Exercise 1

Let $E$ be a Banach space and $x \in E$. Prove that there exists $T \in E^*$ such that

$$T(x) = \|x\|^2$$

and $\|T\|_{E^*} = \|x\|_3$.

Answer of exercise 1

Let $G = \mathbb{R}x$ and $T$ be the linear continuous map defined on $G$ by

$$T(tx) = t\|x\|^2.$$  

From the Hahn-Banach Theorem, there exists an extension of $T$ on $E$ such that $\|T\|_{E^*} = \|T\|_{G^*} = \|x\|_E$.

Exercise 2

Let $E$ be a Banach space and let $A \subset E$ be a subset that is sequentially compact for the weak topology of $E$. Prove that $G$ is bounded.

Answer of exercise 2

Let $(x_n)$ be a sequence of elements of $A$. As $A$ is sequentially compact, there exists $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi$ is increasing and $x \in A$, with

$$x_{\varphi(n)} \to x.$$  

In particular, for all $T \in E^*$, $T(x_{\varphi(n)})$ is bounded and, from the Banach-Steinhaus Theorem, there exists $C$ such that for all $T \in E^*$,

$$T(x_{\varphi(n)}) \leq C\|T\|_{E^*}.$$  

From the Exercise 1, there exists $T$ such $T(x_{\varphi(n)}) = \|x_{\varphi(n)}\|^2$ and $\|T\| = \|x_{\varphi(n)}\|$. It follows that

$$\|x_{\varphi(n)}\| \leq C.$$  

If $A$ was not bounded, we could construct a sequence $x_n$ of elements of $A$ such that $\|x\|_E \geq n$, what is impossible from the last inequality.

Exercise 3

Let $E$ be a Banach space and let $(x_n)$ be a sequence such that $x_n \to x$ in the weak $\sigma(E, E^*)$ topology. Set

$$y_n = \frac{1}{n} \sum_{k \leq n} x_k.$$  

Prove that $y_n \to x$.  

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Answer of exercise 3

Let \( T \in E^* \). We have

\[
T(y_n) = \frac{1}{n} \sum_{k \leq n} T(x_k).
\]

and

\[
||T(y_n) - T(x)|| \leq \frac{1}{n} \sum_{k \leq n} |T(x_n) - (x)|.
\]

For all \( \varepsilon > 0 \), there exists \( N \) such that for all \( n > N \), \( |T(x_n) - T(x)| < \varepsilon \). Thus,

\[
|T(y_n) - T(x)| \leq \frac{1}{n} \sum_{k \leq N} |T(x_n) - T(x)| + \varepsilon,
\]

and for \( n \) great enough,

\[
|T(y_n) - T(x)| \leq 2\varepsilon.
\]

Exercise 4

Let \( \Omega = (0,1) \).

1. Consider the sequence \((f_n)\) of functions defined by \( f_n(x) = ne^{-nx} \). Prove that
   
   1. \( f_n \to 0 \) a.e.
   2. \((f_n)\) is bounded in \( L^1(\Omega) \).
   3. \( f_n \not\to 0 \) in \( L^1(\Omega) \) strongly.
   4. \( f_n \not\to 0 \) weakly in \( \sigma(L^1, L^\infty) \).
   
   More precisely, there is no subsequence that converges weakly \( \sigma(L^1, L^\infty) \).

2. Let \( 1 < p < \infty \) and consider the sequence \((g_n)\) of functions defined by \( g_n(x) = n^{1/p} e^{-nx} \). Prove that
   
   1. \( g_n \to 0 \) a.e.
   2. \((g_n)\) is bounded in \( L^p(\Omega) \).
   3. \( g_n \not\to 0 \) in \( L^p(\Omega) \) strongly.
   4. \( g_n \to 0 \) weakly in \( \sigma(L^p, L^{p'}) \).

