Exercise 1

Let $E$ and $F$ be Banach spaces.

1. Let $T \in \mathcal{K}(E,F)$ and $(x_n)$ be a weakly converging sequence in $E$. Prove that $Tx_n$ is strongly convergent in $F$.

2. Conversely, let us assume that $E$ is reflexive and let $T \in \mathcal{L}(E,F)$ such that for every weakly converging sequence $(x_n)$ in $E$, $(Tx_n)$ is strongly convergent in $F$. Prove that $T$ is compact.

Answer of exercise 1

1. Firstly, as $(x_n)$ is weakly convergent, it is bounded (see previous Tutorial Classes). Thus, as $T$ is compact, there exists $\varphi : \mathbb{N} \to \mathbb{N}$ increasing and $y \in F$ such that $(Tx_n)$ strongly converges toward $y$. Moreover, let $x$ the weak limit of $(x_n)$, for all $f \in F^*$, we have

$$\langle f, Tx \rangle = \langle T^* f, x \varphi(n) \rangle = \lim_n \langle f, T(x_{\varphi(n)}) \rangle = \langle f, y \rangle.$$ 

It follows that $y = T(x)$. We have proved that the subsequence $(x_{\varphi(n)})$ does converge toward $T(x)$. In fact, the whole sequence is converging. Otherwise, there exists $\varepsilon > 0$ and $\psi : \mathbb{N} \to \mathbb{N}$ increasing such that

$$\|x_{\psi(n)} - x\|_E > 0.$$ 

The sequence $(x_{\psi(n)})$ weakly converge toward $x$ in $E$. Thus, it admits a subsequence strongly convergent toward $x$, what is in contradiction with the last inequality.

2. Let $(x_n)$ be a bounded sequence in $E$. As $E$ is reflexive, it admits a subsequence $(x_{\varphi(n)})$ weakly convergent, and $Tx_{\varphi(n)}$ is strongly convergent. Thus, $T$ is compact.

Exercise 2

Let $E = \ell^p$ with $1 \leq p \leq \infty$. Let $\lambda_n$ be a bounded sequence in $\mathbb{R}$ and consider the operator $T \in \mathcal{L}(E)$ defined by

$$Tx = (\lambda_1 x_1, \ldots, \lambda_n x_n, \ldots),$$

where $x = (x_1, \ldots, x_n, \ldots)$. Prove that $T$ is a compact operator iff $\lambda_n \to 0$.

Answer of exercise 2

First of all, we are going to prove that if $T$ is compact, then $\lambda_n$ is a sequence that does converge toward zero. Let $(\lambda_n)$ be a sequence that does not converge toward zero. There exists $M > 0$ and an increasing sequence from $\mathbb{N}^*$ into $\mathbb{N}^*$ such that for all $n$,

$$|\lambda_{\varphi(n)}| > M$$

Let us introduce the sequence $(x^n)$ in $\ell^p$ defined by

$$x^n_k = \begin{cases} 1 & \text{if } k = \varphi(n) \\ 0 & \text{if } k \neq \varphi(n). \end{cases}$$
The sequence $x^n$ is bounded in $\ell^p$ and for all $n > m > 0$, we have

$$
\|T(x^n) - T(x^m)\|_{\ell^p} = \begin{cases} 
(\lambda_{\varphi(n)}^p + |\lambda_{\varphi(m)}|^p)^{1/p} & \text{if } p \neq \infty \\
\max(\lambda_{\varphi(n)}, |\lambda_{\varphi(m)}|) & \text{if } \infty.
\end{cases}
$$

So that for all $n \neq m$,

$$
\|x^n - x^m\|_{\ell^p} > M
$$

It follows that no subsequence of $(T(x^n))$ can be convergent in $\ell^p$, whereas $(x^n)$ is bounded in $\ell^p$. Thus, $T$ is not a compact operator on $\ell^p$.

Now, we have to prove the converse. Let us assume this time that $(\lambda_n)$ is a sequence that does converge toward zero. Let $(x^n)$ be a bounded sequence in $\ell^p$. Using a diagonal process, there exists an increasing map $\varphi: \mathbb{N} \to \mathbb{N}$ such that $x_{\varphi(n)}$ is converging for all $k \in \mathbb{N}^*$ as $n$ goes to infinity.

**Exercise 3**

Let $E$ and $F$ be two Banach spaces, and let $T \in \mathcal{L}(E, F)$.

