Functional analysis and applications
MASTER "Mathematical Modelling"
École Polytechnique and Université Pierre et Marie Curie
October 1st, 2015

See also the course webpage:
http://www.cmap.polytechnique.fr/˜allaire/master/course-funct-analysis.html

Exercise 1

Let $E = \ell^2$. Let λ_n be a bounded sequence in $\mathbb{R}$ and consider the operator $T \in \mathcal{L}(E)$ defined by

$$Tx = (\lambda_1 x_1, \cdots, \lambda_n x_n, \cdots),$$

where $x = (x_1, \cdots, x_n, \cdots)$. Prove that T is a compact operator iff $\lambda_n \to 0$.

Answer of exercise 1

First of all, we are going to prove that if T is compact, then λ_n is a sequence that does converge toward zero. Let (λ_n) be a sequence that does not converge toward zero. There exists $M > 0$ and an increasing sequence from $\mathbb{N}^*$ into $\mathbb{N}^*$ such that for all n,

$$|\lambda_{\varphi(n)}| > M$$

Let us introduce the sequence (x^n) in ℓ^2 defined by

$$x_k^n = \begin{cases} 1 & \text{if } k = \varphi(n) \\ 0 & \text{if } k \neq \varphi(n). \end{cases}$$

The sequence x^n is bounded in ℓ^2 and for all $n > m > 0$, we have

$$\|T(x^n) - T(x^m)\|_{\ell^2} = (|\lambda_{\varphi(n)}|^2 + |\lambda_{\varphi(m)}|^2)^{1/2}.$$

So that for all $n \neq m$,

$$\|T(x^n) - T(x^m)\|_{\ell^2} > M$$

It follows that no subsequence of $(T(x^n))$ can be convergent in ℓ^p, whereas (x^n) is bounded in ℓ^p. Thus, T is not a compact operator on ℓ^p.

Now, we have to prove the converse. Let us assume this time that (λ_n) is a sequence that does converge toward zero. Let (x^n) be a bounded sequence in ℓ^p. Using a diagonal process, there exists an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $x_k^{\varphi(n)}$ is converging for all $k \in \mathbb{N}^*$ as n goes to infinity.

Exercise 2

Let (λ_n) be a sequence of positive numbers ($\lambda_n > 0$) such that $\lim_{n\to\infty} \lambda_n = +\infty$. Let V be the space of sequences $(u_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} \lambda_n |u_n|^2 < \infty.$$

The space V is equipped with the scalar product

$$((u,v)) = \sum_{n=1}^{\infty} \lambda_n u_n v_n.$$

Prove that V is a Hilbert space and that $V \subset \ell^2$ with compact injection.

Answer of exercise 2

First, let us prove that V is a Hilbert space. Obviously, $((\cdot,\cdot))$ defines a scalar product and

$$\|u\|_V = (((u,u)))^{1/2}$$

is a norm on V. It remains to prove that V, endowed with this norm is complete. Let u^n be a Cauchy sequence in V. We have

$$\|u^n - u^m\|_V^2 = \sum_{k=1}^{\infty} \lambda_n |u_k^n - u_k^m|^2.$$

Thus for every $k \in \mathbb{N}^*$, u_k^n is a Cauchy sequence, and is convergent toward an element $u_k \in \mathbb{R}$. Moreover, for every $\varepsilon > 0$,

$$\|u^n - u\|_V^2 = \sum_{k=1}^{\infty} \lambda_k |u_k^n - u_k|^2 \leq \liminf_{m\to\infty} \sum_{k=1}^{\infty} \lambda_k |u_k^n - u_k^m|^2 \leq \varepsilon,$$

for n great enough. Thus, V is indeed a Banach space.

Next, we would like to prove that the $V \subset \ell^2$. This is a straightforward consequence of the inequality

$$\|u\|_{\ell^2} \leq \inf_n \lambda_n^{1/2} \|u\|_V.$$

It remains to prove that the injection is compact. Let (u^n) be a sequence in the unit ball of V, Using a diagonal process, we can extract a subsequence (still denoted (u^n)) such that u_k^n is convergent toward an element $u_k \in \mathbb{R}$. Finally, for every $N > 0$,

$$\begin{aligned}
\sum_k |u_k^n - u_k^m|^2 &\leq \sum_{k=1}^{N} |u_k^n - u_k^m|^2 + \left(\inf_{k\geq N} \lambda_k\right)^{-1} \sum \lambda_k |u_k^n - u_k^m|^2 \\
&\leq \sum_{k=1}^{N} |u_k^n - u_k^m|^2 + 4\left(\inf_{k\geq N} \lambda_k\right)^{-1} (\|u^n\|^2 + \|v^n\|_V^2) \\
&\leq \sum_{k=1}^{N} |u_k^n - u_k^m|^2 + 8\left(\inf_{k\geq N} \lambda_k\right)^{-1}.
\end{aligned}$$

For every $\varepsilon > 0$, there exists N such that

$$\inf_{k\geq N} \lambda_k > \varepsilon/16,$$

and for n and m great enough,

$$\sum_{k=1}^{N} |u_k^n - u_k^m|^2 < \varepsilon/2.$$

It follows, that for n and m great enough,

$$\|u^n - u^m\|_{\ell^2}^2 < \varepsilon,$$

meaning that (u^n) is a Cauchy sequence in ℓ^2. Thus, the injection of V into ℓ^2 is compact as claimed.

Exercise 3

Let $E = L^2(0,1)$. Given $u \in E$, set

$$Tu(x) = \int_0^x u(t)\,dt.$$

1. Prove that $T \in \mathcal{K}(E)$. [**Hint:** Use Ascoli-Arzelà Theorem]
2. Determine the set $EV(T)$ of eigenvalues of T.
3. Determine T^*.

Answer of exercise 3

1. Let (u_n) be a bounded sequence in $L^2(0,1)$. Let $0 < y < x < 1$. We want to prove that Tu_n is compact in E. From Hölder inequality, we have for all $u \in E$,

   $$|Tu(x) - Tu(y)| = \left|\int_y^x u(s)\,ds\right| \leq |x-y|^{1/2} \left(\int_y^x |u|^2\right)^{1/2} \leq |x-y|^{1/2} \|u\|_E.$$

   It follows that the sequence Tu_n is uniformly equicontinuous and from Ascoli-Arzelà Theorem, there exists a subsequence $Tu_{\varphi(n)}$ (where φ is an increasing map from $\mathbb{N}$ into $\mathbb{N}$) converging in $C([0,1])$. In particular, it converges in $L^2(0,1)$ (for the strong topology).
2. Let $\lambda \in EV(T)$, there exists $u \not\equiv 0$ in E such that

   $$\int_0^x u(s)\,ds = \lambda u(x)$$

   a.e. in Ω. Not that Tu admits a weak derivative and that

   $$(Tu)' = u.$$

   It follows that

   $$u = \lambda u'.$$

   The solution of this equation are $u = Ce^{x/\lambda}$. But, a $u(0) = 0$ we get that $u = 0$ is the only possible solution. Thus, $VP(T) = \emptyset$. Finally, as T is compact, we have $\sigma(T) \setminus \{0\} = VP(T) \setminus \{0\}$ and $0 \in \sigma(T)$. Thus, $\sigma(T) = 0$.
3. Let $u, v \in E$, $i(u, T^*v) = \int_t^1 v(x)dx$.