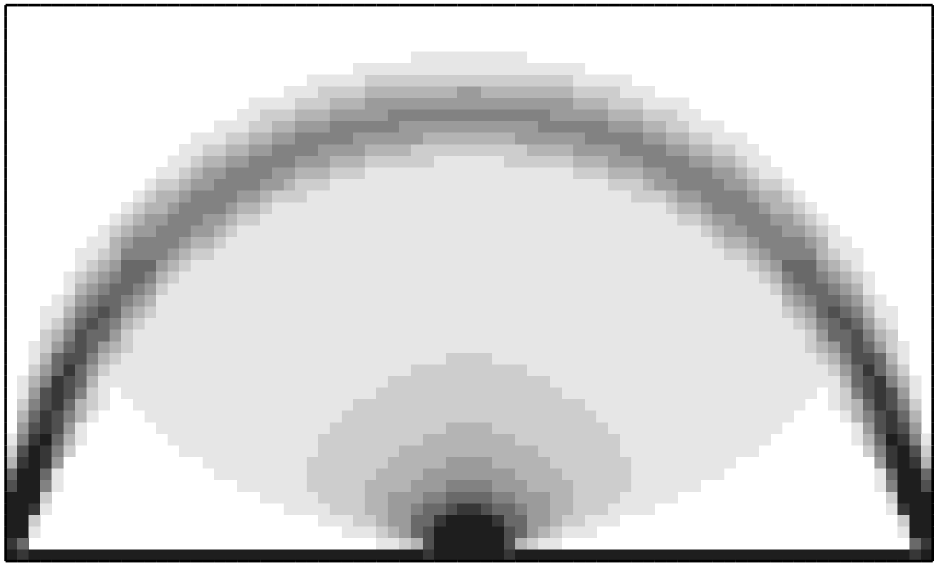
objective function (left), and residual (right),
in terms of the iteration number.

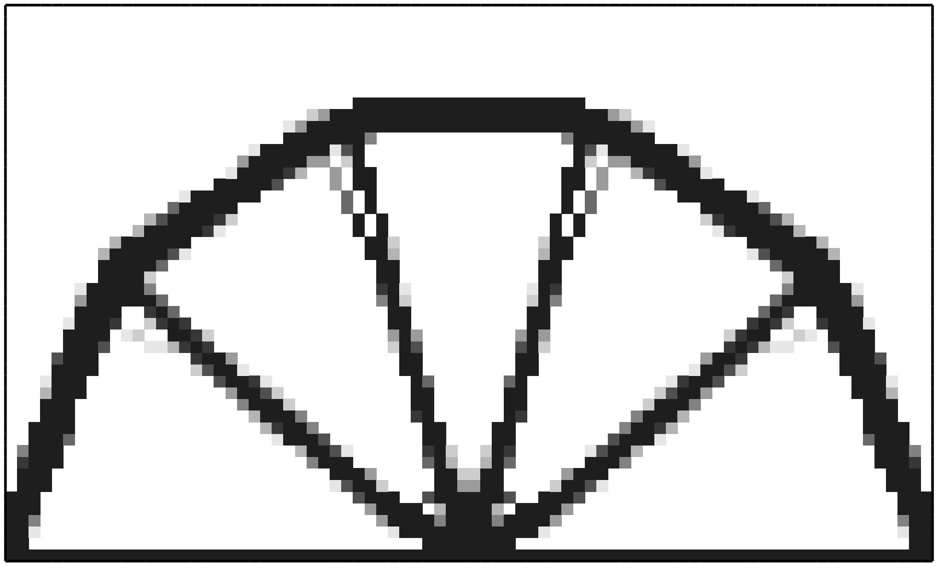




Example: optimal bridge







7.5.6. Convexification and “fictitious materials”

Idea. In the homogenization method composite materials are introduced but discarded at the end by penalization. Can we simplify the approach by introducing merely a density θ ?

A classical shape is parametrized by $\chi(x) \in \{0, 1\}$.

If we **convexify** this admissible set, we obtain $\theta(x) \in [0, 1]$.

The Hooke’s law, which was $\chi(x)A$, becomes $\theta(x)A$. We also call this **fictitious materials** because one can not realize them by a true homogenization process (in general). Combined with a penalization scheme, this methode is called **SIMP** (Solid Isotropic Material with Penalization).

