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HOMOGENIZATION OF THE UNSTEADY STOKES EQUATIONS IN POROUS MEDIA

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Contents

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Grégoire ALLAIRE

0) Introduction.

In [7] J.L. Lions studied the homogenization of the evolution Stokes problem in a periodic porous medium Ω_ε (of period ε)

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \nabla p_\varepsilon - \varepsilon^2 \Delta u_\varepsilon = f \ , \quad div\ u_\varepsilon = 0 \quad in\ \Omega_\varepsilon \\ u_\varepsilon = 0 \quad on\ \partial\Omega_\varepsilon \ , \quad u_\varepsilon(t{=}0,x) = a_\varepsilon(x) \end{cases} \tag{0.1}$$

where u_ε and p_ε denote the velocity and pressure of the fluid, f the density of forces acting on the fluid, and a_ε an initial condition for the velocity. By means of formal asymptotic expansions (see [5], [12]) he derived the homogenized problem for (0.1) as ε goes to zero

$$\begin{cases} u(t,x) \ = \ a(t,x) + \int_0^t A(t-s)[f-\nabla p](s,x)\, ds \quad in\ [0,T]\times\Omega \\ div\ u = 0 \quad in\ \Omega, \quad u.n \ = \ 0 \quad on\ \partial\Omega \end{cases} \tag{0.2}$$

where u and p denote the limit velocity and pressure, a is an initial condition which depends on a_ε and decays exponentially in time, and $A(t)$ is a symmetric permeability tensor. Problem (0.2) is a Darcy's law with memory which generalizes the usual Darcy's law obtained by homogenization of the steady Stokes equations [1], [6], [7], [12], [14].

The purpose of the present paper is to rigorously prove the convergence of the homogenization process, i.e. the convergence of the solutions $(u_\varepsilon, p_\varepsilon)$ of (0.1) to the solution (u, p) of (0.2) (see theorems 3.1 and 3.2). To this end, we use the new "two-scale convergence method" which was first introduced by G. Nguetseng [11], and further developed by the author [3], [4]. Loosely speaking, it is a rigorous justification of two-scale asymptotic expansions (see [5], [6], [12]), and thus, it is an alternative to the so-called "energy method" of L. Tartar [13]. Actually, besides the homogenization result itself, the main interest of the present paper is to demonstrate the power and the simplicity of the two-scale convergence method in the homogenization of a concrete example. The paper is organized as follows : section 1 is devoted to the setting of the problem, basic facts about two-scale convergence are introduced in section 2, while the main results are proved in section 3.

1) Setting of the problem.

As in [5], or [12], a periodic porous medium is defined by a domain Ω and an associated microstructure, or periodic cell $Y = [0;1]^N$, which is made of two complementary parts : the fluid part Y_f, and the solid part Y_s ($Y_f \cup Y_s = Y$ and $Y_f \cap Y_s = \emptyset$). More precisely, we assume that Ω is a smooth, bounded, connected set in $\mathbb{R}^N$, and that Y_f is a subset of Y which is smooth and connected in the unit torus, i.e. Y with periodic boundary condition (equivalently, the Y-periodic subset of $\mathbb{R}^N$, of period Y_f, is smooth and connected). The microscale of a porous medium is a (small) positive number ε. The domain Ω is covered by a regular mesh of size ε : each cell Y_i^ε is of the type $[0;\varepsilon]^N$, and is divided in a fluid part $Y_{f_i}^\varepsilon$ and a solid part $Y_{s_i}^\varepsilon$, i.e. is similar to the unit cell Y rescaled to size ε. The fluid part Ω_ε of a porous medium is defined by

$$\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} Y_{s_i}^\varepsilon = \Omega \cap \bigcup_{i=1}^{N(\varepsilon)} Y_{f_i}^\varepsilon \tag{1.1}$$

where the number of cells is $N(\varepsilon) = |\Omega| \varepsilon^{-N} [1+o(1)]$. Throughout the present paper, we assume that Ω_ε is a smooth, connected set in $\mathbb{R}^N$.

Remark 1.1.

This assumption on Ω_ε is of no fundamental importance in the sequel, but it appeals some comments from a technical point of view. It is automatically satisfied if the solid part Y_s is strictly included in the cell Y, and if we removed the solid parts $Y_{s_i}^\varepsilon$ which meet the boundary $\partial\Omega$ (see [12], and [14]). However, this is not the case when the solid part Y_s meets the boundary of the cell Y (near the boundary $\partial\Omega$, there may be some small connected components of Ω_ε, and the boundary of Ω_ε may be not smooth due to "wild" intersections between $\partial\Omega$ and $\partial Y_{f_i}^\varepsilon$, see [1]). Fortunately, the assumption on Ω_ε, being smooth and connected, is by no means necessary for the sequel, but, since avoiding it introduces some technicalities, we are going to use it anyway, in order to simplify the exposition.