Answer of exercise 4

1. We consider \( f_n = ne^{-nx} \). For all \( x \in (0,1) \), \( f_n(x) = e^{-n(x+\ln(n))/n} \) and converges toward 0.

\[
\|f_n\|_{L^1(\Omega)} = \int_0^1 |f_n| = \int_0^1 ne^{-nx} = - \int_0^1 (e^{-nx})' = -[e^{-nx}]_0^1 = 1 - e^{-n}.
\]
Thus, $f_n$ is bounded in $L^1(\Omega)$ and does not converge toward 0 in $L^1(\Omega)$. Finally, let $u \in C^1([0,1])$,

$$
\int_0^1 f_n u = \int_0^1 (e^{-nx})' u = [e^{-nx} u]_0^1 - \int_0^1 e^{-nx} u' \to u(0).
$$

Thus, $f_n$ does not converge toward 0 in $\sigma(L^1,L^\infty)$ (and even no subsequence).

2. We have defined $g_n$ by

$$
g_n(x) = n^{1/p} e^{-nx},
$$

with $1 < p < \infty$. Obviously, $g_n(x)$ does to zero for every $x \in (0,1)$. Moreover,

$$
\|g_n\|_{L^p}^p = \int_0^1 ne^{-nx} = 1 - e^{-np}.
$$

Thus, $g_n$ is bounded in $L^p(0,1)$ and does not converge toward 0 in $L^p(0,1)$. Now, let $u \in C_0^\infty(0,1)$. As $g_n$ does converge uniformly toward 0 on the support of $u$, we have

$$
\int_0^1 g_n u \to 0.
$$

Let $v \in L^p(0,1)$. For all $\varepsilon > 0$, there exists $u_\varepsilon \in C_0^\infty(0,1)$ such that $\|u_\varepsilon - v\|_{L^p(\Omega)} < \varepsilon$. It follows that

$$
\left| \int_0^1 g_n v \right| \leq \int_0^1 g_n (u_\varepsilon - v) + \int_0^1 g_n u_\varepsilon \leq \|g_n\|_{L^p} \|u_\varepsilon - v\|_{L^p} + \int_0^1 g_n u_\varepsilon.
$$

As $g_n$ is bounded in $L^p(0,1)$, we get that

$$
\left| \int_0^1 g_n v \right| \leq C\varepsilon + \int_0^1 g_n u_\varepsilon.
$$

For $n$ great enough, we obtain that

$$
\left| \int_0^1 g_n v \right| \leq (C + 1)\varepsilon.
$$

It follows that $\int_0^1 g_n v$ converges toward 0 as $n$ goes to infinity, that is $g_n$ converges weak toward 0 in $L^p(0,1)$.

**Exercise 5**

Let $1 < p < \infty$. Let $(f_n)$ be a sequence in $L^p(\Omega)$ such that

1. $(f_n)$ is bounded in $L^p(\Omega)$,
2. \( f_n \to f \text{ a.e. on } \Omega \).

1. Prove that \( f_n \rightharpoonup f \) weakly in \( \sigma(L^p, L^p') \).
2. Same conclusion if assumption (2) is replaced by 
   \[ \|f_n - f\|_1 \to 0. \]
3. Assume now (1) and (2) and \( |\Omega| < \infty \). Prove that 
   \[ \|f_n - f\|_q \to 0 \text{ for every } q \text{ with } 1 \leq q < p. \]

