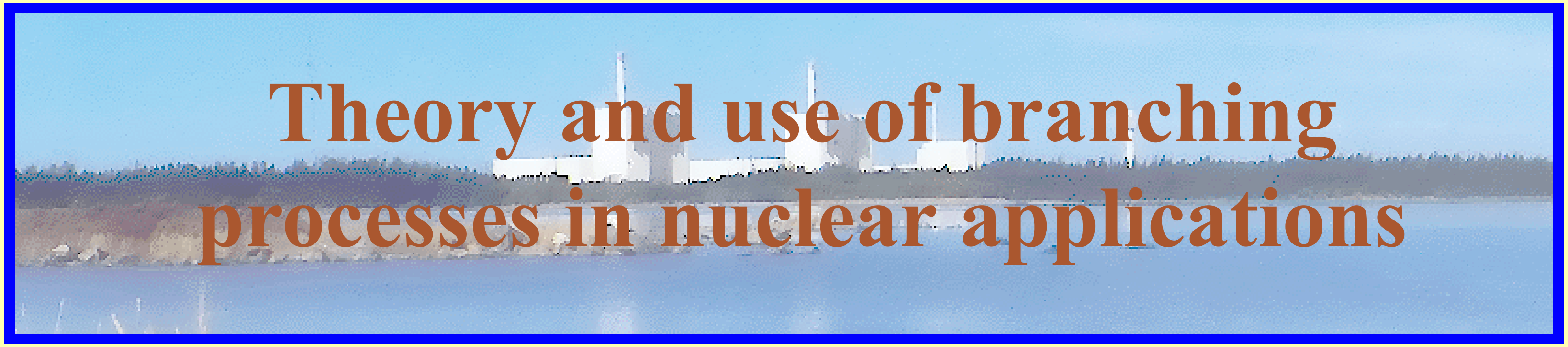

Theory and use of branching processes in nuclear applications

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Talk at the Miniworkshop on Stochastic Processes and Transport
Chalmers, 29 May 2008

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The transport and of neutrons in a multiplying system is an area of branching processes with an aesthetically pleasing and clear formalism. The theory has very concrete and useful applications for diagnostics of nuclear systems. Yet, this beautiful theory has never been compiled in a self-contained and complete monograph. The lack of such a treatise has become apparent recently when new fields of applications were opened with the appearance of new reactor concepts. This monograph is intended to fill this vacancy.

This book was written with two objectives in mind, and correspondingly it consists of two parts. The first part presents an account of the mathematical tools used in describing branching processes, which are then used to derive a large number of properties of the neutron distribution in multiplying systems with or without an external source. The emphasis is, however, not so much on giving lengthy derivations or a complete collection of formulae, rather to expose the reader to the methodology of setting up and solving master equations of particle transport through the completeness of the treatment.

In the second part the theory is applied to the description of the neutron fluctuations in nuclear reactor cores as well as in small samples of fissile material. The question of how to extract information about the system in question is discussed. In particular the measurement of the reactivity of subcritical cores, driven with various Poisson and non-Poisson (pulsed) sources, and the identification of fissile material samples, is illustrated. This part of the book gives pragmatic information for those planning and executing and evaluating experiments on such systems.

This book will be of interest for graduate students as well as researchers in the fields of nuclear engineering, reactor physics, safeguards, as well as physicists and biologists working with branching processes.

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Neutron Fluctuations
A TREATISE ON THE PHYSICS OF BRANCHING PROCESSES

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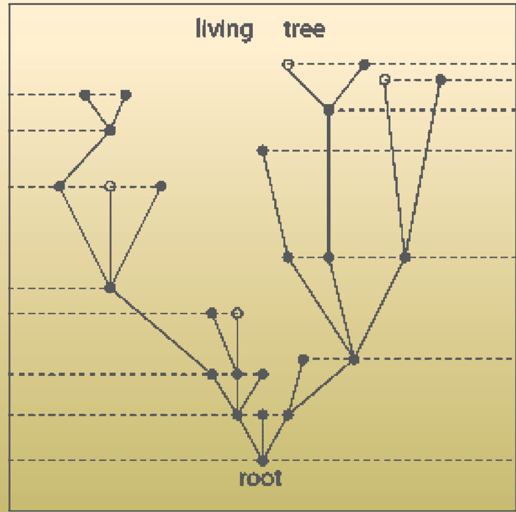