1. Assume that $E$ is reflexive. Prove that $T(B_E)$ is closed (strongly).
2. Assume that $E$ is reflexive and that $T \in \mathcal{K}(E, F)$. Prove that $T(B_E)$ is compact.
3. Let $E = F = C([0, 1])$ and $T(u) = \int_0^1 u(s) \, ds$ Check that $T \in \mathcal{K}(E)$. Prove that $T(B_E)$ is not closed.

**Answer of exercise 3**

1. Let $x_n$ be a sequence in $B_E$ such that $y_n = T(x_n)$ converges in $F$ toward an element $F$. As $E$ is reflexive, the unit ball is weakly compact and without lost of generality, we can assume that it is weakly convergent toward an element $x \in B_E$. For all $L \in F^*$, $\langle L, T(\cdot) \rangle$ is a continuous linear form, so

$$
\langle L, T(x_n) \rangle \to \langle L, T(x) \rangle.
$$

Moreover, as $T(x_n)$ converges strongly toward $y$, we have

$$
\langle L, T(x) \rangle = \langle L, y \rangle,
$$

for all $L \in F^*$, so that $T(x) = y$. Finally, $x \in B_E$, so that $y \in T(B_E)$ and $T(B_E)$ is closed for the strong topology.

2. If $T$ is compact, $T(B_E)$ is relatively compact. Moreover, as $E$ is reflexive, $T(B_E)$ is closed. Thus, it is compact.

3. Let $u$ be an element of the unit ball of $C([0, 1])$. We have

$$
|T(u)(s) - T(u)(t)| \leq \int_s^t |u(x)| \, dx \leq |s - t|.
$$

Thus, $T(B_E)$ is uniformly equi-continuous and thus is relatively compact in $C([0, 1])$ (from the Ascoli Theorem). It is easy to see that $v(x) = |x - 1/2|$ belongs to the closure of $T(B_E)$ but does not belong to $T(B_E)$. 

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Exercise 4

Let $E$ and $F$ be two Banach spaces and let $T \in K(E,F)$. Assume that $\dim E = \infty$. Prove that there exists a sequence $(u_n)$ in $E$ such that $\|u_n\|_E = 1$ and $\|Tu_n\|_F \to 0$. [Hint: Argue by contradiction]

Answer of exercise 4

Assume that it is not the case, then we claim that there exists $r$ such that the ball $B_r(F)$ of $F$ of radius $r$ centered in the origin is such that $B_r(E) \cap T(S_E) = \emptyset$, where $S_E$ is the unit sphere of $E$. Thus, $T(B_E)$ contains the ball $B_r$ of $F$. As $T(B_E)$ is relatively compact, the ball of $F$ is also relatively compact. It follows that $F$ is of finite dimension. Finally, as $E$ is of non finite dimension, the kernel of $T$ is not empty and there exists $u \in S_E$ such that $T(u) = 0$ what is contradictory.

Exercise 5

Let $1 \leq p < \infty$. Check that $\ell^p \subset c_0$ with continuous injection (we recall that $c_0$ is the set of sequences $(x_n) \in \mathbb{R}^\mathbb{N}$ such that $\lim x_n = 0$. Is the injection compact?

Answer of exercise 5

1. Let $x \in \ell^p$. We have

$$\sum_{n} |x_n|^p < \infty.$$ 

Thus, $x_n \to 0$ as $n$ goes to infinity. Moreover,

$$\|x\|_{c_0} = \sup_n |x_n| \leq \left( \sum_{n} |x_n|^p \right)^{1/p} = \|x\|_{\ell^p}.$$ 

2. The injection is not compact. Indeed, let $(x^n)$ be the sequence in $\ell^p$ defined by

$$x^n_k = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

We have $\|x^n\|_{\ell^p} = 1$ for all $n$ and no subsequence of $x^n$ can be a Cauchy sequence of $c_0$ as for every $n \neq m$,

$$\|x^n - x^m\|_{c_0} = 1.$$ 

Exercise 6

Let $(\lambda_n)$ be a sequence of positive numbers such that $\lim_{n \to \infty} = +\infty$. Let $V$ be the space of sequences $(u_n)_{n \geq 1}$ such that

$$\sum_{n=1}^{\infty} \lambda_n |u_n|^2 < \infty.$$
The space $V$ is quipped with the scalar product

$$(u, v) = \sum_{n=1}^{\infty} \lambda_n u_n v_n.$$ 

Prove that $V$ is a hilbert space and that $V \subset \ell^2$ with compact injection.

