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A NEW FINITE ELEMENT SCHEME FOR LANDAU-LIFCHITZ EQUATIONS

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Abstract. In this paper we describe a new implicit finite element scheme for the discretization of Landau-Lifchitz equations. A proof of convergence of the numerical solution to a (weak) solution of the original equations is given and numerical tests showing the applicability of the method are also provided.

1. **Introduction.** Landau-Lifchitz equations describe the evolution of the magnetization m (a three dimensional vectorfield) inside a ferromagnetic body Ω (typically an open bounded subset of $\mathbb{R}^d$). After adimensioning these equations write down as

$$\left[\begin{array}{l} \partial_t m = m \times H_{\text{eff}} - \alpha m \times (m \times H_{\text{eff}}) \text{ in } \Omega, \\ \partial_n m = 0 \text{ on } \partial\Omega, \end{array}\right. \tag{1}$$

where H_{eff} stands for the effective magnetic field, '$\times$' is the three dimensional cross product, and the magnitude of the magnetization (which is constant in space and time) has been scaled to one

$$|m(x,t)| = 1. \tag{2}$$

In (1), $\alpha > 0$ is a damping parameter, and

$$H_{\text{eff}}(m) = -\frac{\partial \mathcal{E}}{\partial m}$$

is the (opposite of the) functional derivative of the free energy $\mathcal{E}$. Typical expressions for $\mathcal{E}$ that are usually used in practice take into account several different physical phenomena, and can be found in [9] for instance.

The purpose of this paper is to derive an implicit finite element scheme for solving numerically (1). As we are only interested into the main description of the method (and not yet into its complete physical relevance), we will focus on the case where $H_{\text{eff}}(m)$ contains only its highest order term, namely

$$H_{\text{eff}}(m) = \Delta m.$$

The generalization to classical models of the method can be done by following exactly the arguments exposed in [3].

The model Cauchy problem we have in mind therefore writes down as

$$\left[\begin{array}{l} \partial_t m = m \times \Delta m - \alpha m \times (m \times \Delta m) \text{ in } \Omega, \\ \partial_n m = 0 \text{ on } \partial\Omega, \\ m(x,0) = m_0(x), \end{array}\right. \tag{3}$$

where the initial data $m_0 \in H^1(\Omega, S^2)$. It is easily seen that, in that case, the energy

$$\mathcal{E}(m) = \frac{1}{2}\int_\Omega |\nabla m|^2 \,,$$

formally decays along the trajectories thanks to the (still formal) relation

$$d_t\mathcal{E}(m) = -\alpha \int_\Omega |m \times \Delta m|^2 = -\frac{\alpha}{1+\alpha^2}\int_\Omega |\partial_t m|^2 \,.$$

These equations have already been substantially studied in the literature and solutions to (3) have been found:

- either in a strong sense [8] but only locally in time, or globally but for initial data of small energy and in 2D;
- either in a weak sense [4, 14]. In that case, solutions to (3) are shown to be global in time but may not be unique (explicit cases of nonuniqueness are provided in [4]).

As far as the discretization of (3) by finite elements methods is concerned, a lot of papers have been recently published. Besides the book by Prohl [10] where the discretization was obtained via a penalization strategy (and convergence to local strong solutions was proved), a completely novel approach was given in [3], where the authors used the definition of weak solutions of (3) given in [4] to get a finite element discretization of the problem which does not use any penalization parameter (always difficult to tune in practice), and which satisfies the non-linear constraint (2) at the nodes of the triangularization. Although based on an explicit scheme, the authors have been able to prove the weak convergence of their numerical solutions to a weak global solution of (3) when first the time step k and then the space step h tend to 0. Later on, their convergence results have been improved in [6] by requiring

$$\frac{k}{h^{1+\frac{d}{2}}} \to 0, \tag{4}$$

where d is the dimension in which the problem is posed.

Obviously, the scheme being explicit, there exists a stability condition (like for the heat equation) in order to get convergence of the method. To avoid this very restrictive condition, one usually has to go to implicit schemes. Such schemes have been proposed [7, 11], but are all based on a non-linear iteration. This non-linear iteration is usually solved via a kind of Newton's method for which the authors prove the convergence under an assumption close to (4).