**Answer of exercise 5**

1. To simplify the proof, we assume that \( |\Omega| < \infty \). It can be easily adapt to 
   the case where \( \Omega \) is \( \sigma \)-finite. First, let us notice that that from Fatou's 
   Lemma, \( f \in L^p(\Omega) \). In a first step, we are going to prove that up to a 
   subsequence, \( f_n \) weakly converges toward \( f \) in \( L^p(\Omega) \). As \( f_n \) is bounded 
   in \( L^p(\Omega) \), it admits a weakly convergent subsequence. That is there 
   exists \( \varphi \) monotone map from \( \mathbb{N} \) into \( \mathbb{N} \) and \( \tilde{f} \in L^p(\Omega) \) such that 
   \( f_{\varphi(n)} \) weakly converges toward \( \tilde{f} \). Moreover, from the Egorov's Theorem, for 
   all integer \( m > 0 \), there exists a measurable subset \( A_m \) of \( \Omega \) such that 
   \( f_{\varphi(n)} \) converges toward \( f \) uniformly. It follows that for all \( g \in L^{p'}(\Omega) \), 
   \[ \int_{\Omega \setminus A_m} f_{\varphi(n)} g \to \int_{\Omega \setminus A_m} fg \]
   and 
   \[ \int_{\Omega \setminus A_m} f_{\varphi(n)} g \to \int_{\Omega \setminus A_m} \tilde{f} g. \]
   Thus, 
   \[ \int_{\Omega \setminus A_m} (f - \tilde{f}) g = 0, \]
   for every \( g \in L^{p'}(\Omega) \). Choosing \( g = \text{sign}(f - \tilde{f}) \), it follows that \( f = \tilde{f} \text{ a.e.} \)
   in \( \Omega \setminus A_m \). In particular, \( f = \tilde{f} \text{ a.e. in } \Omega \setminus (\cap_m A_m) \). As \( |\cap_m A_m| = 0 \), 
   we deduce that \( f = \tilde{f} \) almost everywhere. It remains to prove that the whole sequence \( f_n \) weakly converges toward \( f \) in \( L^p(\Omega) \). Assume this is 
   not the case. Then, there exists \( h \in L^{p'}(\Omega) \) and \( \psi : \mathbb{N} \to \mathbb{N} \) monotone 
   such that 
   \[ \left| \int_{\Omega} (f_{\psi(n)} - f) h \right| > \delta > 0. \]
   Replacing \( f_n \) by \( f_{\psi(n)} \) in the first part of the proof, we conclude that 
   there exists \( \varphi : \mathbb{N} \to \mathbb{N} \) monotone such that 
   \( f_{\varphi\psi(n)} \rightharpoonup f \) in \( L^p(\Omega) \)
   and 
   \[ \left| \int_{\Omega} (f_{\varphi\psi(n)} - f) h \right| > \delta > 0, \]
what is contradictory. We conclude as the whole sequence \( f_n \) weakly converges toward \( f \) in \( L^p(\Omega) \).

2. The proof is exactly the same as in the previous case. It departs only in to establish that \( \tilde{f} = f \). In this case, we have immediately that

\[
\int_{\Omega} (f - \tilde{f}) g = 0,
\]

for all \( g \in L^\infty(\Omega) \). Choosing once again \( g = \text{sign}(f - \tilde{f}) \), we get that \( f = \tilde{f} \) a.e.

3. For every \( \varepsilon > 0 \), from the Egorov’s Theorem, there exists a measurable subset \( A \) of \( \Omega \), such that \( |A| < \varepsilon \) and \( f_n \) converges uniformly toward \( f \) in \( \Omega \setminus A \). We have

\[
\int_{\Omega} |f_n - f|^q = \int_{A} |f_n - f|^q + \int_{\Omega \setminus A} |f_n - f|^q.
\]

From Hölder’s inequality, we have

\[
\int_{A} |f_n - f|^q \leq \left( \int_{A} |f_n - f|^p \right)^{q/p} |A|^{\frac{q-p}{p}} < C \varepsilon^{\frac{p-q}{p}}.
\]

Moreover, as \( f_n \) uniformly converges toward \( f \) on \( \Omega \setminus A \), for \( n \) great enough, we have

\[
\int_{\Omega \setminus A} |f_n - f|^q < \varepsilon.
\]

We conclude that for \( n \) great enough,

\[
\int_{\Omega} |f_n - f|^q < C \varepsilon^{\frac{p-q}{p}} + \varepsilon,
\]

and that \( f_n \) converges toward \( f \) in \( L^q(\Omega) \) for all \( 1 \leq q < p \).