Convexified formulation with $0 \leq \theta(x) \leq 1$

$$\left\{ \begin{array}{ll} \sigma = \theta(x) A e(u) & \text{with } e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right), \\ \operatorname{div}\sigma = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N). \end{array} \right.$$

Compliance minimization

$$\min_{0 \leq \theta(x) \leq 1} \left(c(\theta) + \ell \int_D \theta(x) \right).$$

with

$$c(\theta) = \int_{\Gamma_N} g \cdot u = \int_D (\theta(x) A)^{-1} \sigma \cdot \sigma = \min_{\substack{div\tau = 0 \text{ in } D \\ \tau n = g \text{ on } \Gamma_N \\ \tau n = 0 \text{ on } \partial D \setminus \Gamma_N \cup \Gamma_D}} \int_D (\theta(x) A)^{-1} \tau \cdot \tau \, dx.$$

Now, there is **only one single** design parameter: the material density θ (the microstructure A^* has disappeared).

Existence of solutions

Theorem 7.33. The convexified formulation

$$\min_{0\le\theta(x)\le 1} \quad \min_{\substack{div\tau=0 \text{ in } D\\ \tau n=g \text{ on } \Gamma_N\\ \tau n=0 \text{ on } \partial D\setminus\Gamma_N\cup\Gamma_D}} \int_D (\theta(x)A)^{-1}\tau\cdot\tau\,dx + \ell\int_D \theta\,dx$$

admits at least one solution.

Proof. The function, defined on $\mathbb{R}^+ \times \mathcal{M}_n^s$,

$$\phi(a,\sigma) = a^{-1}A^{-1}\sigma\cdot\sigma,$$

is convex because

$$\phi(a,\sigma) = \phi(a_0,\sigma_0) + D\phi(a_0,\sigma_0)\cdot(a-a_0,\sigma-\sigma_0) + \phi(a,\sigma - aa_0^{-1}\sigma_0),$$

where the derivative $D\phi$ is given by

$$D\phi(a_0,\sigma_0)\cdot(b,\tau) = -\frac{b}{a_0^2}A^{-1}\sigma_0\cdot\sigma_0 + 2a_0^{-1}A^{-1}\sigma_0\cdot\tau.$$

Optimality condition

If we exchange the minimizations in τ and in θ, we can compute the optimal θ which is

$$\theta(x) = \begin{cases} 1 & \text{if } A^{-1}\tau\cdot\tau \geq \ell \\ \sqrt{\ell^{-1}A^{-1}\tau\cdot\tau} & \text{if } A^{-1}\tau\cdot\tau \leq \ell \end{cases}$$

Again we can use an “alternating” double minimization algorithm.

Numerical algorithm

- intialization of the shape θ_0
- iterations $k \geq 1$ until convergence
  - given a shape θ_{k-1}, we compute the stress σ_k by solving an elasticity problem (by a finite element method)
  - given a stress field σ_k, we update the new material density θ_k with the explicit optimality formula in terms of σ_k.

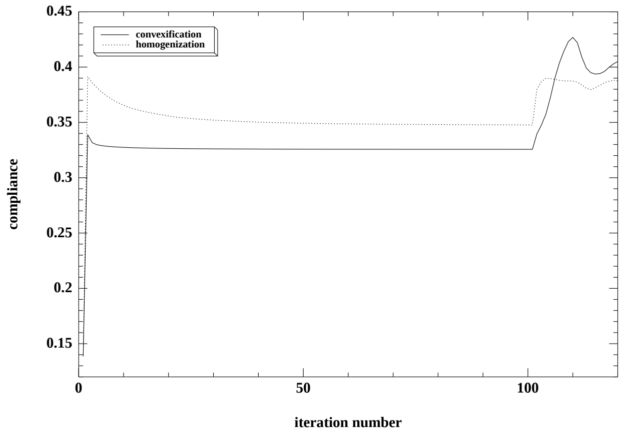
Penalization: we use a penalized density

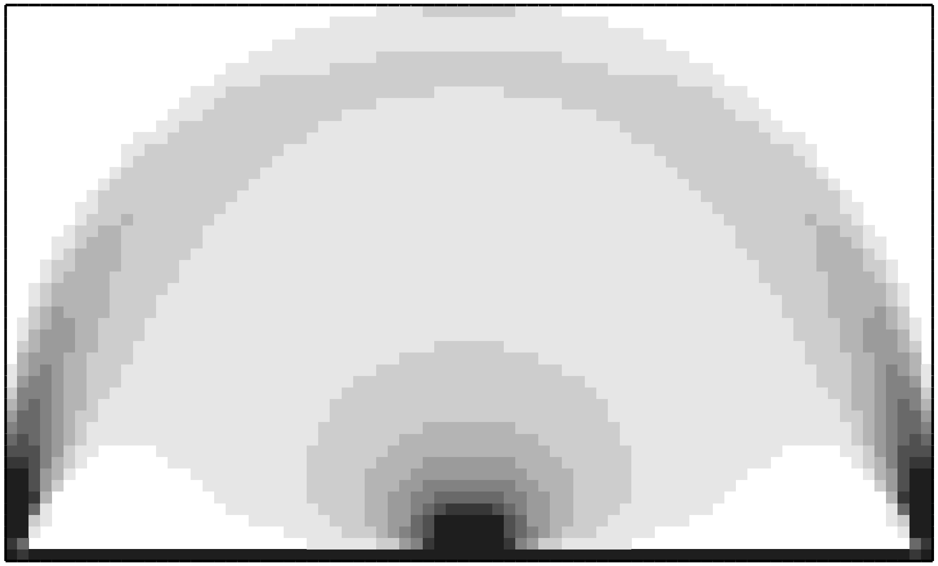
$$\theta_{pen} = \frac{1 - \cos(\pi \theta_{opt})}{2} \quad \text{or (SIMP)} \quad \theta_{pen} = \theta^p \quad p > 1.$$