Therefore, there is a need for an implicit finite element scheme which would require only to solve linear systems at each iteration, be unconditionally stable and convergent to a weak solution of (3) when $k, h \to 0$. This is the purpose of this paper.

The content of this paper is as follows: Section 2 is devoted to the description of our scheme, section 3 gives the convergence theorem, and numerical experiments are provided in section 4.

2. **A $\theta-$scheme for Landau-Lifchitz equations.** We use hereafter the following notations for the finite element spaces. Namely, for $(\mathcal{T}_h)_h$ a regular family of conformal triangulations of the domain Ω parameterized by the space step h, we call $(x_i^h)_i$ the vertices of $\mathcal{T}_h$ and $(\phi_i^h)_{1\le i\le N_h}$ the set of associated basis functions of the so-called $P^1(\mathcal{T}_h)$ discretization. That is to say the functions $(\phi_i^h)_i$ are continuous

and linear on each triangle (or tetrahedron in 3D), and satisfy $\phi_i^h(x_j^h) = \delta_{i,j}$ the Kronecker symbol. We also call (see also [3])

$$V_h = \left\{ u = \sum_i u_i \phi_i^h, \ \text{ s. t. } \forall i,\, u_i \in \mathbb{R}^3 \right\},\ M_h = \left\{ u \in V_h, \ \text{ s. t. } \forall i,\, u_i \in \mathbb{S}^2 \right\},$$

the finite element space and the subset of V_h where the solution will be sought. For any $u = \sum_i u_i \phi_i^h \in M_h$, we call

$$K_u = \left\{ v = \sum_i v_i \phi_i^h, \ \text{ s. t. } \forall i,\, v_i \cdot u_i = 0 \right\}.$$

We also denote by $\mathcal{I}_h$ the classical interpolation operator

$$\begin{array}{rcl} \mathcal{I}_h : C^0(\Omega, \mathbb{R}^3) & \to & V_h \\ u & \mapsto & \displaystyle\sum_i u(x_i^h)\phi_i^h. \end{array}$$

In [3] the finite element scheme proposed to approximate (3) was based upon the observation that one may (at least formally) rewrite (3) into two different equivalent forms

$$\partial_t m + \alpha m \times \partial_t m = (1+\alpha^2) m \times \Delta m, \tag{5}$$

or

$$\alpha \partial_t m - m \times \partial_t m = (1+\alpha^2)(\Delta m - |\nabla m|^2 m). \tag{6}$$

The first form is the so-called Gilbert form and is useful to define weak solutions [4] (see also section 3) while the second one is used to build the scheme in [3]. Namely, calling $v = \partial_t m$, one has from (6)

$$\alpha \int_\Omega v \cdot \psi - \int_\Omega m \times v \cdot \psi = -(1+\alpha^2) \int_\Omega \nabla m \cdot \nabla \psi, \tag{7}$$

for every test function $\psi \in H^1(\Omega, \mathbb{R}^3)$ which furthermore verifies $\psi(x) \cdot m(x) = 0$ for a.e. x in Ω. Finding a discretized version of (7) is now straightforward [3]

$$\begin{array}{l} \text{Start with an initial } m^0 \in M_h, \\ \text{For } n = 0, 1, \cdots, \\ \left[\begin{array}{l} \text{Find } v^n \in K_{m^n} \text{ such that } \forall \psi \in K_{m^n}, \\ \qquad \alpha \displaystyle\int_\Omega v^n \cdot \psi - \int_\Omega m^n \times v^n \cdot \psi = -(1+\alpha^2) \int_\Omega \nabla m^n \cdot \nabla \psi, \\ \text{Set } m^{n+1} = \displaystyle\sum_i m_i^{n+1} \phi_i^h, \text{ with } \forall i,\ m_i^{n+1} = \frac{m_i^n + k v_i^n}{|m_i^n + k v_i^n|}, \text{ and iterate.} \end{array} \right. \end{array} \tag{8}$$

Due to the construction, it is clear that $m^n \in M_h$ for all $n \in \mathbb{N}$. In [3], it was explained that this scheme follows the spirit of the explicit scheme for solving the heat equation and in [6], it was shown that (4) is sufficient to making converge the solution towards a (global) weak solution of (3) after suitable interpolation in space and time (of course $u^n \sim u(nk)$).