**Exercise 6  Rademacher’s functions**

Let \( 1 \leq p \leq \infty \) and let \( f \in L_{\text{loc}}^p(\mathbb{R}) \). Assume that \( f \) is \( T \)-periodic, i.e., \( f(x + T) = f(x) \), a.e. on \( \mathbb{R} \). Set

\[
\overline{f} = |T|^{-1} \int_0^T f(t) dt.
\]

Consider the sequence \( \{u_n\} \) in \( L^p(0,1) \) defined by

\[
u_n(x) = f(nx), \quad x \in (0,1).
\]

1. Prove that \( u_n \to \overline{f} \) with respect to the topology \( \sigma(L^p, L^p) \).
2. Determine \( \lim_{n \to \infty} \|u_n - \mathcal{J}\|_p \).

3. Examine the following examples:

1. \( u_n(x) = \sin(nx) \).
2. \( u_n(x) = f_n(x) \) where \( f \) is 1-periodic and
   \[
   f(x) = \begin{cases} 
   \alpha & \text{for } x \in (0, 1/2), \\
   \beta & \text{for } x \in (1/2, 1).
   \end{cases}
   \]

The functions of (2) are called \textit{Rademacher’s functions}.

\textbf{Answer of exercise 6}

1. Let \( 0 < a < b < 1 \) and \( v \) be the indicator function of \( (a, b) \) on \( (0, 1) \), that is
   \[
   v(x) = \begin{cases} 
   1 & \text{if } a < x < b, \\
   0 & \text{if } x \in (0, 1) \setminus (a, b).
   \end{cases}
   \]

We have
   \[
   \int_0^1 u_n v = \int_0^1 f(nx)v(x)\,dx = n^{-1} \int_0^n f(x)v(x/n)\,dx = n^{-1} \int_{na}^{nb} f(x)\,dx.
   \]

We set \( k \) and \( l \) to be the integers such that
   \[
   (k - 1)T < na \leq kT, \quad lT < nb \leq (l + 1)T.
   \]

We have
   \[
   \int_0^1 u_n v = \frac{1}{n} \left[ \int_{na}^{kT} f + \sum_{k \leq i \leq l-1} \int_{iT}^{(i+1)T} f + \int_{IT}^{nb} f \right]
   \]
   \[
   = \frac{1}{n} \left[ \sum_{k \leq i \leq l-1} \int_0^T f \right] + \frac{1}{n} \left[ \int_{na}^{kT} f + \int_{IT}^{nb} f \right]
   \]
   \[
   = \frac{l - k}{n} \int_0^T f + \frac{1}{n} \left[ \int_{na}^{kT} f + \int_{IT}^{nb} f \right].
   \]

From the definition of \( k \) and \( l \), we have
   \[
   \frac{l - k}{n} \leq \frac{b - a}{T} \leq \frac{l - k + 2}{n}.
   \]

Thus, \( (l - k)/n \to (b - a)/T \). Moreover,

\[
\frac{1}{n} \left| \int_{na}^{kT} f + \int_{IT}^{nb} f \right| \leq \frac{2}{n} \|f\|_{L^1(0, T)} \to 0.
\]

It follows that
   \[
   \int_0^1 u_n v \to \mathcal{J}(b - a),
   \]
as $n$ goes to infinity. We deduce that for any step function $v$, we have

$$\int_0^1 u_n v \rightarrow \mathcal{T} \int_0^1 v.$$ 

As the set of step functions is dense in $L^{p'}(0, 1)$, with $1 \leq p' < \infty$, we deduce that if $f \in L^p_{loc}(0, T)$, with $1 < p \leq \infty$, $u_n$ does converge toward $\mathcal{T}$ in $\sigma(L^p, L^{p'})$. Indeed, for every $v \in L^{p'}(0, 1)$ and for every $\varepsilon > 0$, there exists a step function $w$ such that $\|v - w\|_{L^{p'}(0, 1)} \leq \varepsilon$. We then have

$$\left| \int_0^1 u_n v - \mathcal{T} \int_0^1 v \right| \leq \left| \int_0^1 u_n w - \mathcal{T} \int_0^1 w \right| + \int_0^1 |u_n||v - w| + |\mathcal{T}| \int_0^1 |v - w|.$$ 