We consider the unsteady Stokes equations in the fluid domain Ω_ε with a Dirichlet boundary condition. We denote by u_ε and p_ε the velocity and pressure of the fluid, f the density of forces acting on the fluid, and a_ε an initial condition for the velocity. We assume that the density of the fluid is equal to 1, while its viscosity is very small, and indeed is exactly ε^2 (where ε is the microscale). The system of equations is

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \nabla p_\varepsilon - \varepsilon^2 \Delta u_\varepsilon = f \ , \quad div\ u_\varepsilon = 0 \quad in\ \Omega_\varepsilon \\ u_\varepsilon = 0 \quad on\ \partial\Omega_\varepsilon \ , \quad u_\varepsilon(t{=}0,x) = a_\varepsilon(x). \end{cases} \tag{1.2}$$

Remark 1.2.

The scaling ε^2 of the viscosity is not surprising : indeed it is well-known (see [6], [7], and [12]), that it is the precise scaling which gives a non-zero limit for the velocity u_ε as ε goes to zero. The scaling 1 of the density is the precise one that keeps a dependence on time for the limit problem. With these scalings, system 1.2 was studied by J.L. Lions [7], using formal asymptotic expansions. A. Mikelic [10] studied (1.2) with an ε^2 scaling for the density, leading to a limit problem, different from ours, and with no inertial terms.

In (1.2), the force $f(t,x)$ is given in $[L^2([0,T]\times\Omega)]^N$, and the initial condition $a_\varepsilon(x)$ belongs to $[H_0^1(\Omega_\varepsilon)]^N$. Furthermore, denoting by $\tilde{\cdot}$ the extension operator by zero in $\Omega-\Omega_\varepsilon$, we assume that $\tilde{a}_\varepsilon$ satisfies

$$\|\tilde{a}_\varepsilon\|_{L^2(\Omega)} + \varepsilon\,\|\nabla\tilde{a}_\varepsilon\|_{L^2(\Omega)} \le C \quad and \quad div\ a_\varepsilon = 0 \quad in\ \Omega_\varepsilon . \tag{1.3}$$

Proposition 1.3.

The Stokes equations (1.2) admits a unique solution $u_\varepsilon \in L^2([0,T]\,;H_0^1(\Omega_\varepsilon)^N)$, and $p_\varepsilon \in L^2([0,T]\,;L^2(\Omega_\varepsilon)/I\!R)$. Furthermore, the extension by zero of the velocity $\tilde{u}_\varepsilon$ satisfies the a priori estimates

$$\|\tilde{u}_\varepsilon\|_{L^\infty([0,T];L^2(\Omega))} + \varepsilon\,\|\nabla\tilde{u}_\varepsilon\|_{L^\infty([0,T];L^2(\Omega))} \le C\ ,\ and\ \left\|\frac{\partial\tilde{u}_\varepsilon}{\partial t}\right\|_{L^2([0,T]\times\Omega)} \le C \tag{1.4}$$

where the constant C does not depend on ε. (The proof is left to the reader.)

Proposition 1.4.

There exists an extension P_ε of the pressure defined in $L^2([0,T]\,;L^2(\Omega)/I\!R)$ by

$$P_\varepsilon = p_\varepsilon \quad in\ \Omega_\varepsilon\ ,\ and\ P_\varepsilon = \frac{1}{|Y_{f_i}^\varepsilon|}\int_{Y_{f_i}^\varepsilon} p_\varepsilon \quad in\ each\ Y_{s_i}^\varepsilon \tag{1.5}$$

and a constant C, which does not depend on ε, such that

$$\|P_\varepsilon\|_{L^2([0,T];L^2(\Omega)/I\!R)} \le C. \tag{1.6}$$

Proof.

Proposition 1.4 is a mere combination of previous results of [14], [1], and [9]. We briefly sketch its proof. Introducing a projection operator R_ε from $H_0^1(\Omega)^N$ in $H_0^1(\Omega_\varepsilon)^N$, the extension P_ε is defined, a.e. in time, by

$$<\nabla P_\varepsilon, v>_{H^{-1},H_0^1(\Omega)} = <\nabla p_\varepsilon, R_\varepsilon v>_{H^{-1},H_0^1(\Omega_\varepsilon)} \quad \text{for any } v \in H_0^1(\Omega)^N. \tag{1.7}$$

Due to properties of the operator R_ε (see [14] in the case of isolated obstacles, and [1] in the case of connected obstacles), definition (1.7) makes sense. Estimate (1.6) is deduced from (1.7) by integration by parts, and using the estimates (1.4) on the velocity. Finally, the equivalent definition (1.5) is obtained from (1.7) by choosing suitable functions v with compact support in Y_i^ε and $Y_{s_i}^\varepsilon$ (see [9]). We point out that the assumption on Ω_ε, being smooth and connected, is used only here (without that assumption, the extension P_ε would be merely defined and bounded in $L^2_{loc}(\Omega)$).

Since (extensions of) the velocity u_ε and the pressure p_ε are bounded sequences as ε goes to zero, we can extract a subsequence such that they converge to a limit velocity u and pressure p. The homogenization process amounts to find a system of equations (the homogenized problem) satisfied by u and p. For this purpose, we introduce in the next section a new method of homogenization, called the two-scale convergence method.