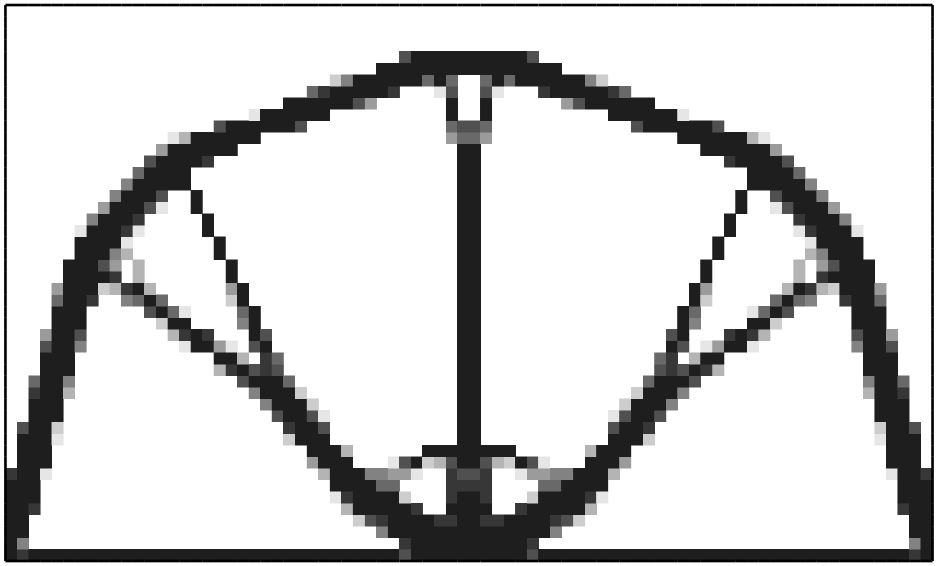
In practice: it is extremely simple ! But the numerical results are not as good ! An explanation is the lack of a relaxation theorem.

Be careful: very delicate monitoring of the penalization...

Optimal bridge by the convexification method







Conclusion

- SIMP (or convexification, or "fictitious materials") is very simple and **very popular** (many commercial codes are using it).
- SIMP uses very few informations on composites !
- On the contrary to the homogenization method, SIMP **is not a relaxation method**: it changes the problem !
- There is a gap between the true minimal value of the objective function and that of SIMP.
- SIMP can be delicate to monitor: how to increase the penalization parameter ?

Generalizations of the homogenization method

- ☞ multiple loads
- ☞ vibration eigenfrequency
- ☞ general criterion of the least square type

The two first cases are self-adjoint and we have a complete understanding and justification of the relaxation process. However, the third case is not self-adjoint and only a partial relaxation is known.

Multiple loads

For n loads $(f_i)_{1\leq i\leq n}$, the homogenized formulation is

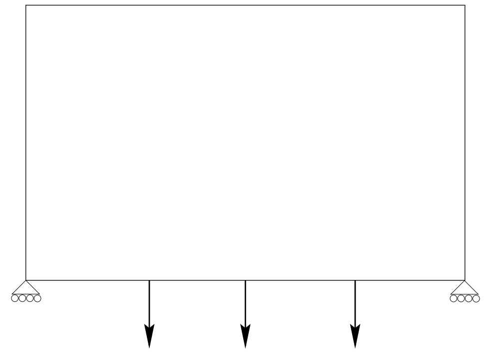
$$\min_{\substack{\mathrm{div}\sigma_i=0 \text{ in } D\\ \sigma_i n=g_i \text{ on } \Gamma_N}} \int_D \min_{0\leq\theta\leq 1} \min_{A^*\in L_\theta} \left(\sum_{i=1}^n {A^*}^{-1}\sigma_i\cdot\sigma_i + \ell\theta \right) dx$$