From Hölder inequality,

$$\int_0^1 |u_n||v - w| \leq \|u_n\|_{L^p(0, 1)} \|v - w\|_{L^{p'}(0, 1)}$$

and

$$|\mathcal{T}| \int_0^1 |v - w| \leq |\mathcal{T}| \|v - w\|_{L^{p'}(0, 1)}.$$ 

Moreover, from the previous analysis, we have

$$\|u_n\|^p_{L^p(0, 1)} = \int_0^1 u_n^p dx \rightarrow T^{-1} \int_0^T f^p.$$ 

In particular, $u_n$ is bounded in $L^p(0, 1)$. We have obtained that

$$\left| \int_0^1 u_n v - \mathcal{T} \int_0^1 v \right| \leq \left| \int_0^1 u_n w - \mathcal{T} \int_0^1 w \right| + C \|v - w\|_{L^{p'}(0, 1)}.$$ 

Finally, as $w$ is a step function, for $n$ great enough,

$$\left| \int_0^1 u_n w - \mathcal{T} \int_0^1 w \right| \leq \varepsilon$$

and

$$\left| \int_0^1 u_n v - \mathcal{T} \int_0^1 v \right| \leq (1 + C)\varepsilon.$$ 

It remains to consider the case $p = 1$ and $f \in L^1_{loc}(\mathbb{R})$. For every $\varepsilon > 0$, there exists a $T$ periodic function, $g \in L^\infty$ such that

$$T^{-1} \|f - g\|_{L^1(0, T)} \leq \varepsilon.$$ 

For every $v \in L^\infty(0, 1)$, we have from the previous analysis,

$$\int_0^1 g(nx)v(x) dx \rightarrow \mathcal{T} \int_0^1 v.$$ 

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On the hand, we have
\[ \left| \int_0^1 u_n v - T \int_0^1 v \right| \leq \| f(nx) - g(nx) \|_{L^1(0,1)} \| v \|_{\infty} + \left| \int_0^1 g(nx)v - T \int_0^1 v \right| \]

We have
\[ \| f(nx) - g(nx) \|_{L^1(0,1)} \to \frac{1}{T} \int_0^T |f - g| \leq \varepsilon. \]

Thus, for \( n \) great enough, we have
\[ \| f(nx) - g(nx) \|_{L^1(0,1)} \leq 2\varepsilon \]
and
\[ \left| \int_0^1 g(nx)v - T \int_0^1 v \right| \leq \varepsilon + |T - \mathcal{T}| \| v \|_{L^1(0,1)}. \]

Furthermore
\[ |T - \mathcal{T}| \leq T^{-1} \int_0^T |f - g| \leq \varepsilon. \]

We thus have proved that for \( n \) great enough
\[ \left| \int_0^1 u_n v - T \int_0^1 v \right| \leq 2\varepsilon \| v \|_{\infty} + \varepsilon + \varepsilon \| v \|_1, \]
and that \( \int u_n v \to \int f v \) as claimed.

2. We set \( g(s) = |f(s) - \mathcal{T}|^p \). As \( g \) is \( T \)-periodic ans \( g \in L^1_{\text{loc}}(\mathbb{R}) \), we have from the previous question
\[ \int_0^1 g(nx) dx \to \mathcal{g}, \]
that is
\[ \lim ||u_n - \mathcal{T}||^p = \frac{1}{|T|^p} \left| \int_0^T \left| \int_0^T (f(s) - f(t)) ds \right|^p dt \right|. \]

3. 1. \( u_n = \sin(nx) \). We have \( u_n \to 0 \) weakly-* in \( L^\infty \),
2. \( u_n = f(nx) \) where \( f \) is one periodic and
\[ f(x) = \begin{cases} \alpha & \text{if } x \in (0, 1/2) \\ \beta & \text{if } x \in (1/2, 1). \end{cases} \]

Then \( u_n \to (\alpha + \beta)/2 \) for the weak-* topology of \( L^\infty(0,1) \).