We now consider the following generalization of (8). Considering a parameter $\theta \in [0,1]$, we modify the previous scheme as follows:

$$
\begin{array}{l}
\text{Start with an initial } m^0 \in M_h,\\
\text{For } n = 0, 1, \cdots,\\
\left[\begin{array}{l}
\text{Find } v^n \in K_{m^n} \text{ such that } \forall \psi \in K_{m^n},\\
\alpha \displaystyle\int_\Omega v^n \cdot \psi - \int_\Omega m^n \times v^n \cdot \psi = -(1+\alpha^2) \int_\Omega \nabla(m^n + \theta k v^n) \cdot \nabla \psi,\\
\text{Set } m^{n+1} = \displaystyle\sum_i m_i^{n+1} \phi_i^h, \text{ with } \forall i,\ m_i^{n+1} = \frac{m_i^n + k v_i^n}{|m_i^n + k v_i^n|}, \text{ and iterate.}
\end{array}\right.
\end{array}
\tag{9}
$$

This new scheme looks like the classical $\theta-$scheme for solving the heat equation and therefore, we expect that it is unconditionally stable as soon as $\frac{1}{2} \le \theta \le 1$. In particular, $\theta = 0$ gives back (8), $\theta = \frac{1}{2}$ gives a Crank-Nicolson like scheme while $\theta = 1$ is a kind of fully implicit scheme. Of course, the renormalization stage a priori forbids the scheme to be of order 2 in time (even with $\theta = \frac{1}{2}$).

We also remark that, as for (8), the problem that needs to be solved at each iteration is always linear and can be written as:

$$
\text{Find } v^n \in K_{m^n}, \text{ such that } \forall \psi \in K_{m^n},\ a(v^n, \psi) + b^n(v^n, \psi) = L^n(\psi), \tag{10}
$$

where the bilinear forms

$$
a(\phi, \psi) = \alpha \int_\Omega \phi \cdot \psi + \theta k (1+\alpha^2) \int_\Omega \nabla \phi \cdot \nabla \psi, \quad b^n(\phi, \psi) = -\int_\Omega m^n \times \phi \cdot \psi
$$

are respectively symmetric positive definite, and skew symmetric on K_{m^n}, and the linear form L^n is defined by

$$
L^n(\psi) = -(1+\alpha^2) \int_\Omega \nabla m^n \cdot \nabla \psi.
$$

Therefore, the problem (10) possesses a unique solution $v^n \in K_{m^n}$.

The renormalization stage has been extensively used for related problem. In [1] for instance, it was one of the fundamental arguments to build a finite element scheme for the problem of finding minimizing harmonic maps into the unit sphere. Moreover, it was also remarked that for maps $w \in H^1(\Omega, \mathbb{R}^3)$, such that $|w(x)| \ge 1$ a.e. $x \in \Omega$, one has

$$
\int_\Omega \left| \nabla \frac{w}{|w|} \right|^2 \le \int_\Omega |\nabla w|^2, \tag{11}
$$

and hence this renormalization stage is expected to be energy decreasing. Other applications more related to finite element approximation of micromagnetic configurations can be found in [2].

However, if (11) holds for H^1 maps, it does not necessary hold at the discrete level: if $v = \sum_i v_i \phi_i^h \in V_h$ is such that $\forall i \in \{1, \cdots, N_h\}$, $|v_i| \ge 1$, do we have

$$
\int_\Omega \left| \nabla \mathcal{I}_h \left(\frac{v}{|v|} \right) \right|^2 \le \int_\Omega |\nabla v|^2\,? \tag{12}
$$

This particular question has been studied in [5], and the following result was proven.

Theorem 1. [5] *For the P^1 approximation, if*

$$\forall i \neq j, \ \int_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \leq 0, \tag{13}$$

then (12) holds.

Theorem 1 is remarkable, since it relates a condition on the mesh to an analytic property. In particular, there were given in [5] a map v and a mesh for which (12) does not hold. Moreover, it turns out that (13) is related to discrete maximum principle and is satisfied in the two following cases [13].