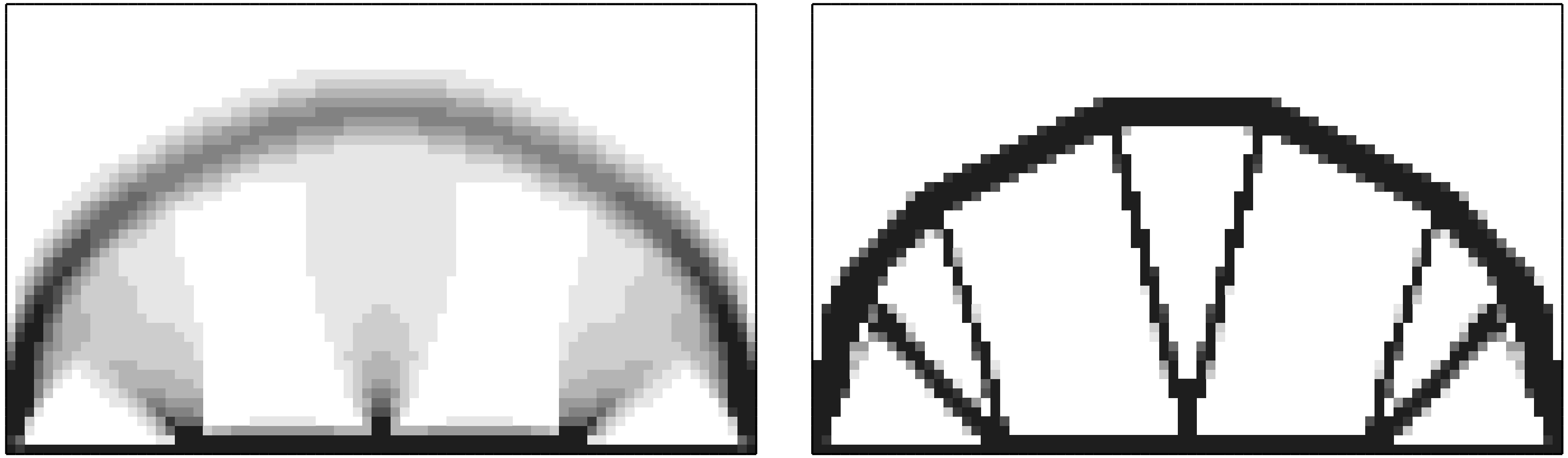
with $A^*\in L_\theta$ and

$$(1-\theta)\left({A^*}^{-1} - A^{-1}\right)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} + \theta \sum_{i=1}^p m_i f_A^c(e_i)$$

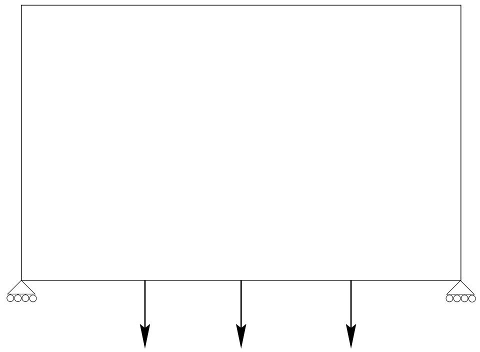
The optimal laminate is no more of rank N. The m_i's optimization is now done numerically (with numerous enough lamination directions).

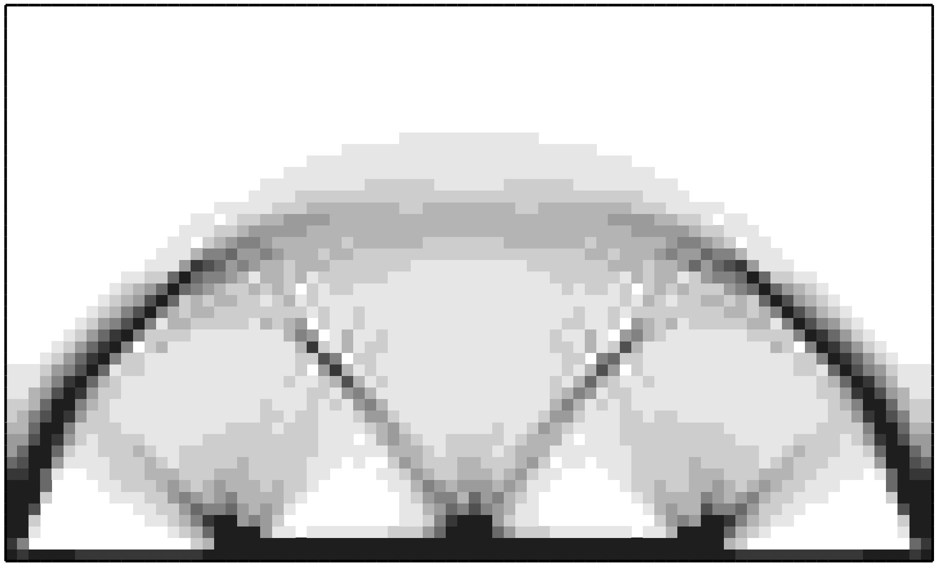
Optimal bridge for 3 simultaneously applied loads