IMRE PÁZSIT AND LÉNÁRD PÁL

Neutron fluctuations in a nuclear reactor

Two different types:

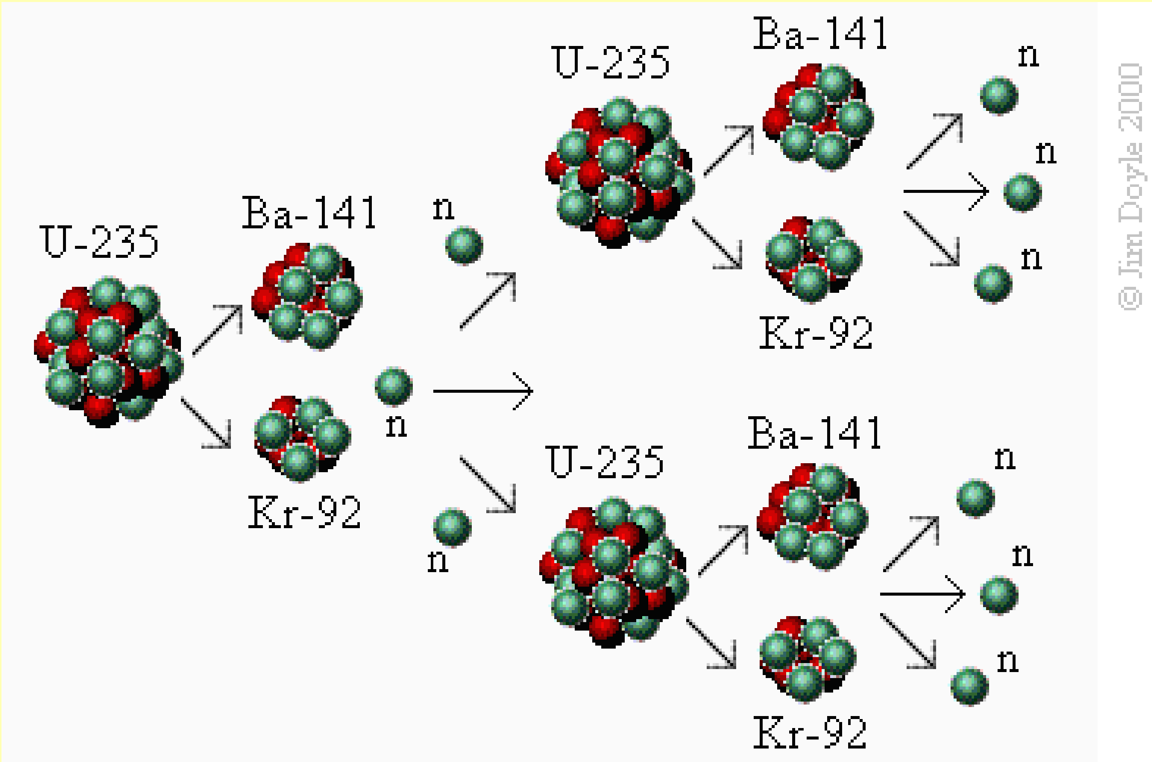
- zero power systems: -> the inherent randomness of the neutron chain dominates, through the branching.
- power reactors:
  -> technological processes in the core (vibrations of control rods, boiling of the coolant in a BWR etc) influence the neutron distribution -> power reactor noise.