2) Two-scale convergence.

Let $C_\#^\infty(Y)$ be the space of infinitely differentiable functions in $\mathbb{R}^N$ which are periodic of period Y. Denote by $L^2_\#(Y)$ (resp. $H^1_\#(Y)$) its completion for the norm of $L^2(Y)$ (resp. $H^1(Y)$). (Remark that $L^2_\#(Y)$ actually coincides with the space of functions in $L^2(Y)$ extended by Y-periodicity to the whole of $\mathbb{R}^N$.)

Following the lead of G. Nguesteng [11], we introduce the following

Definition 2.1.

A sequence of functions u_ε in $L^2(\Omega)$ is said to *two-scale converge* to a limit $u_0(x,y)$ belonging to $L^2(\Omega\times Y)$ if, for any function $\psi(x,y)$ in $D[\Omega;C_\#^\infty(Y)]$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\,dx = \iint_{\Omega Y} u_0(x,y)\psi(x,y)\,dxdy\ . \tag{2.1}$$

This new notion of "two-scale convergence" makes sense because of the next compactness theorem.

Theorem 2.2.

From each bounded sequence u_ε in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y)\in L^2(\Omega\times Y)$ such that this subsequence two-scale converges to u_0.

Theorem 2.2 is proved in [3], [4], [11]. The main idea of two-scale convergence is that, if a sequence $u_\varepsilon(x)$ is given as an expansion of the type

$u_0(x,\frac{x}{\varepsilon}) + \varepsilon u_1(x,\frac{x}{\varepsilon}) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon}) + \cdots$, where the functions $u_i(x,y)$ are Y-periodic in y, then the first term of the expansion actually coincides with the two-scale limit of u_ε. Loosely speaking, two-scale convergence captures the oscillations of a sequence which are in resonance with that of the test functions $\psi(x,\frac{x}{\varepsilon})$. For a given sequence u_ε, there is more information in its two-scale limit u_0 than in its weak-L^2 limit u, since u_0 contains some knowledge on the periodic oscillations of u_ε, while u is just an "average" of u_ε. These claims are made rigorous in the next proposition which establishes a link between two-scale and weak-L^2 convergences.

Proposition 2.3.

Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega\times Y)$. Then u_ε converges also to $u(x) = \int_Y u_0(x,y)\, dy$ in $L^2(\Omega)$ weakly. Furthermore, we have

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega\times Y)} \geq \|u\|_{L^2(\Omega)} . \tag{2.2}$$

Proof.

By taking test functions $\psi(x)$, which depends only on x, in (2.1), we immediately obtain that u_ε weakly converges to $u(x) = \int_Y u_0(x,y)\, dy$ in $L^2(\Omega)$. Let $\psi(x,y)$ be a smooth Y-periodic function

$$\int_\Omega [u_\varepsilon(x) - \psi(x,\frac{x}{\varepsilon})]^2\, dx \;=\; \int_\Omega u_\varepsilon(x)^2\, dx + \int_\Omega \psi(x,\frac{x}{\varepsilon})^2\, dx - 2\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \;\geq\; 0.$$

Passing to the limit as ε goes to zero yields

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)^2\, dx \;\geq\; 2\int_\Omega\int_Y u_0(x,y)\psi(x,y)\, dxdy - \int_\Omega\int_Y \psi(x,y)^2\, dxdy .$$

Then, using a sequence of smooth functions which converges strongly to u_0 in $L^2(\Omega\times Y)$ leads to the desired result.

The next theorem shows that, if a two-scale limit contains all the oscillations of a sequence (condition (2.3)), then one obtains a corrector-type result, i.e. a strong convergence for $u_\varepsilon(x) - u_0(x,\frac{x}{\varepsilon})$.

Theorem 2.4.

Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega\times Y)$. Assume that

$$\lim_{\varepsilon\to 0} \| u_\varepsilon \|_{L^2(\Omega)} = \| u_0 \|_{L^2(\Omega\times Y)} \tag{2.3}$$

and that $u_0(x,y)$ is sufficiently smooth (see remark 2.5), then

$$\lim_{\varepsilon\to 0} \| u_\varepsilon(x) - u_0(x,\frac{x}{\varepsilon}) \|_{L^2(\Omega)} = 0. \tag{2.4}$$

A proof of theorem 2.4 may be found in [3], [4].

Remark 2.5.

In the definition 2.1 of two-scale convergence, we consider very smooth test functions $\psi(x,y)$ (which are also Y-periodic in y). Their regularity can be weakened, but not too much since $\psi(x,\frac{x}{\varepsilon})$ needs to be measurable. We emphasize that this problem of measurability is not purely technical, but is linked to possible counter-examples of the well-known convergence result for periodic functions which says that $\psi(x,\frac{x}{\varepsilon})$ converges to $\int_Y \psi(x,y)\, dy$ in a suitable weak topology. For more details, we refer the interested reader to [4]. Here, it is enough to know that the regularity assumption on the test function $\psi(x,y)$ in definition 2.1, or on the two-scale limit $u_0(x,y)$ in theorem 2.4, can be, e.g., either $L^2[\Omega;C_\#(Y)]$, or $L^2_\#[Y;C(\overline{\Omega})]$ (roughly speaking, continuity is needed in only one variable).