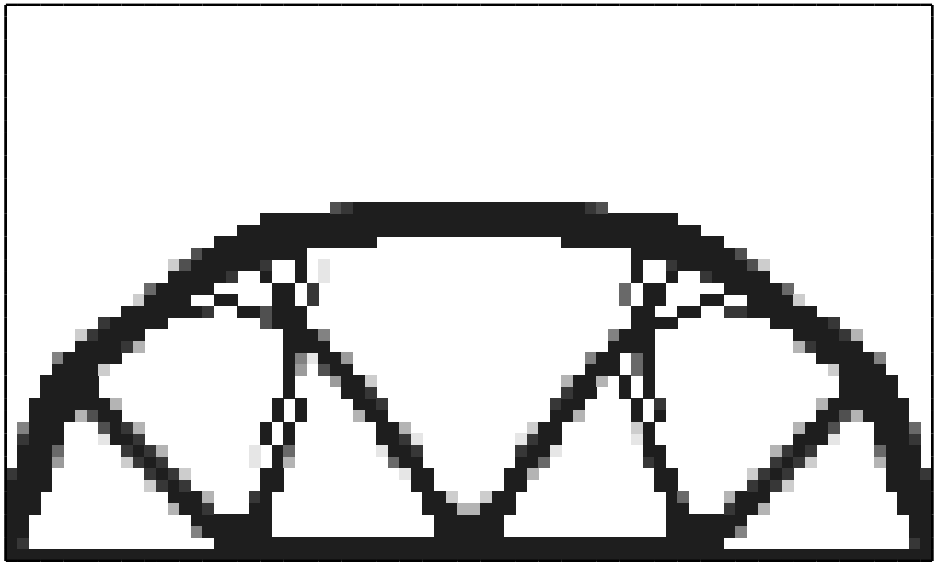




Optimal bridge for 3 independently applied loads







Vibration eigenfrequencies

We maximize the first vibration eigenfrequency

$$\omega_1^2(\theta, A^*) = \min_{u \in \mathcal{H}} \frac{\displaystyle\int_D A^* e(u) \cdot e(u) dx}{\displaystyle\int_D \overline{\rho} |u|^2 dx}.$$

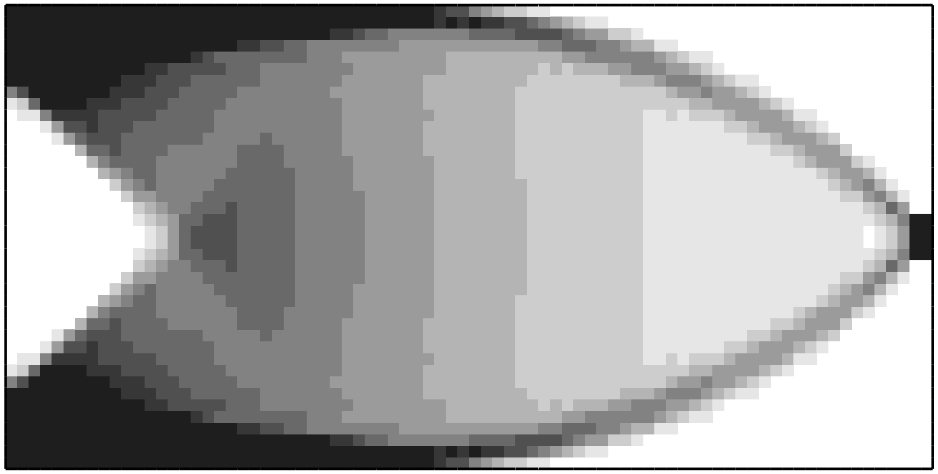
with the density $\overline{\rho} = \theta \rho_A + (1-\theta)\rho_B$, and the space of admissible displacements $\mathcal{H} = \left\{ u \in H^1(D)^N \text{ such that } u = 0 \text{ on } \Gamma_D \right\}$.