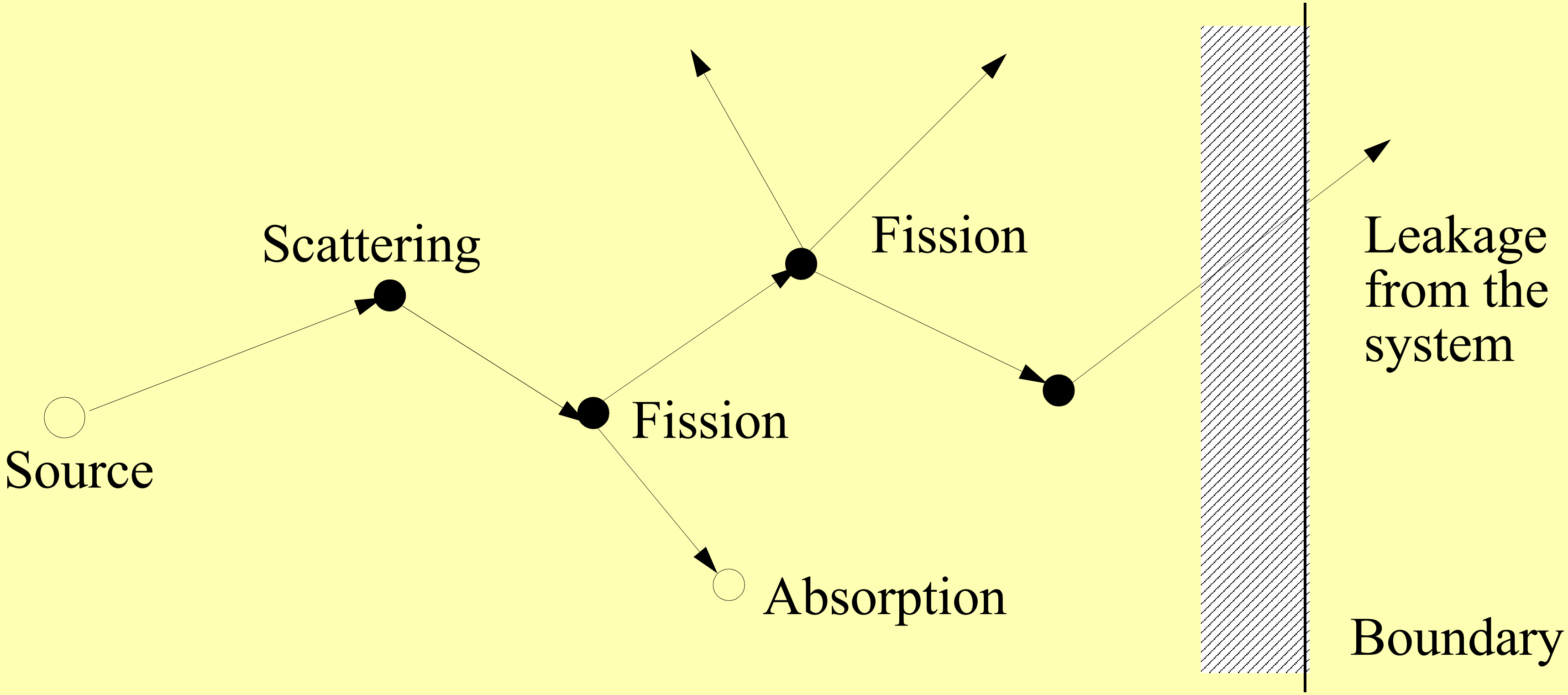
  These processes can be diagnosed by analysis of the induced neutron noise in a non-intrusive way during operation.

Branching processes

Nuclear chain reaction -> family trees

Follows statistical rules

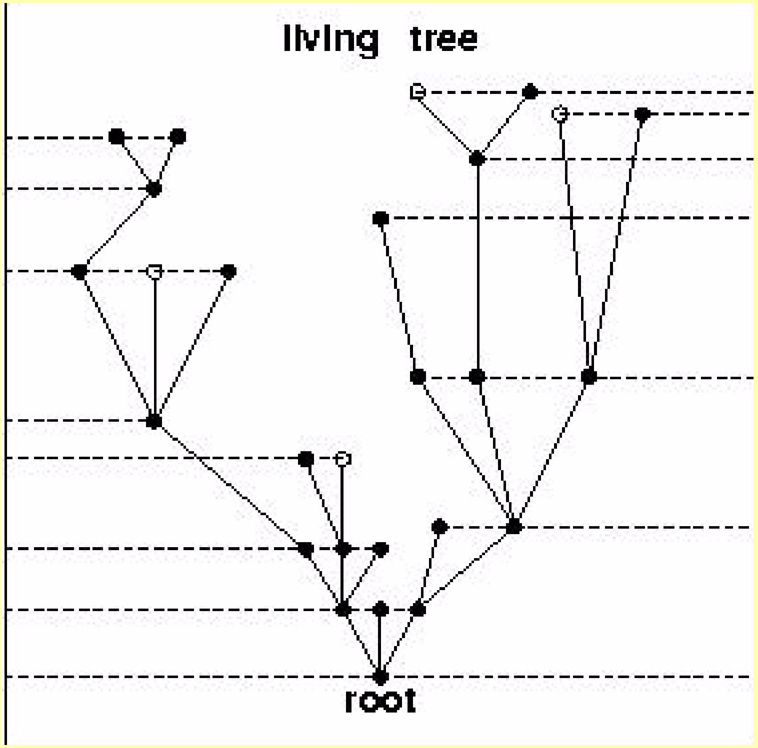




A schematic view of neutron transport

A random tree as a branching process.

