ECOLE POLYTECHNIQUE
Applied Mathematics Master Program
MAP 562 Optimal Design of Structures (G. Allaire)
Written exam, March 20th, 2013 (2 hours)

1 Parametric optimization: 12 points

We consider an elastic membrane with a variable thickness $h(x)$, occupying at rest a plane domain Ω (a smooth bounded open set of $\mathbb{R}^2$). The displacement at the boundary $\partial\Omega$ is imposed to be equal to u_0 where $u_0 \in H^1(\Omega)$. The membrane is loaded by a given force $f(x) \in L^2(\Omega)$. Its vertical displacement $u(x)$ is the unique solution in $H^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}\,(h\nabla u) = f & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The thickness belongs to the following space of admissible designs

$$\mathcal{U}_{ad} = \left\{ h \in L^\infty(\Omega)\ , \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega \right\}.$$

The goal is to minimize the objective function

$$\inf_{h \in \mathcal{U}_{ad}} \left\{ J(h) = \int_\Omega j(u)\, dx \right\}, \tag{2}$$

where j is a smooth function satisfying

$$|j(v)| \leq C(|v|^2 + 1), \quad |j'(v)| \leq C(|v| + 1).$$

1. By making the change of variables $v = u - u_0$ write the standard variational formulation for $v \in H^1_0(\Omega)$. We define the affine space $\mathcal{A}$ by

   $$\mathcal{A} = \{\psi \in H^1(\Omega) \text{ such that } \psi = u_0 + \phi \text{ with } \phi \in H^1_0(\Omega)\}.$$

   Deduce that a variational formulation for (1) is: find $u \in \mathcal{A}$ such that

   $$\int_\Omega h\nabla u \cdot \nabla\phi\, dx = \int_\Omega f\phi\, dx \quad \forall\, \phi \in H^1_0(\Omega).$$

2. Write the Lagrangian associated to the minimization problem (2) and give the adjoint problem, the solution of which shall be denoted by p. Define explicitly the p.d.e. and the boundary condition satisfied by p.

3. Compute (formally) the derivative of $J(h)$.

4. We consider the special case

   $$J(h) = \int_\Omega f(x)\, u(x)\, dx.$$

   Check that, if $u_0 \neq 0$, the problem is not self adjoint, i.e., p is not a multiple of u.

5. Instead of (2) we now consider an objective function of the type

$$J(h) = \int_\Omega j(h, u, \nabla u)\, dx$$

where $j : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function satisfying

$$|j(h, v, \zeta)| \le C(|v|^2 + |\zeta|^2 + 1), \qquad \left|\frac{\partial j}{\partial v}(h, v, \zeta)\right| + \left|\frac{\partial j}{\partial \zeta}(h, v, \zeta)\right| \le C(|v| + |\zeta| + 1).$$

Give the associated Lagrangian and compute the adjoint problem .

6. For the particular choice

$$J(h) = \int_\Omega \left(f\, u - \frac{1}{2} h \nabla u \cdot \nabla u \right) dx \tag{3}$$

show that the solution of the adjoint problem is $p = 0$. Compute the derivative of $J(h)$ and check that, as in the "usual" self adjoint case, it has a precise sign. To minimize (3) should we increase or decrease the thickness ? Which case is recovered when $u_0 = 0$?

2 Geometric optimization: 8 points

We consider a thermal conductivity problem in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, occupied by a fluid flowing with a given incompressible velocity $V(x) : \mathbb{R}^N \mapsto \mathbb{R}^N$, smooth and such that $\mathrm{div} V = 0$ in Ω. For a given source term $f \in L^2(\mathbb{R}^N)$, the temperature, assumed to vanish on the boundary, is the solution $u \in H^1_0(\Omega)$ of

$$\begin{cases} V \cdot \nabla u - \nu \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\nu > 0$ is the constant thermal conductivity. For a given temperature target $u_0 \in L^2(\mathbb{R}^N)$, the goal is to minimize the objective function

$$\min_{\Omega \subset \mathbb{R}^N} \left\{ J(\Omega) = \frac{1}{2} \int_\Omega |u - u_0|^2 \, dx \right\}. \tag{5}$$

We use Hadamard's method of shape variations.

1. Write the Lagrangian corresponding to (5), taking care of the Dirichlet boundary condition on $\partial\Omega$.

2. Deduce the adjoint problem. Is the differential operator for the adjoint similar to that in (4) ?

3. Compute (formally) the shape derivative of (5).