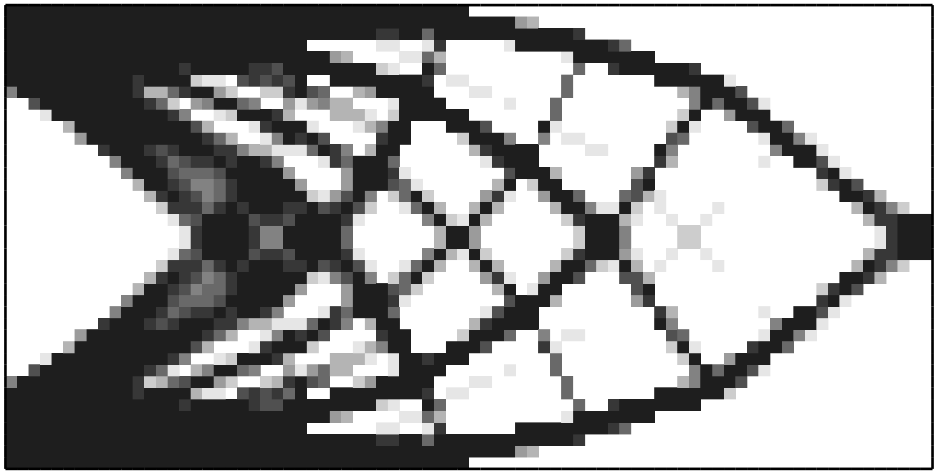
The homogenized formulation is

$$\max_{0 \leq \theta \leq 1} \left\{ \min_{u \in \mathcal{H}} \frac{\displaystyle\int_D \left(\max_{A^* \in L_\theta} A^* e(u) \cdot e(u) \right) dx}{\displaystyle\int_D \overline{\rho} |u|^2 dx} + \ell \int_D \theta(x) dx \right\},$$

with L_θ the set of sequential laminates.

Be careful: there is a max-min which can not be exchanged...





Least square objective functions

Classical two-phase formulation:

$$\inf_{\chi \in L^\infty(\Omega;\{0,1\})} J(\chi) = \int_\Omega k(x)|u_\chi(x) - u_0(x)|^2\,dx + \ell \int_\Omega \chi(x)dx$$

where u_χ is solution of

$$\begin{cases} -\operatorname{div}\left(A_\chi e(u_\chi)\right) = f & \text{in } \Omega \\ u_\chi = 0 & \text{on } \partial\Omega, \end{cases}$$

with a Hooke's law $A_\chi = \chi A + (1-\chi)B$.

Homogenized formulation:

$$\min_{(\theta,A^*)} J^*(\theta,A^*) = \int_\Omega \Big(k|u-u_0|^2 + \ell\theta \Big) dx$$

with u solution of

$$\begin{cases} -\operatorname{div}\left(A^* e(u)\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Difficulty: we don't know G_θ and we cannot replace it by L_θ. In other words, we don't know which microstructures are optimal...

Partial relaxation: we nevertheless replace G_θ by L_θ. We thus loose the existence of an optimal solution but we keep the link with the original problem.

Partial relaxation

We restrict ourselves to sequential laminates A^* with matrix A and inclusions B. The number of laminations and their directions are fixed. We merely optimize with respect to θ and the proportions $(m_i)_{1\leq i\leq p}$

$$(1-\theta)\left(A-A^*\right)^{-1} \;=\; \left(A-B\right)^{-1} - \theta\sum_{i=1}^{q} m_i f_A(e_i),$$

with $\forall e \in \mathbb{R}^N,\ |e|=1,\ \forall\xi$ symmetric matrix

$$f_A(e)\xi\cdot\xi = \frac{1}{\mu_A}\left(|\xi e|^2 - (\xi e\cdot e)^2\right) + \frac{1}{\lambda_A+2\mu_A}(\xi e\cdot e)^2.$$

Thus, the objective function is

$$J^*(\theta, A^*) \equiv J^*(\theta, m_i)$$

with the constraints $0\leq\theta\leq 1$, $m_i\geq 0$, $\sum_{i=1}^p m_i = 1$.

We compute its gradient with the help of an adjoint state.

Adjoint state

Typical example of an objective function

$$J^*(\theta, A^*) = \int_\Omega k(x)|u(x) - u_0(x)|^2 dx + \ell \int_\Omega \theta \, dx$$

Adjoint state

$$\left\{ \begin{array}{rcll} -\operatorname{div}(A^* e(p)) & = & 2k(x)(u(x) - u_0(x)) & \text{in } \Omega \\ p & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

$\boxed{\text{Gradient}}$

$$
\begin{aligned}
\nabla_\theta J^*(x) &= \ell + \frac{\partial A^*}{\partial \theta} e(u) \cdot e(p), \\
\nabla_{m_i} J^*(x) &= \frac{\partial A^*}{\partial m_i}(x) e(u) \cdot e(p),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial A^*}{\partial \theta}(x) &= T^{-1}\left((A-B)^{-1} - \sum_{i=1}^{q} m_i f_A(e_i) \right) T^{-1}, \\
\frac{\partial A^*}{\partial m_i}(x) &= -\theta(1-\theta) T^{-1} f_A(e_i) T^{-1}, \\
T &= (A-B)^{-1} - \theta \sum_{i=1}^{q} m_i f_A(e_i)\,.
\end{aligned}
$$