The word "branching process" was coined by Kolmogorov and Dmitriev in 1947.



Watson and Galton, extinction of family names (1874) (Galton-Watson process).

The story of the extinction of families

Sources:

David G. Kendall: "Branching processes since 1873", Journal of London Math. Society, **41**, (1966), 385 - 406.

Address delivered on the occasion of the Centenary of the Society.

T.E. Harris: The Theory of Branching Processes. Springer Verlag, Berlin, 1963.

Conjecture by M. Alphonse de Candolle:

"The decay of the families of men who occupied conspicuos positions (peerages) in the past times has been a subject of frequent remark, and has given rise to various conjectures... The tendency is universal, and in explanation of it, the conclusion has been hastily drawn that a rise in physical comfort and intellectual capacity is necessarily accompanied by diminution of fertility."
(cited by Galton).

Galton: formulated the problem in the Educational Times in 1873:

- Follow a large population N of adult males.
- Let f_n be the probability that in each generation, an adult male will have exactly n male children.
- Let $p_k^{(n)}$ be the probability of having k males in the nth generation
- Find what proportion of the surnames will have become extinct after r generations.

Galton had a solution which he was not satisfied with, and persuaded the reverend Watson to deal with the problem.

Watson's solution

Generating functions:

$$\boxed{q(s) = \sum_{0}^{\infty} f_k\, s^k}$$ -> gen fct of males born in one generation

$$\boxed{g_n(s) = \sum_{0}^{\infty} p_k^{(n)}\, s^k}$$ -> gen fct of tot. number of males in *n*th gen.

One has $q(1) = g_n(1) = 1$

Probabilities: Taylor coefficients of $g(s)$

Factorial moments: derivatives of $g(s)$ at $s = 1$

In particular, the average number of males born in 1 gen:

$$E\{k\} \equiv m = \sum_{0}^{\infty} f_k s^k = q'(1)$$

Then if $g_n(s)$ is the gen fct. of the distribution of males in the nth generation, the original male being the zeroth, it can be shown that

$$g_0(s) = s, \; g_1(s) \equiv g(s) = q(s), \quad g_2(s) = g(g(s)) = q(q(s))$$

and in general

$$g_{n+1}(s) = q(g_n(s)) = g_n(q(s))$$

This follows from the following relationship:

$$p_k^{(n+1)} = \sum_{l=0}^{\infty} p_l^{(n)} p_{lk} =$$

$$= \sum_{l=0}^{\infty} p_l^{(n)} \sum_{k_1+k_2+\ldots k_l = k}^{\infty} f_{k_1} f_{k_2} \ldots f_{k_l}$$

Hence

$$g_{n+1}(s) = \sum_{l=0}^{\infty} p_l^{(n)} \sum_{k=0}^{\infty} \sum_{k_1+k_2+\ldots k_l = k}^{\infty} f_{k_1} f_{k_2} \ldots f_{k_l} s^k$$