Remark 2.6.

Two-scale convergence also applies to sequences $u_\varepsilon(t,x)$ which depends on a dummy variable t (here, t stands for the time variable, and dummy means that the test functions do not oscillate with respect to t). Theorem 2.2 is easily generalized as follows: for any sequence $u_\varepsilon(t,x)$ bounded in $L^2([0,T]\times\Omega)$, there exists a function $u_0(t,x,y)$ in $L^2([0,T]\times\Omega\times Y)$ such that, up to a subsequence and for any $\phi(t) \in C^\infty([0,T])$ and $\psi(x,y) \in D[\Omega;C^\infty_\#(Y)]$, one has

$$\lim_{\varepsilon\to 0} \int_0^T\int_\Omega u_\varepsilon(t,x)\phi(t)\psi(x,\frac{x}{\varepsilon})\, dtdx = \int_0^T\int_\Omega\int_Y u_0(t,x,y)\phi(t)\psi(x,y)\, dtdxdy . \tag{2.5}$$

In the two-scale limit (2.5), the variable t is merely a parameter, and the two-scale limit $u_0(t,x,y)$ does not capture any possible oscillations in t of the sequence u_ε.

3) Main results.

This section is devoted to the homogenization of the unsteady Stokes equations (1.2). The proof of convergence of the homogenization process is based on the two-scale convergence results obtained in section 2. In theorem 3.1, the limit problem is presented as a "two-scale homogenized" problem. In theorem 3.2, the same limit problem is proved to be equivalent to the "usual" homogenized problem combined with the cell problem. Both formulations of the limit problem have their pros and cons as discussed in remark 3.3. All the results of this section are proved under the assumption that the entire sequence $\tilde{a}_\varepsilon$ (the initial conditions of the Stokes problem (1.2)) two-scale converges to a unique limit $a_0(x,y)$. Remark that the only point in this assumption is the uniqueness of the two-scale limit. This is a very natural assumption, which is automatically satisfied if a_ε is itself the unique solution of a steady Stokes problem in Ω_ε (with a given force independent of ε).

Theorem 3.1.

The extension $(\tilde{u}_\varepsilon, P_\varepsilon)$ of the solution of (1.2) two-scale converges to the unique solution $(u_0(x,y), p(x))$ of the two-scale homogenized problem

$$\begin{cases} \dfrac{\partial u_0}{\partial t}(x,y) + \nabla_y p_1(x,y) + \nabla_x p(x) - \Delta_{yy} u_0(x,y) = f(x) \quad in\ [0,T]\times\Omega\times Y_f \\ div_y\, u_0(x,y) = 0 \quad in\ \Omega\times Y_f \quad \text{and} \quad div_x \left[\int_Y u_0(x,y)\, dy \right] = 0 \quad in\ \Omega \\ u_0(x,y) = 0 \quad in\ \Omega\times Y_s \quad \text{and} \quad \left[\int_Y u_0(x,y)\, dy \right].n = 0 \quad on\ \partial\Omega \\ y \to u_0, p_1 \quad Y\text{-periodic} \\ u_0(t=0) = a_0(x,y). \end{cases} \tag{3.1}$$

Theorem 3.2.

The extension $(\tilde{u}_\varepsilon, P_\varepsilon)$ of the solution of (1.2) converges, weakly in $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, to the unique solution (u,p) of the homogenized problem

$$\begin{cases} u(t,x) = a(t,x) + \int_0^t A(t-s)[f - \nabla p](s,x)\, ds \quad in\ [0,T]\times\Omega \\ div\ u(t,x) = 0 \quad in\ [0,T]\times\Omega \\ u(t,x).n = 0 \quad on\ [0,T]\times\partial\Omega \end{cases} \tag{3.2}$$

where $a(t,x)$ is an initial condition which depends only on the sequence a_ε and on the microstructure Y_f, and $A(t)$ is a symmetric, positive definite, (permeability) tensor which depends only on the microstructure Y_f (their precise form is to be found in the proof of the present theorem). Furthermore, the two-scale homogenized problem (3.1) is equivalent to (3.2) complemented with the cell problems (3.13)-(3.14), and $u(t,x) = \int_{Y_f} u_0(t,x,y)\, dy$, while the pressure $p(t,x)$ is the same in (3.1) and (3.2).

Remark 3.3.

The two-scale homogenized problem is also called a two pressures Stokes system (see [7]). The homogenized problem (3.2) is a Darcy's law with memory (due to the convolution in time). It is not difficult to check that both $a(t,x)$ and $A(t,x)$ decay exponentially in time. Thus, if the force f is steady (i.e. does not depend on t), asymptotically, for large time t, we recover the usual steady Darcy's law for u and p. In homogenization, the limit problem is usually presented as (3.2) (i.e. only macroscopic variables are used). However, in the present case, the elimination of the microscopic variable y induces a complicate, integro-differential, type for (3.2). Thus, for establishing that the limit problem is well-posed (i.e. existence and uniqueness of solutions), the "two-scale" form (3.1) of the limit problem is preferable. Furthermore, compared to (3.2), (3.1) contains some supplementary informations (namely, the so-called cell problem is included in (3.1)), which yields a corrector result for the velocity (theorem 3.5). The two approaches (3.1) or (3.2) of the limit problem were also discussed earlier by J.L. Lions (see chapter 2.5 in [7]).