- In 2D, when the mesh is of Delaunay type,
- In 3D when all dihedral angles of the tetrahedra are smaller than $\pi/2$.

As most of the mesh generators are building Delaunay meshes, this is therefore not a serious constraint (at least in 2D).

3. **The convergence result.** Before stating the convergence theorem, we recall the definition of a weak solution of (3).

Definition 1. Let $m_0 \in H^1(\Omega)^3$ be such that $|m_0| = 1$ a.e., we say that m is a weak solution to (3) if for all $T > 0$

1. $m \in H^1(Q_T)^3$ $(Q_T = \Omega \times (0,T))$; $|m| = 1$ a.e.;
2. for all ϕ in $H^1(Q_T)^3$, there holds :

$$\begin{aligned}\int_{Q_T} \partial_t m \cdot \phi \, dx\, dt \quad &+ \quad \alpha \int_{Q_T} (m \times \partial_t m) \cdot \phi \, dx\, dt \\ &= \quad -(1+\alpha^2) \sum_{i=1}^{d} \int_{Q_T} (m \times \partial_{x_i} m) \cdot \partial_{x_i}\phi \, dx\, dt.\end{aligned} \tag{14}$$

3. $m(x,0) = m_0(x)$ in the trace sense ;
4. we have the energy inequality

$$\frac{1}{2}\int_\Omega |\nabla m(T)|^2 \, dx + \frac{\alpha}{1+\alpha^2}\int_{Q_T} |\partial_t m|^2 \, dx\, dt \leq \frac{1}{2}\int_\Omega |\nabla m_0|^2 \, dx. \tag{15}$$

We also need to interpolate in time the discrete solutions constructed via algorithm (9). We therefore introduce $T > 0$, and $J = \left[\frac{T}{k}\right]$.

Definition 2. The discrete solutions are interpolated in time in different ways: for all $x \in \Omega$ and all $t \in [0,T]$, calling $j \in \{0, \cdots, J\}$ such that $t \in [jk, (j+1)k)$, we set

$$\begin{aligned} m_{h,k}(x,t) &= \frac{t - jk}{k} m^{j+1}(x) + \frac{(j+1)k - t}{k} m^j(x), \\ m_{h,k}^-(x,t) &= m^j(x), \ v_{h,k}(x,t) = v^j(x). \end{aligned}$$

Theorem 2. *Let $m_0 \in H^1(\Omega, S^2)$. Suppose $m^0 \to m_0$ in $H^1(\Omega)$ as $h \to 0$, and $\theta \in (\frac{1}{2}, 1]$. If the regular sequence of conformal triangulations $(\mathcal{T}_h)_h$ satisfies condition (13), then $(m_{h,k})$ converges (up to the extraction of a subsequence) weakly in $H^1(\Omega \times (0,T))$ to a weak solution m of (3) as h and k tend to 0.*

Proof. As we have already observed, the variational formulation (10) possesses a unique solution v^n. Taking $\phi = v^n$, leads then to the following estimate

$$\frac{\alpha}{1+\alpha^2}\int_\Omega |v^n|^2 + \theta k \int_\Omega |\nabla v^n|^2 = -\int_\Omega \nabla m^n \cdot \nabla v^n.$$

Now, using the fact that $\mathcal{T}_h$ satisfies (12) leads to

$$\begin{aligned}\frac{1}{2}\int_\Omega \left|\nabla m^{n+1}\right|^2 &\leq \frac{1}{2}\int_\Omega \left|\nabla(m^n + kv^n)\right|^2\\ &\leq \frac{1}{2}\int_\Omega |\nabla m^n|^2 + k\int_\Omega \nabla m^n \cdot \nabla v^n + \frac{k^2}{2}\int_\Omega |\nabla v^n|^2\\ &\leq \frac{1}{2}\int_\Omega |\nabla m^n|^2 - \frac{\alpha k}{1+\alpha^2}\int_\Omega |v^n|^2\\ &\qquad - \left(\theta - \frac{1}{2}\right)k^2\int_\Omega |\nabla v^n|^2 . \end{aligned} \tag{16}$$