$$= \sum_{l=0}^{\infty} p_l^{(n)} q^l(s) = g_n[q(s)] = g_n(g(s))$$

Extinction probability in the nth generation: $h_n = g_n(0)$

$$h_1 = f_0 = q(0)$$

$$h_{n+1} = q(h_n)$$

and if $h_n \to h$ when $n \to \infty$, then

$$\boxed{h = q(h)}$$

or

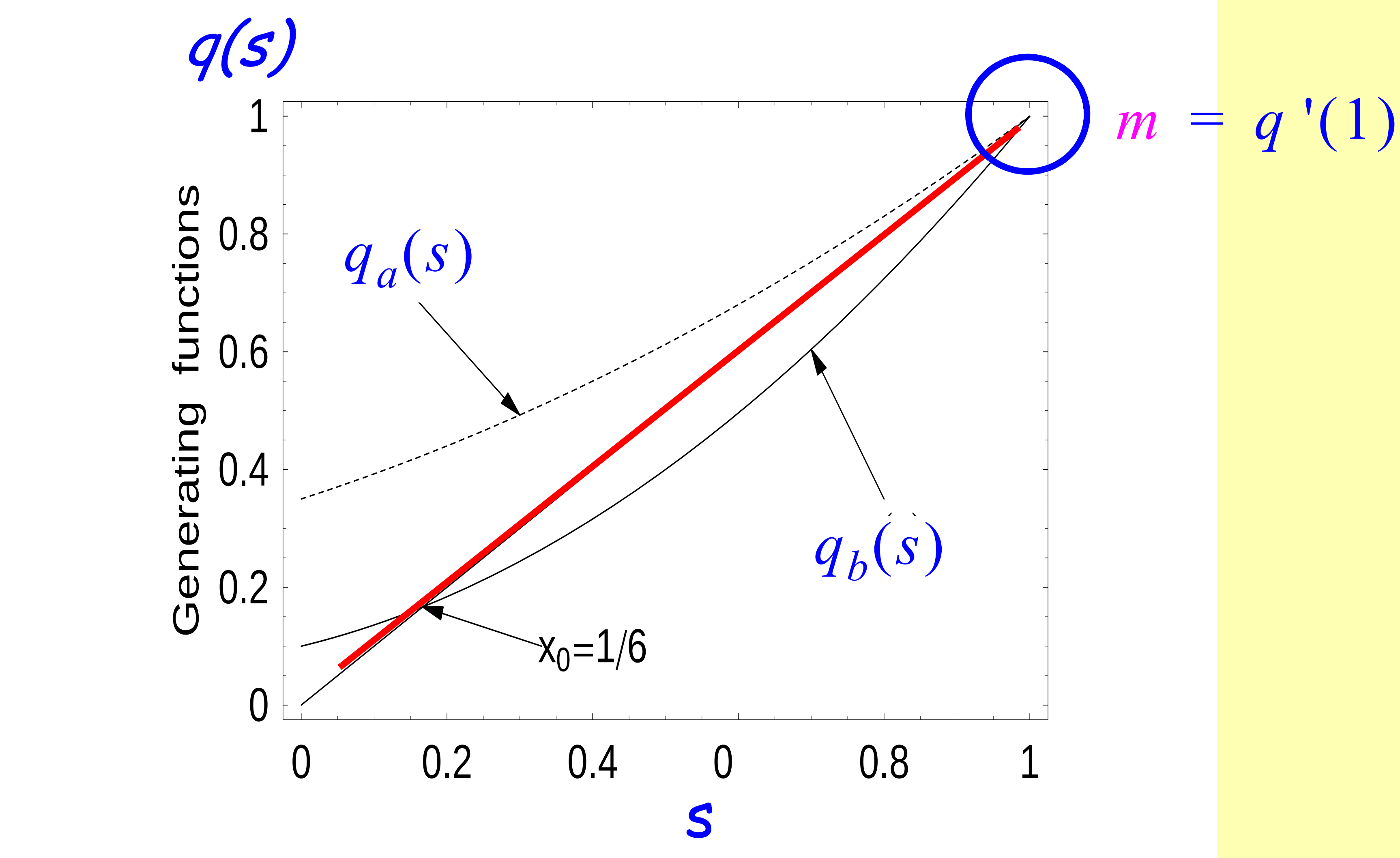
$$\boxed{s = q(s)}$$

Roots of the equation $\boxed{s = q(s)}$

Since $q(1) = 1$, $h = 1$ is always a root.

The eqation has a further root if the average number of male descendants per family is larger than unity, i.e. $m > 1$. The smallest root is the "real" one.

Graphical representation



Hence the extinction probability is unity if $m \leq 1$. I. e. even a "critical" population, m = 1, will almost certainly die out.

But:

Expectation of the number of males in the nth generation:

$$E\{N_n\} = m^n$$

In a "critical" system, i.e. for $m = 1$, the average population is constant.

How are the above two things possible simultaneously?

Because there are even more “pathological” facts.

V<u>ariance:</u>

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$$

for the number of males born in one generation.

Variance of the <u>population</u> in the nth generation:

$$\sigma_n^2 = \langle N_n{}^2 \rangle - \langle N_n \rangle^2 = n\sigma^2$$

That is, with the increase of the number of the generations, the variance diverges.

This means that almost all family lines will die out, except a vanishingly small fraction, which will grow unbounded.