Remark 3.4.

The homogenization of the evolution Stokes problem (1.2) can also be considered in a domain Ω_ε with isolated obstacles $Y_{s_i}^\varepsilon$ of size a_ε much smaller than the period ε. Using our previous results [2], it is easily seen that, when the obstacles are smaller than ε, but also larger than a given critical size (in 3-D, we require $\varepsilon^3 \ll a_\varepsilon \ll \varepsilon$), the corresponding homogenized system is a time dependent Darcy's law

$$\begin{cases} \dfrac{\partial u}{\partial t} + Mu + \nabla p = f \ , \quad div\ u = 0 \ \ in\ [0,T]\times\Omega \\ u.n = 0 \quad on\ [0,T]\times\partial\Omega \ , \quad u(t{=}0,x) = \bar{a}(x) \end{cases} \tag{3.3}$$

where M is a constant tensor, and $\bar{a}$ is an initial condition. We emphasize that the two situations (obstacles of size, either ε, or much smaller than ε) are completely different : in particular, the homogenized problem (3.2) can not be written under the form (3.3).

Theorem 3.5.

Assume that the initial condition satisfies $\lim_{\varepsilon\to 0}\int_\Omega |\tilde{a}_\varepsilon(x)|^2 dx = \iint_{\Omega Y} |a_0(x,y)|^2 dxdy$. Then, the convergence of the velocity is improved : $\lim_{\varepsilon\to 0} \| u_\varepsilon(t,x) - u_0(t,x,\frac{x}{\varepsilon}) \|_{L^2([0,T]\times\Omega)} = 0$.

The remaining part of this section is devoted to the proofs of the previous results. In view of the estimates (1.4) on the velocity u_ε, we can state the following

Lemma 3.6.

There exists a limit $u_0(t,x,y) \in L^2([0,T]\times\Omega\ ;\ H^1_\#(Y)^N)$ such that, up to a subsequence, the sequences $\tilde{u}_\varepsilon$, $\varepsilon\nabla\tilde{u}_\varepsilon$, and $\partial\tilde{u}_\varepsilon/\partial t$ two-scale converge to u_0, $\nabla_y u_0$, and $\partial u_0/\partial t$ respectively. Furthermore, u_0 satisfies

$$\left\{\begin{array}{ll} div_y u_0(t,x,y) = 0 \ \ in\ \Omega\times Y, \ \ and & div_x[\int_Y u_0(t,x,y)\, dy] = 0 \ \ in\ \Omega \\ u_0(t,x,y) = 0 \ \ in\ \Omega\times Y_s\ ,\ \ and & [\int_Y u_0(t,x,y)\, dy].n = 0 \ \ on\ \partial\Omega. \end{array}\right. \tag{3.4}$$

Proof.

By application of theorem 2.2 and remark 2.6, there exists three functions $u_0(t,x,y)$, $\xi_0(t,x,y)$, and $\zeta_0(t,x,y)$ in $L^2([0,T]\times\Omega\times Y)$ such that

$$\left\{\begin{array}{lcl} \lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon(t,x).\phi(t)\psi(x,\frac{x}{\varepsilon})\, dtdx &=& \int_0^T\int_\Omega\int_Y u_0(t,x,y).\phi(t)\psi(x,y)\, dtdxdy \\ \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varepsilon\nabla u_\varepsilon(t,x).\phi(t)\Xi(x,\frac{x}{\varepsilon})\, dtdx &=& \int_0^T\int_\Omega\int_Y \xi_0(t,x,y).\phi(t)\Xi(x,y)\, dtdxdy \\ \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}(t,x).\phi(t)\psi(x,\frac{x}{\varepsilon})\, dtdx &=& \int_0^T\int_\Omega\int_Y \zeta_0(t,x,y).\phi(t)\psi(x,y)\, dtdxdy \end{array}\right. \tag{3.5}$$

for any $\psi(x,y) \in D[\Omega;C^\infty_\#(Y)]^N$, $\Xi(x,y) \in D[\Omega;C^\infty_\#(Y)]^{N^2}$, and $\phi(t) \in D([0,T])$. Integrating by parts and passing to the two-scale limit in the two last lines of (3.5) yields

$$\left\{\begin{array}{l} \lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon\phi(t).div_y\Xi(x,\frac{x}{\varepsilon})\, dtdx = -\int_0^T\int_\Omega\int_Y \xi_0.\phi(t)\Xi(x,y)\, dtdxdy = \int_0^T\int_\Omega\int_Y u_0.\phi(t)div_y\Xi(x,y)\, dtdxdy \\ \lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon.\frac{\partial\phi(t)}{\partial t}\psi(x,\frac{x}{\varepsilon})\, dtdx = -\int_0^T\int_\Omega\int_Y \zeta_0.\phi(t)\psi(x,y)\, dtdxdy = \int_0^T\int_\Omega\int_Y u_0.\frac{\partial\phi(t)}{\partial t}\psi(x,y)\, dtdxdy \end{array}\right.$$