Numerical algorithm of gradient type

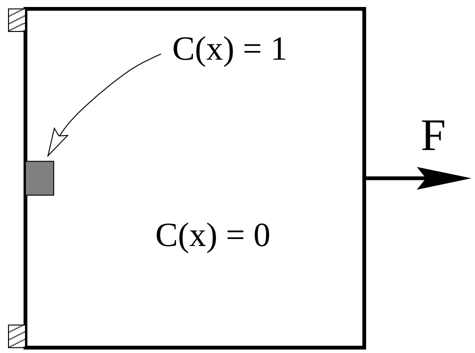
Projected gradient with a variable step:

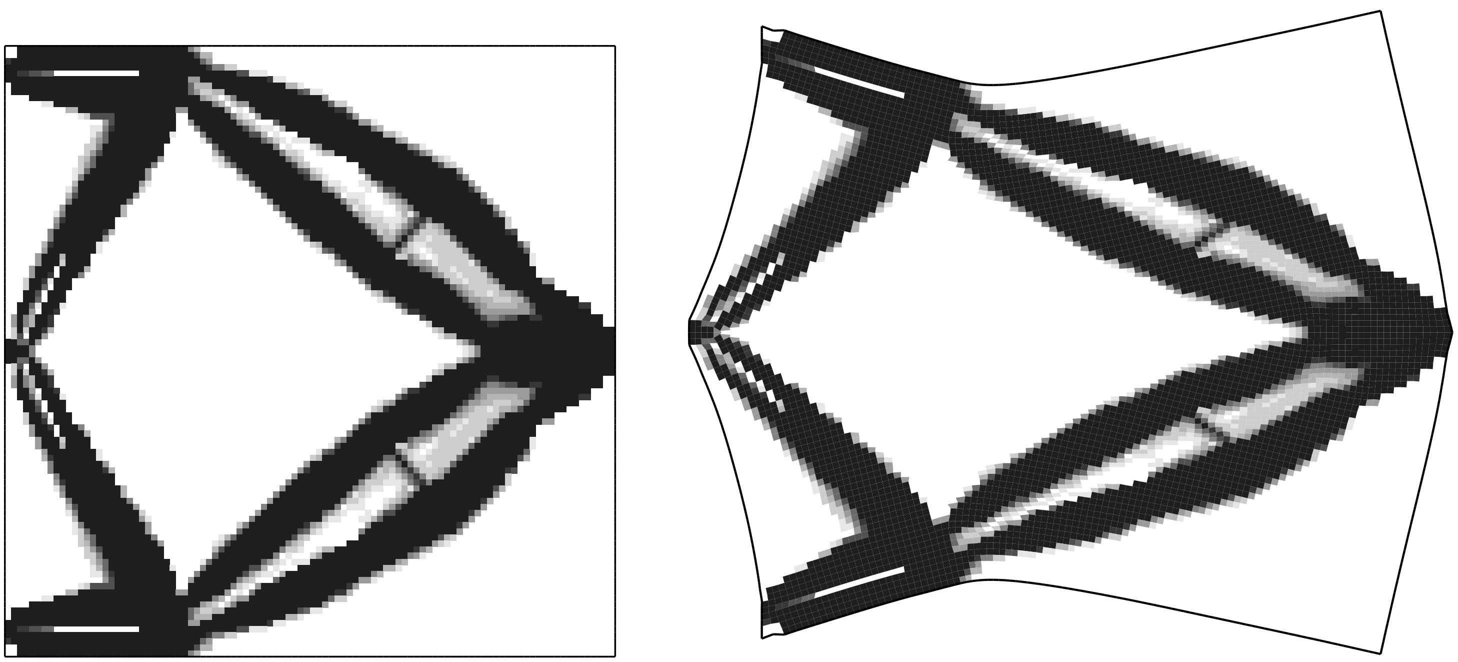
1. Initialization of the design parameters $\theta_0, m_{i,0}$ (for example, constants satisfying the constraints).
2. Iterations until convergence, for $k \geq 0$:
   (a) Computation of the state u_k and the adjoint p_k, with the previous design parameters $\theta_k, m_{i,k}$.
   (b) Update of the design parameters :

$$\begin{array}{rcl} \theta_{k+1} & = & \max\left(0, \min\left(1, \theta_k - t_k \nabla_\theta J_k^*\right)\right), \\ m_{i,k+1} & = & \max\left(0, m_{i,k} - t_k \nabla_{m_i} J_k^* + \ell_k\right), \end{array}$$

where ℓ_k is a Lagrange multiplier for the constraint $\sum_{i=1}^q m_{i,k} = 1$, iteratively updated, and $t_k > 0$ is a descent step such that $J^*(\theta_{k+1}, m_{k+1}) < J^*(\theta_k, m_k)$.