For the case of a "supercritical" population, $m > 1$, only a finite fraction will die out, and the rest will diverge.

The same is valid for the random walk (Brownian motion; Einstein 1905).

Corollary: the process is not ergodic (time averages are not equal to ensemble averages).

In other words, continuing or re-starting the process is not the same.

This is why certain games are played in several short sets instead of one long (tennis, table-tennis etc).

Application to particle transport

Cosmic radiation, electron-photon showers.

Problem: to calculate the statistics of the number of electrons that slowed past down some given energy.

This is a continuous parametric discrete process.

H. J. Bhabha, W. Heitler, L. Jánossy (late 40's and early 50's)

S.K. Srinivasan, Vasudevan etc.

"Regeneration point technique".

Purpose: to find out whether the mean value is meaningsful (i.e. the fluctuations are not too large).

The relative variance of an electron-photon shower is in the order of unity, that is closely to Poisson, but has values both under and above unity, i.e. $\sigma_N^2 / \langle N \rangle <> 1$

In case of ionisation, ion pair production (detectors!) and defect generation in an infinite medium:

$$\frac{\sigma_N^2}{\langle N \rangle} \equiv F \ll 1 \qquad F = \text{Fano factor}$$

This is due to <u>conservation relations</u>.

The role of correlations in the variance:

$$\frac{\sigma_N^2}{\langle N \rangle} = 1 + \frac{1}{\langle N \rangle} \int C_{NN}(\boldsymbol{R}_1, \boldsymbol{R}_2) d\boldsymbol{R}_1 d\boldsymbol{R}_2$$

where

$$C_{NN}(\boldsymbol{R}_1, \boldsymbol{R}_2) \equiv \langle N(\boldsymbol{R}_1) N(\boldsymbol{R}_2) \rangle - \langle N(\boldsymbol{R}_1) \rangle \langle N(\boldsymbol{R}_2) \rangle$$

Negative correlations (conservation relations): sub-Poisson variance

Positive correlations (such as generated by branching): over-Poisson variance.

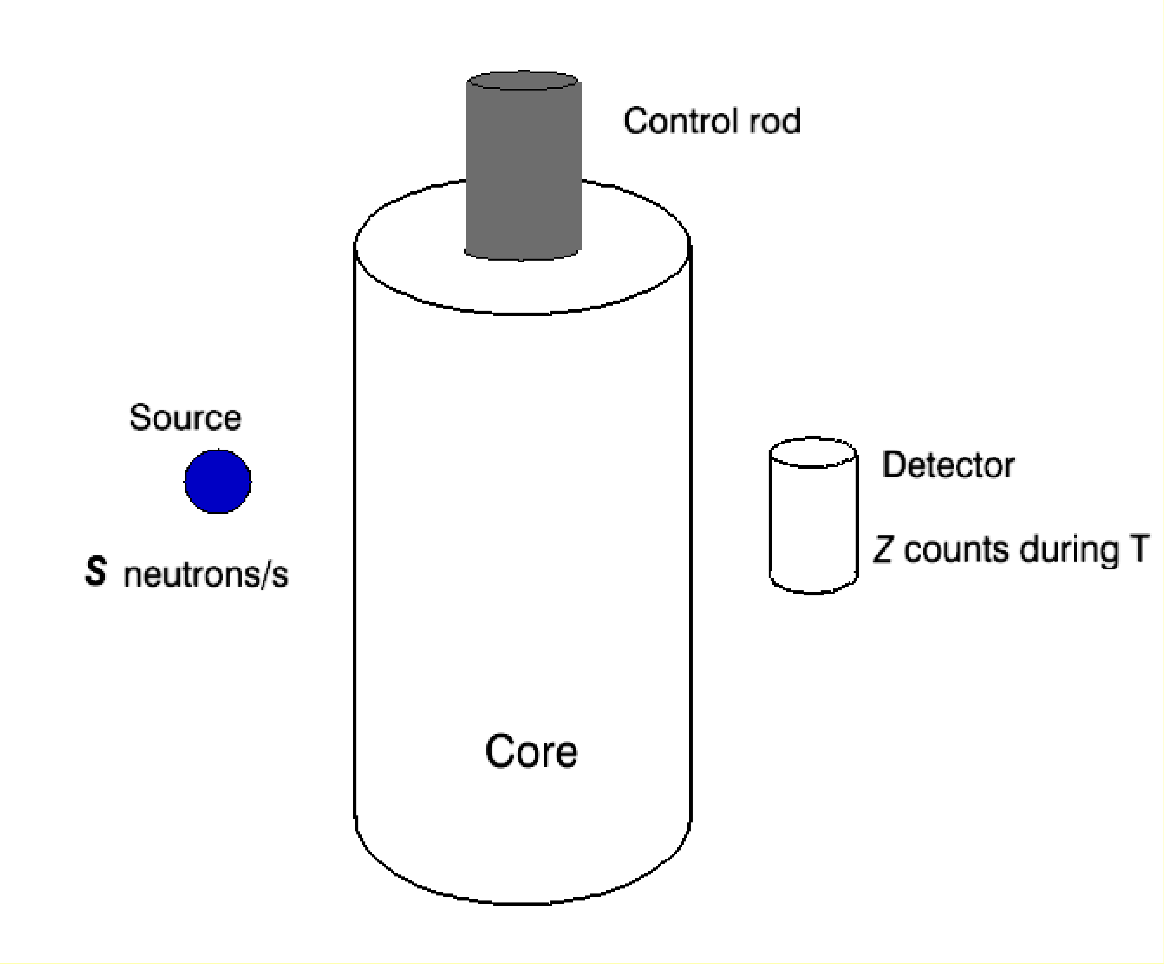
No correlations (independent entities): Poisson variance.