Desintegrating by parts leads to $\xi_0 = \nabla_y u_0$ and $\zeta_0 = \partial u_0/\partial t$. Moreover, the incompressibility condition $div\ u_\varepsilon = 0$ yields $div_y u_0(x,y) = 0$ and $div_x [\int_Y u_0(x,y) dy] = 0$, by integrating by parts the first line of (3.5) with $\psi(x,y)$ successively equal to $\nabla_y \theta(x,y)$ and $\nabla_x \theta(x)$. The other properties (3.4) are also easily obtained by a proper choice of test functions in (3.5).

Proof of theorem 3.1.

Let $\phi(t) \in C^\infty([0,T])$ with $\phi(T) = 0$. Let $\psi(x,y) \in D[\Omega;C_\#^\infty(Y)]^N$ with $\psi(x,y) \equiv 0$ in $\Omega \times Y_s$ (thus, $\psi(x,\frac{x}{\varepsilon}) \in [H_0^1(\Omega_\varepsilon)]^N$). Multiplying equation (1.2) by $\varepsilon\phi(t)\psi(x,\frac{x}{\varepsilon})$ and integrating by parts in the space variable x gives a single non-zero term when passing to the limit

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega P_\varepsilon \phi(t) div_y \psi(x,\frac{x}{\varepsilon})\ dtdx \ = \ 0. \tag{3.6}$$

Since P_ε is a bounded sequence in $L^2([0,T]\ ;\ L^2(\Omega)/\mathbb{R})$ (see proposition 1.4), it admits a two-scale limit $p_0(t,x,y)$. Passing to the limit in (3.6), we deduce

$$\int_0^T \int_\Omega \int_Y p_0(t,x,y) \phi(t) div_y \psi(x,y)\ dtdxdy \ = \ 0,$$

which implies that p_0 does not depend on y in Y_f. Using the particular form (1.5) of the extension P_ε in $\Omega - \Omega_\varepsilon$, we obtain the same result in Y_s, namely $p_0(t,x,y) = p(t,x)$.

Next, we add to the previous assumptions on $\psi(x,y)$ the incompressibility condition $div_y \psi(x,y) = 0$. Multiplying equation (1.2) by $\phi(t)\psi(x,\frac{x}{\varepsilon})$, integrating by parts, and passing to the two-scale limit yields

$$\begin{aligned} &\int_\Omega \int_Y a_0(x,y).\phi(0)\psi(x,y)\ dxdy \ - \int_0^T \int_\Omega \int_Y u_0(t,x,y).\frac{\partial \phi}{\partial t}(t)\psi(x,y)\ dtdxdy \\ &- \int_0^T \int_\Omega \int_Y p(t,x)\phi(t) div_x \psi(x,y)\ dtdxdy + \int_0^T \int_\Omega \int_Y \nabla_y u_0(t,x,y).\phi(t) \nabla_y \psi(x,y)\ dtdxdy \\ &= \int_0^T \int_\Omega \int_Y f(t,x)\phi(t).\psi(x,y)\ dtdxdy. \end{aligned} \tag{3.7}$$

Since (3.7) holds for any functions ϕ, with $\phi(T) = 0$, and ψ, with $\psi(x,y) \equiv 0$ in $\Omega \times Y_s$ and $div_y \psi(x,y) = 0$, and recalling that the orthogonal of divergence-free vector-functions is the set of all gradients (see lemma 3.8), there exists a pressure $p_1(t,x,y)$ such that

$$\frac{\partial u_0}{\partial t}(t,x,y) + \nabla_y p_1(t,x,y) + \nabla_x p(t,x) - \Delta_{yy} u_0(t,x,y) = f(t,x) \quad in \ [0,T]\times\Omega\times Y_f \ . \quad (3.8)$$

Together with (3.4) equation (3.8) is just the two-scale homogenized system (3.1). If (3.1) admits a unique solution, then the entire sequence ($\tilde{u}_\varepsilon, P_\varepsilon$) converges to its unique solution ($u_0(x,y)$, $p(x)$). Thus, the proof of theorem 3.1 is completed by the next lemma 3.7.

Lemma 3.7.

There exists a unique solution (u_0, p, p_1) of the two-scale homogenized system (3.1).

Proof.