**Answer of exercise 6**

First, let is prove that $V$ is a Hilbert space. Obviously, $(\cdot, \cdot)$ defines a scalar product and $\|u\|_V = ((u, u))^{1/2}$ is a norm on $V$. It remains to prove that $V$, endowed with this norm is complete. Let $u^n$ be a Cauchy sequence in $V$. We have

$$\|u^n - u^m\|_V^2 = \sum_{k=1}^{\infty} \lambda_n |u^n_k - u^m_k|^2.$$ 

Thus for every $k \in \mathbb{N}^*$, $u_k^n$ is a Cauchy sequence, and is convergent toward an element $u_k \in \mathbb{R}$. Moreover, for every $\varepsilon > 0$,

$$\|u^n - u\|_V^2 = \sum_{k=1}^{\infty} \lambda_n |u^n_k - u_k|^2 \leq \liminf_{m \to \infty} \sum_{k=1}^{\infty} \lambda_n |u^n_k - u^m_k|^2 \leq \varepsilon,$$

for $n$ great enough. Thus, $V$ is indeed a Banach space.

Next, we would like to prove that the $V \subset \ell^2$. This is a straightforward consequence of the inequality

$$\|u\|_{\ell^2} \leq \inf_n \lambda_n^{1/2} \|u\|_V.$$ 

It remains to prove that the injecton is compact. Let $(u^n)$ be a sequence in the unit ball of $V$. Using a diagonal process, we can extract a subsequence (still denoted $(u^n)$) such that $u^n_k$ is convergent toward an element $u_k \in \mathbb{R}$. Finally, for every $N > 0$,

$$\sum_k |u^n_k - u_k|^2 \leq \sum_{k=1}^{N} |u^n_k - u_k|^2 + \left( \inf_{k \geq N} \lambda_k \right)^{-1} \sum_{k=1}^{\infty} \lambda_k |u^n_k - u_k|^2$$

$$\leq \sum_{k=1}^{N} |u^n_k - u_k|^2 + 4 \left( \inf_{k \geq N} \lambda_k \right)^{-1} (\|u^n\|^2 + \|u^n\|^2)$$

$$\leq \sum_{k=1}^{N} |u_k^n - u_k|^2 + 8 \left( \inf_{k \geq N} \lambda_k \right)^{-1}.$$ 

For every $\varepsilon > 0$, there exists $N$ such that

$$\inf_{k \geq N} \lambda_k > \varepsilon/16.$$
and for \( n \) and \( m \) great enough,

\[
\sum_{k=1}^{N} |u_{nk} - u_{mk}|^2 < \varepsilon/2.
\]

It follows, that for \( n \) and \( m \) great enough,

\[
\|u^n - u^m\|_{\ell^2} < \varepsilon,
\]

meaning that \((u^n)\) is a Cauchy sequence in \( \ell^2 \). Thus, the injection of \( V \) into \( \ell^2 \) is compact as claimed.

**Exercise 7**

Let \( 1 \leq q \leq p \leq \infty \). Prove that the canonical injection from \( L^p(0, 1) \) into \( L^q(0, 1) \) is continuous but not compact. [**Hint:** Use Rademacher’s functions]

**Answer of exercise 7**

First, from the generalized Hölder inequality,

\[
\int_0^1 |f|^q \leq \int_0^1 |f|^p,
\]

proving that the injecton of \( L^p(0, 1) \) into \( L^q(0, 1) \) is continuous. We st \( u_n(x) = f(nx) \), where \( f \) is the 1-periodic function defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \geq 1/2, \\
0 & \text{if } x < 1/2.
\end{cases}
\]

We already know that \( f \) is weakly convergent in \( \sigma(L^\infty, L^1) \) toward the constant map \( 1/2 \). If the injection was compact, \( u_n \) would admit a subsequence \( u_{\varphi(n)} \) converging toward an element \( u \) in \( L^q(0, 1) \). As the strong convergence does imply the weak convergence, we would have \( u = 1/2 \). Finally, as

\[
\|u_n - u\|_q = \left( \int_0^1 |u_n - 1/2|^q \right)^{1/q} = 1/2,
\]

\( u_n \) can not converge toward \( u \), leading to a contradiction. Thus, the injection is not compact.