Summing this inequality from $n = 0$ to $n = j-1$ gives (since $\theta > \frac{1}{2}$)

$$\frac{1}{2}\int_\Omega |\nabla m^j|^2 + \frac{\alpha}{1+\alpha^2}\sum_{n=0}^{j-1} k\int_\Omega |v^n|^2 \leq \frac{1}{2}\int_\Omega |\nabla m^0|^2. \tag{17}$$

Now, we use the fact that

$$\forall n \leq J,\ \forall i \in \{1, \cdots, N_h\},\ \left|\frac{m_i^{n+1} - m_i^n}{k}\right| \leq |v_i^n|,$$

and since there exists $c > 0$ such that for all $1 \leq p < +\infty$ and all $\phi_h \in V_h$ there holds

$$\frac{1}{c}||\phi_h||^p_{L^p(\Omega)} \leq h^d \sum_i |\phi_h(x_i^h)|^p \leq c||\phi_h||^p_{L^p(\Omega)}, \tag{18}$$

we obtain that

$$\left\|\frac{m^{n+1} - m^n}{k}\right\|_{L^2} \leq c^2||v^n||_{L^2}.$$

Hence, from the bound (17), $m_{h,k}$ is uniformly bounded in $H^1(Q_T)$, and $v_{h,k}$ is bounded in $L^2(Q_T)$. Extracting possibly subsequences, there exist $m \in H^1(Q_T)$ and $v \in L^2(Q_T)$ such that

$$m_{h,k} \rightharpoonup_{(h,k)\to 0} m \text{ weakly in } H^1(Q_T), \tag{19}$$

$$m_{h,k} \to_{(h,k)\to 0} m \text{ strongly in } L^2(Q_T), \tag{20}$$

$$v_{h,k} \rightharpoonup_{(h,k)\to 0} v \text{ weakly in } L^2(Q_T). \tag{21}$$

Now, since for all $j = 0, \cdots, J$ and all $t \in [jk, (j+1)k)$

$$|m_{h,k}(x,t) - m^-_{h,k}(x,t)| = \left|(t - jk)\left(\frac{m^{j+1}(x) - m^j(x)}{k}\right)\right| \leq k\left|\partial_t m_{h,k}(x,t)\right|,$$

we get

$$||m_{h,k} - m^-_{h,k}||_{L^2(Q_T)} \leq k\left\|\partial_t m_{h,k}\right\|_{L^2(Q_T)} \to_{(h,k)\to 0} 0.$$

Therefore

$$m^-_{h,k} \to_{(h,k)\to 0} m \text{ strongly in } L^2(Q_T).$$

Moreover, on any triangle (tetrahedron in 3D) K of $\mathcal{T}_h$, and any $u \in M_h$ one has, x_i^h being any vertex of K

$$\left||u(x)| - |u(x_i^h)|\right|^2 \leq Ch^2|\nabla u|^2,$$

(recall that ∇u is constant on K), from which one deduces (since $|m^-_{h,k}(x_i^h)| = 1$)

$$\int_{Q_T}\left|1 - |m^-_{h,k}|\right|^2 \leq Ch^2||\nabla m^-_{h,k}||^2_{L^2(Q_T)}, \tag{22}$$

which shows that $|m(x,t)| = 1$ a.e. $(x,t) \in Q_T$.

Eventually, from the fact that $\forall i \in \{1, \cdots, N_h\}$

$$|m_i^{n+1} - m_i^n - kv_i^n| = |m_i^n + kv_i^n| - 1 \leq \frac{1}{2}k^2|v_i^n|^2,$$

we have

$$\left|\frac{m_i^{n+1} - m_i^n}{k} - v_i^n\right| \leq \frac{1}{2}k|v_i^n|^2,$$

which using (18) leads to

$$\|\partial_t m_{h,k} - v_{h,k}\|_{L^1(Q_T)} \leq c^2 k||v_{h,k}||^2_{L^2(Q_T)} \to_{(h,k)\to 0} 0.$$

This is sufficient to conclude that $v = \partial_t m$ in (21).