Denote by $H^1_{0\#}(Y_f)$ the subspace of $H^1_\#(Y_f)$ composed of the functions which are zero on $\partial Y_f \cap \partial Y_s$. Let us define the Hilbert spaces

$$V = \left\{ v(x,y) \in L^2(\Omega; H^1_{0\#}(Y_f)^N) \ / \ div_y v = 0, \ div_x [\int_{Y_f} v dy] = 0, \ [\int_{Y_f} v dy].n_x = 0 \ on \ \partial\Omega \right\}, \quad (3.9)$$

H, the completion of V in $[L^2(\Omega\times Y_f)]^N$, and, denoting by V' the dual space of V,

$$E = \left\{ v(t,x,y) \ / \ v \in L^2([0,T];V), \ \frac{\partial v}{\partial t} \in L^2([0,T];V') \right\}, \text{ and } \ E_0 = \left\{ v \in E \ / \ v(T) = 0 \right\}.$$

Multiplying the equation (3.1) by a function $v \in E_0$, and integrating by parts leads to

$$\int_0^T\int_\Omega\int_{Y_f} \nabla_y u_0 . \nabla_y v \ dtdxdy - \int_0^T\int_\Omega\int_{Y_f} u_0 . \frac{\partial v}{\partial t} \ dtdxdy = \int_0^T\int_\Omega\int_{Y_f} f.v \ dtdxdy + \int_\Omega\int_{Y_f} a_0 . v(0) \ dxdy. \quad (3.10)$$

Since the left hand side of (3.10) is coercive on E_0, by application of the Lions lemma (see theorem 1.1, chapter 3, [8]), there exists a unique solution u_0 in $E \cap C([0,T];H)$ of the variational formulation (3.10). Furthermore, since $a_0 \in V$ and $f \in [L^2([0,T]\times\Omega)]^N$, the regularity of the solution is improved : $u_0 \in L^2([0,T]\times\Omega; H^2_\#(Y_f)^N)$ and $\partial u_0/\partial t \in L^2([0,T];H)$. It remains to prove that the variational formulation (3.10) is actually equivalent to the two-scale homogenized problem (3.1). The only difficulty is to obtain the two pressures $\nabla_x p(t,x) + \nabla_y p_1(t,x,y)$ when desintegrating by parts (3.10) : this is the purpose of the next lemma 3.8.

Lemma 3.8.

The orthogonal $V^\perp$ of the Hilbert space V, defined in (3.9), has the following characterization

$$V^\perp = \left\{ v(x,y) = \nabla_x \phi(x) + \nabla_y \phi_1(x,y) \ \text{ with } \phi \in H^1(\Omega), \ \text{ and } \phi_1 \in L^2(\Omega; L^2_\#(Y_f)) \right\}. \tag{3.11}$$

Proof.

Remark that $V = V_1 \cap V_2$ with

$$V_1 = \left\{ v(x,y) \in L^2(\Omega; H^1_{0\#}(Y_f)^N) \ / \ div_y v = 0 \right\}$$

$$V_2 = \left\{ v(x,y) \in L^2(\Omega; H^1_{0\#}(Y_f)^N) \ / \ div_x [\int_{Y_f} v dy] = 0, \ [\int_{Y_f} v dy].n_x = 0 \ on \ \partial\Omega \right\}.$$

It is a well-known result (see, e.g., [15], [16]) that

$$V_1^\perp = \left\{ \nabla_y \phi_1(x,y) \ / \ \phi_1 \in L^2(\Omega; L^2_\#(Y_f)) \right\}, \ \text{ and } \ V_2^\perp = \left\{ \nabla_x \phi(x) \ / \ \phi \in H^1(\Omega) \right\}.$$

Since V_1 and V_2 are two closed subspaces, it is equivalent to say $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$ or $V_1 + V_2 = \overline{V_1 + V_2}$. Indeed, we are going to prove that $V_1 + V_2$ is equal to $L^2(\Omega; H^1_{0\#}(Y_f)^N)$, which establishes that $V_1 + V_2$ is closed, and thus (3.11).

For $1 \le i \le N$, denote by $v_i(y)$ the unique solution in $[H^1_{0\#}(Y_f)]^N$ of the steady Stokes problem

$$\begin{cases} \nabla s_i - \Delta v_i = e_i \ , \ div \ v_i = 0 \ in \ Y_f \\ v_i = 0 \ on \ \partial Y_f \cap \partial Y_s \ , \ s_i, v_i \ Y\text{-}periodic. \end{cases}$$

For a given $v(x,y) \in L^2(\Omega; H^1_{0\#}(Y_f)^N)$, there exists a unique solution $p(x)$ in $H^1(\Omega)/\mathbb{R}$ of the Neuman problem

$$\begin{cases} div_x \left[\sum_{i=1}^N \int_{Y_f} v_i(y) dy \ \frac{\partial p}{\partial x_i}(x) - \int_{Y_f} v(x,y) dy \right] = 0 \quad in \ \Omega \\ \left[\sum_{i=1}^N \int_{Y_f} v_i(y) dy \ \frac{\partial p}{\partial x_i}(x) - \int_{Y_f} v(x,y) dy \right].n = 0 \quad on \ \partial\Omega \end{cases}$$

Remark that the constant matrix $(\int_{Y_f} v_i(y) \, dy)_{1 \le i \le N}$ is positive definite since $\int_{Y_f} v_i.e_j = \int_{Y_f} \nabla v_i . \nabla v_j$. Then, decomposing v as

$$v(x,y) \;=\; \sum_{i=1}^{N} v_i(y)\frac{\partial p}{\partial x_i}(x) + \left[v(x,y) - \sum_{i=1}^{N} v_i(y)\frac{\partial p}{\partial x_i}(x) \right], \tag{3.12}$$

it is easy to see that the first term in the right hand side of (3.12) belongs to V_1, while the second one belongs to V_2.