Example: force inverter





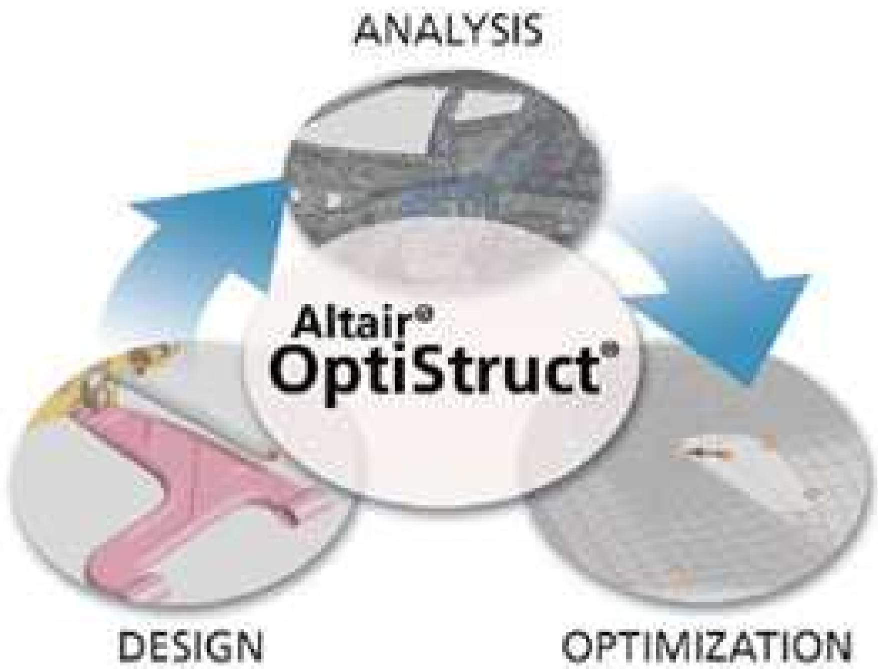
Other methods of topology optimization

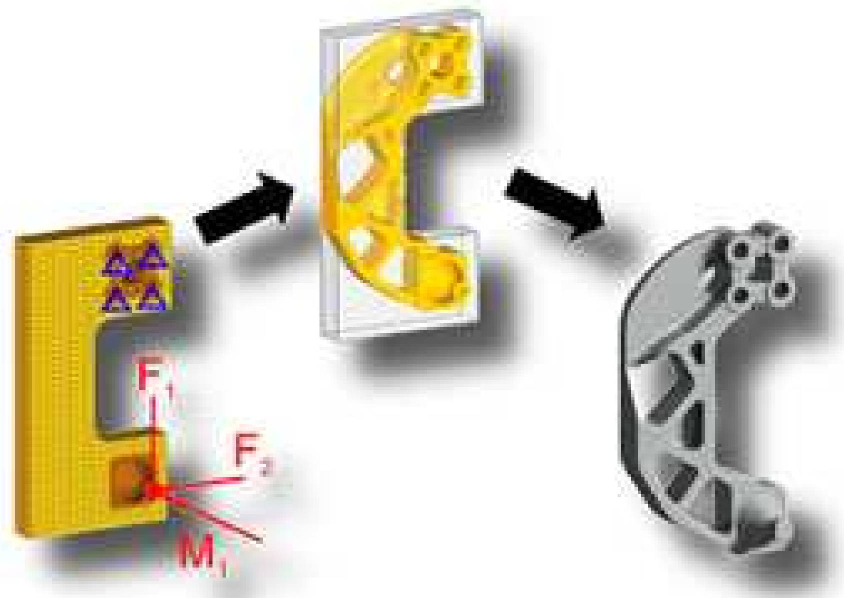
- Discrete 0/1 optimization: genetic algorithms.
- Level set methods based on geometric optimization.
- Topological derivative: sensitivity to the nucleation of a small hole.
- Phase-field methods.

Commercial softwares and industrial applications

See the web page:

`http://www.cmap.polytechnique.fr/~optopo/links.html`





Industrial applications

- ☞ Automotive industry.
- ☞ Aerospace industry.
- ☞ Civil engineering, architecture.
- ☞ Nano-technologies, MEMS.
- ☞ Optics, wave guides.