It remains now to prove that m satisfies both (14) and (15). For (15), we simply rewrite (17) as $\forall j \in \{0, \cdots, N_h\}$

$$\frac{1}{2}\int_\Omega |\nabla m_{h,k}(jk)|^2 + \frac{\alpha}{1+\alpha^2}\int_{\Omega\times(0,jk)} |v_{h,k}|^2 \leq \frac{1}{2}\int_\Omega |\nabla m^0|^2,$$

Passing to the weak convergence on the left-hand side and strong convergence on the right-hand side leads the result (recall that m^0 is a finite element approximation of the initial data m_0).

As far as (14) is concerned, we take a smooth test function $\Psi \in \mathcal{C}_0^\infty(Q_T)$, and test for all $t \in [nk, (n+1)k)$, the weak formulation in (9) with $\psi = \mathcal{I}_h(m_{h,k}^- \times \Psi(x,t)) \in K_{m^n}$. We get after suitable integration in time and summation on n

$$\int_{Q_T} (\alpha v_{h,k} - m_{h,k}^- \times v_{h,k}) \cdot \mathcal{I}_h(m_{h,k}^- \times \Psi)$$
$$= -(1+\alpha^2)\int_{Q_T} \nabla(m_{h,k}^- + \theta k v_{h,k}) \cdot \nabla\mathcal{I}_h(m_{h,k}^- \times \Psi). \quad (23)$$

Now, we use that in dimension $d \leq 3$, for any function $\varphi \in H^2(\Omega) \subset \mathcal{C}^0(\bar{\Omega})$, one has

$$||\varphi - \mathcal{I}_h(\varphi)||_{H^1(\Omega)} \leq Ch||\nabla^2\varphi||_{L^2(\Omega)}. \quad (24)$$

This leads, taking any triangle (tetrahedron in 3D) $K \subset \mathcal{T}_h$, to

$$||m_{h,k}^-(\cdot,t) \times \Psi(\cdot,t) - \mathcal{I}_h\left(m_{h,k}^-(\cdot,t) \times \Psi(\cdot,t)\right)||^2_{H^1(K)}$$
$$\leq Ch^2||\nabla^2(m_{h,k}^-(\cdot,t) \times \Psi(\cdot,t))||^2_{L^2(K)}$$
$$\leq Ch^2||m_{h,k}^-(\cdot,t)||^2_{H^1(K)}||\Psi||^2_{W^{2,\infty}(K)}, \quad (25)$$

since $m_{h,k}^-$ is linear on each triangle. Summing over all the triangles of $\mathcal{T}_h$, and integrating in time gives

$$||m_{h,k}^- \times \Psi - \mathcal{I}_h(m_{h,k}^- \times \Psi)||_{L^2([0,T],H^1)} \leq Ch||m_{h,k}^-||_{H^1(Q_T)}||\Psi||_{W^{2,\infty}(Q_T)}, \quad (26)$$

Using (24) and (26) into (23) leads to

$$\int_{Q_T} (\alpha v_{h,k} - m_{h,k}^- \times v_{h,k}) \cdot (m_{h,k}^- \times \Psi) + \theta k(1+\alpha^2)\int_{Q_T} \nabla v_{h,k} \cdot \nabla(m_{h,k}^- \times \Psi)$$
$$+(1+\alpha^2)\int_{Q_T} \nabla m_{h,k}^- \cdot \nabla(m_{h,k}^- \times \Psi) = O(h). \quad (27)$$

We now examine each term separately. From (20) and (21), we get

$$\int_{Q_T} (\alpha v_{h,k} - m_{h,k}^- \times v_{h,k}) \cdot (m_{h,k}^- \times \Psi) \to_{(h,k)\to(0,0)} \int_{Q_T} (\alpha\partial_t m - m \times \partial_t m) \cdot (m \times \Psi).$$

For the third term, we have

$$\begin{aligned}\int_{Q_T} \nabla m_{h,k}^- \cdot \nabla(m_{h,k}^- \times \Psi) &= \int_{Q_T} \nabla m_{h,k}^- \cdot m_{h,k}^- \times \nabla\Psi \\ &\to \int_{Q_T} \nabla m \cdot m \times \nabla\Psi,\end{aligned}$$

as $(h,k) \to (0,0)$ from (19) and (20). Eventually, the second term is slightly more subtle. Going back to (16), we get

$$\left(\theta - \frac{1}{2}\right) k^2 \int_\Omega \left|\nabla v^j\right|^2 + \frac{1}{2}\int_\Omega \left|\nabla m^{j+1}\right|^2 + \frac{\alpha k}{1+\alpha^2}\int_\Omega \left|v^j\right|^2 \leq \frac{1}{2}\int_\Omega \left|\nabla m^j\right|^2$$