Q.E.D.

Proof of theorem 3.2.

First, by virtue of proposition 2.3, the sequence $(\tilde{u}_\varepsilon, P_\varepsilon)$ converges to $\left(\int_Y u_0(x,y)dy \ , p(x) \right)$ (the average, with respect to y, of its two-scale limit) in $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$ weakly. Second, to obtain the homogenized problem (3.2), we separate the variables x and y in the two-scale homogenized problem (3.1). We decompose its solution u_0 in two parts $u_1 + u_2$ where u_1 is just the evolution (without any forcing term) of the initial condition a_0. Thus u_1 is the unique solution of

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(t,x,y) + \nabla_y q(t,x,y) - \Delta_{yy} u_1(t,x,y) \;=\; 0 \quad in \ \Omega \times Y_f \\ div_y u_1(t,x,y) = 0 \quad in \ \Omega \times Y_f \\ u_1 \;=\; 0 \quad on \ \partial Y_f \cap \partial Y_s \ , \ y \to u_1, q \quad Y\text{-}periodic \\ w_i(t{=}0,x,y) = a_0(x,y). \end{cases} \tag{3.13}$$

The average of u_1 in y is just $a(t,x)$ (the initial condition in the homogenized system (3.2)). On the other hand, u_2 is given by

$$u_2(t,x,y) \;=\; \int_0^t \sum_{i=1}^{N} [f_i - \frac{\partial p}{\partial x_i}](s,x) \frac{\partial w_i}{\partial t}(t-s,y) \, ds \tag{3.14}$$

where, for $1 \le i \le N$, w_i is the unique solution of the cell problem

$$\begin{cases} \dfrac{\partial w_i}{\partial t}(t,y) + \nabla_y q_i(t,y) - \Delta_{yy} w_i(t,y) \;=\; e_i \quad in \ Y_f \\ div_y w_i \;=\; 0 \quad in \ Y_f \\ w_i = 0 \quad on \ \partial Y_f \cap \partial Y_s \ , \ y \to w_i, q_i \quad Y\text{-}periodic \\ w_i(t{=}0,y) = 0. \end{cases} \tag{3.15}$$

Introducing the matrix A defined by

$$A_{ij}(t) \;=\; \int_{Y_f} \frac{\partial w_i}{\partial t}(t,y) e_j \, dy, \tag{3.16}$$

we deduce (3.2) from (3.1), by averaging u_1 and u_2 with respect to y (actually, (3.1) is equivalent to (3.2) combined with (3.13)-(3.16)). Eventually, using semi-group theory, one can prove that A is symmetric, positive definite, and decays exponentially in time.

Proof of theorem 3.5.

Multiplying the Stokes equation (1.2) by u_ε leads to

$$\frac{1}{2}\int_\Omega |\tilde{u}_\varepsilon(T)|^2 - \frac{1}{2}\int_\Omega |\tilde{a}_\varepsilon|^2 + \varepsilon^2 \int_0^T\int_\Omega |\nabla \tilde{u}_\varepsilon|^2 = \int_0^T\int_\Omega f.\tilde{u}_\varepsilon . \tag{3.17}$$

Multiplying the two-scale homogenized equation (3.1) by u_0 yields

$$\frac{1}{2}\int_\Omega\int_Y |u_0(T)|^2 - \frac{1}{2}\int_\Omega\int_Y |a_0|^2 + \int_0^T\int_\Omega\int_Y |\nabla_y u_0|^2 = \int_0^T\int_\Omega\int_Y f.u_0 . \tag{3.18}$$

The right hand side of (3.17) converges to that of (3.18), and by assumption so does $\int_\Omega |\tilde{a}_\varepsilon|^2$ to $\int_\Omega\int_Y |a_0|^2$. Thus, as ε goes to zero,

$$\frac{1}{2}\int_\Omega |\tilde{u}_\varepsilon(T)|^2 + \varepsilon^2 \int_0^T\int_\Omega |\nabla \tilde{u}_\varepsilon|^2 \to \frac{1}{2}\int_\Omega\int_Y |u_0(T)|^2 + \int_0^T\int_\Omega\int_Y |\nabla_y u_0|^2. \tag{3.19}$$

By virtue of proposition 2.3, the limit of each term on the left hand side of (3.19) is greater than its corresponding term in the right hand side. Thus

$$\lim_{\varepsilon\to 0} \int_\Omega |\tilde{u}_\varepsilon(T)|^2 = \int_\Omega\int_Y |u_0(T)|^2.$$

By application of theorem 2.4, we obtain the desired result.

Note added in proof. After this work has been completed, I learned that similar results have been recently and independently obtained by M. Avellaneda and A. Mikelic.

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