Fluctuations in multiplying nuclear systems

Originally, the question was posed as whether it can happen that the bomb will not explode, due to the randomness of the fission chain.

Feynman, Fermi and others.

"Peaceful" output: the Feynman-formula for measuring the subcritical reactivity of a core.



Feynman, de Hoffmann, Serber (1956)

Variance to mean of the neutron counts as a function of the measurement time T (Feymnan-alpha formula):

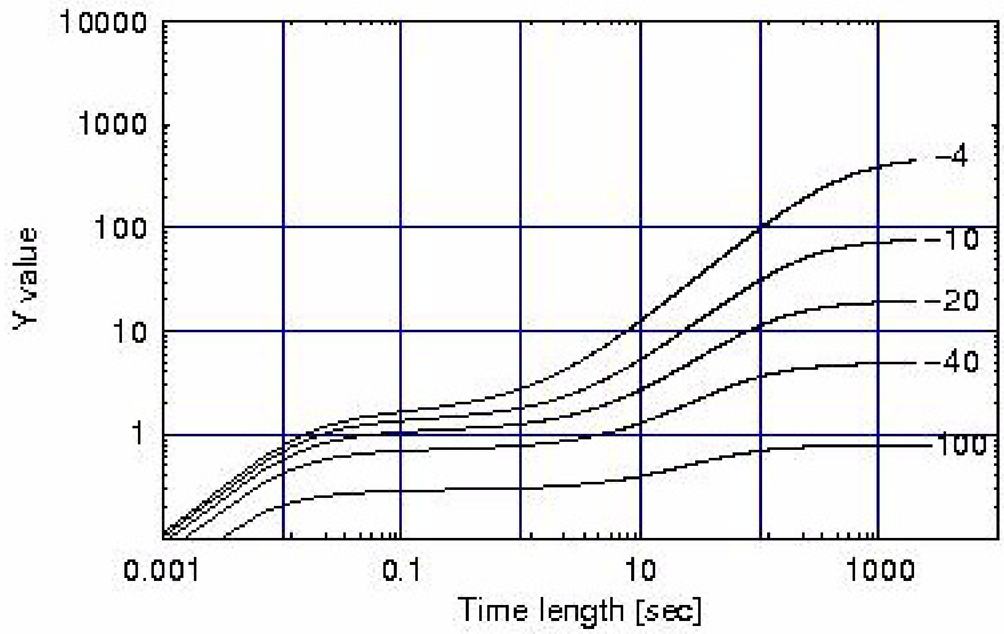
$$\boxed{\frac{\sigma_Z^2(t)}{\langle Z(t)\rangle} = 1 + \varepsilon A\left(1 - \frac{1 - e^{-\alpha t}}{\alpha t}\right)}$$

i.e. it is over-Poisson.

The over-Poisson character is due to the correlations in the neutrons in the same chain.