Summing from $j=0$ to J leads to

$$\left(\theta - \frac{1}{2}\right) k \int_{Q_T} \left|\nabla v_{h,k}\right|^2 + \frac{\alpha}{1+\alpha^2}\int_{Q_T} \left|v_{h,k}\right|^2 \leq \frac{1}{2}\int_{Q_T} \left|\nabla m^0\right|^2,$$

from which we deduce that $(\sqrt{k}\nabla v_{h,k})$ is bounded in $L^2(Q_T)$. We then deduce

$$\begin{aligned}\left|k\int_{Q_T} \nabla v_{h,k} \cdot \nabla(m_{h,k}^- \times \Psi)\right| &\leq \sqrt{k}||\sqrt{k}\nabla v_{h,k}||_{L^2}||m_{h,k}^-||_{L^2(0,T;H^1)}||\Psi||_{W^{1,\infty}} \\ &\to 0, \end{aligned} \tag{28}$$

as $(h,k) \to (0,0)$ which ends the proof. □

Remark 1. 1. When $\theta \in [0, \frac{1}{2}]$ one has to bound the term $\int_\Omega |\nabla v^n|^2$ in (16). Since, on regular sequences of triangulations there exists $c > 0$ such that

$$\forall v \in V_h, \ \int_\Omega |\nabla v|^2 \leq \frac{c}{h^2}\int_\Omega |v|^2,$$

one has

$$-\frac{\alpha k}{1+\alpha^2}\int_\Omega |v^n|^2 - \left(\theta - \frac{1}{2}\right) k^2 \int_\Omega |\nabla v^n|^2 \leq -\frac{\alpha k}{(1+\alpha^2)}(1 - c(h,k))\int_\Omega |v^n|^2$$

where $c(h,k) = c\frac{1+\alpha^2}{\alpha}(\frac{1}{2} - \theta)\frac{k}{h^2}$. This gives the following theorem which already improves the result of [6] in dimension 3 for instance.

Theorem 3. *Let $m_0 \in H^1(\Omega, S^2)$. Suppose $m^0 \to m_0$ in $H^1(\Omega)$ as $h \to 0$, and $\theta \in [0, \dfrac{1}{2}]$. If the regular sequence of conformal triangulations $(\mathcal{T}_h)_h$ satisfies condition (12), then provided $\dfrac{k}{h^2}$ tends to zero as h and k go to 0, $(m_{h,k})$ converges (up to the extraction of a subsequence) weakly in $H^1(\Omega \times (0,T))$ to a weak solution m of (3) as h and k tend to 0.*

2. In the case of Crank-Nicolson scheme ($\theta = \frac{1}{2}$), everything is identical but the proof of (28) which is wrong in this way since there is no more bound on ∇v. However, using the remark (that was proposed by the unknown referee) $||k\nabla v||_{L^2(Q_T)} \leq \frac{k}{h}||v||_{L^2(Q_T)}$, one obtains the same result (convergence to a weak solution) as soon as $\frac{k}{h}$ tends to 0 as k and h go to 0.
3. The same method applies with only slight modifications to closely related problems like the heat-flow of harmonic maps into spheres.

4. **Numerical experiments.** We propose here a few numerical experiments in 2D. The code has been written using a MATLAB finite element toolbox developed by the author and the linear systems have been solved using the MINRES technique [12] (the system is not symmetric). A typical mesh of the domain $\Omega = B(0,1)$ is given in Fig. 3, and the scheme was initialized with $m^0(x,y) = \left(-\frac{y}{r}\sin\left(\frac{\pi r}{2}\right), \frac{x}{r}\sin\left(\frac{\pi r}{2}\right), \cos\left(\frac{\pi r}{2}\right)\right)$, where $r = \sqrt{x^2+y^2}$. The time step used was $k = 0.01$ and the energy was plotted in Fig. 1 for different values of α and $\theta = 1$ (fully implicit). The same pictures obtained for $\theta = 0.5$ are shown in Fig. 2. We see that as expected from the convergence proof and (16), the closer θ to 0.5, the less diffusive the scheme (recall that for $\alpha = 0$ the energy should be conserved). Although the proof we have given fails in many points for P^2 approximation (in particular (13) is wrong in $P2$), we show the same cases but in P^2 in Fig. 3. There is no instability, but the improvement is unclear on this test-case.

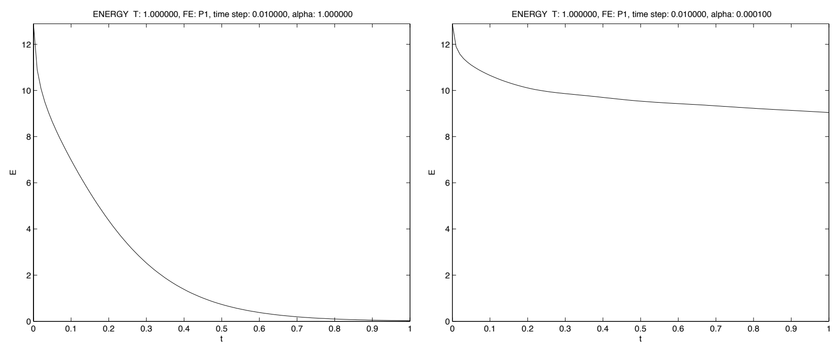

FIGURE 1. Energy versus time for P^1 elements, $\theta = 1$, $\alpha = 1$ (left), and $\alpha = 0.0001$ (right).

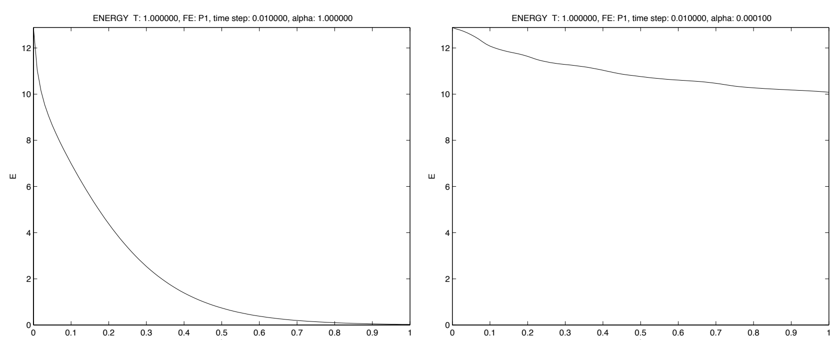

FIGURE 2. Energy versus time for P^1 elements, $\theta = 0.5$, $\alpha = 1$ (left), and $\alpha = 0.0001$ (right).

5. **Conclusion.** We have given a new implicit finite element formulation for Landau-Lifshitz equations. A convergence theorem is also given improving known results in the literature. Numerical experiments are also provided showing the practical applicability of the method.

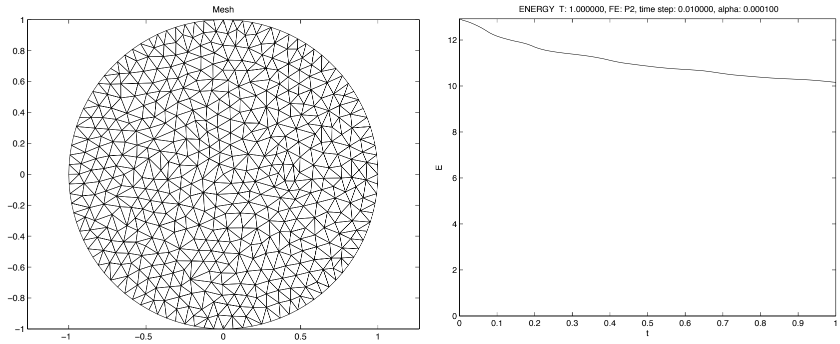

FIGURE 3. Mesh used and Energy versus time for P^2 elements, $\theta = 0.5$, $\alpha = 0.0001$.

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