Neutrons which are born simultaneously, die also nearly simultaneously.

Measuring of reactivity with the Feynman-alpha method



multiplica-
tion factor

The parameter α, which containes the searched informa-
tion (reactivity) is obtained by curve-fitting.

Theoretical foundation

L. Pál: On the Theory of Stochastic Processes in Nuclear Reactors, *Nuovo Cimento*, N. 1 del *Supplemento* al Vol. 7, Serie X, 25 - 42 (1958)

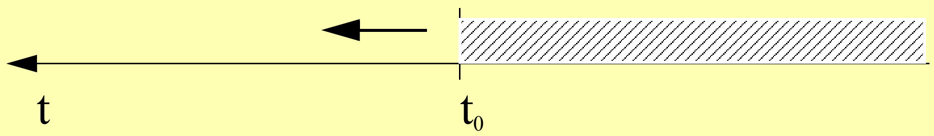
Acta Physica Hungarica (1962)

G.I. Bell, Nucl. Sci. Engng (1965)

The “Pál - Bell equation”.

Markov processes

A process is called Markovian if the state of the process at a certain time point t_0 determines its future evolution uniquely, independent of its past (its state for any $t < t_0$).



Note: time flows from right to left in the following, both in the figures as well as the arguments and indices of the functions! Thus in $P(N, t, M, t_0)$ we assume $t > t_0$; likewise, $w_{N,M}$ will indicate a transition from state M to state N.

The Chapman-Kolmogorov equation (or master equation) for stationary Markovian processes

Taking $t > t' > t_0$, the following probability balance equation can be formulated:

$$P(N, t|M, t_0) = \sum P(N, t|L, t')P(L, t'|M, t_0) \tag{16}$$

This can be interpreted like this: in its transition from state M at time t_0 to state N at time t, at an intermediate time t' the system must go through *some* of its possible states.

A differential equation from (16) can be obtained in two ways, either by letting $t' \to t$ or $t' \to t_0$. This will give two differential equations for the same quantity. They are called the forward and the backward Chapman-Kolmogorov or master equations, respectively.

For the differential equation to be useful, we need to assume that the transitional probabilities for infinitesimal times are known. For different states, i.e. for $N \neq M$

$$W_{N,M}(t)dt \equiv P(N, t + dt | M, t) \tag{17}$$

From (15) we have the normalisation condition

$$\sum_N W_{N,M}(t)dt = 1 \tag{18}$$

Since the probability of no transition at all tends to unity with $dt \to 0$, with the above definition $W_{N,N}(t)$ would diverge. Thus it is expressed through (18) as

$$W_{N,N}(t)dt = 1 - \sum_{L \neq N} W_{L,N}dt \tag{19}$$

The transition probabilities are usualy known from the physics, as we shall see soon in examples.

Forward master equation: $t' = t - dt,\ dt \to 0$

$$\frac{d}{dt}P(N, t|M, t_0) = \sum_{\neq} W_{N,L}P(L, t|M, t_0) - P(N, t|M, t_0)\sum_{\neq} W_{L,N} \qquad (20)$$

Operates on the final (detection) co-ordinates.

Backward master equation: $t' = t_0 + dt$

$$-\frac{d}{dt_0}P(N, t|M, t_0) = \sum_{\neq} W_{L,M}P(N, t|L, t_0) - P(N, t|M, t_0)\sum_{\neq} W_{L,M} \qquad (21)$$

Operates on the initial (source) co-ordinates.

These two equations refer to the same problem, and thus the solutions are identical. This gives a possibility to choose any of the wo forms for a given problem - one selects the form which is easier to solve.

Another form of the backward equation (mixed equation):

If $W_{N,M}$ is constant (independent of time), then $P(N, t|M, t_0)$ will only depend on $t - t_0$. Thus

$$\frac{dP}{dt} = -\frac{dP}{dt_0} \tag{22}$$

Choosing $t_0 = 0$ and $P(N, t|M, t_0) \Rightarrow P(N, t|M)$ gives

$$\frac{d}{dt}P(N, t|M) = \sum_{L \neq M} W_{L,M} P(N, t|L) - P(N, t|M) \sum_{L \neq M} W_{L,M} \tag{23}$$

This is however a mixed type equation. Both initial and final co-ordinates are operated upon.

This form is frequently used in concrete work.

Current renewed interest in neutron fluctuations

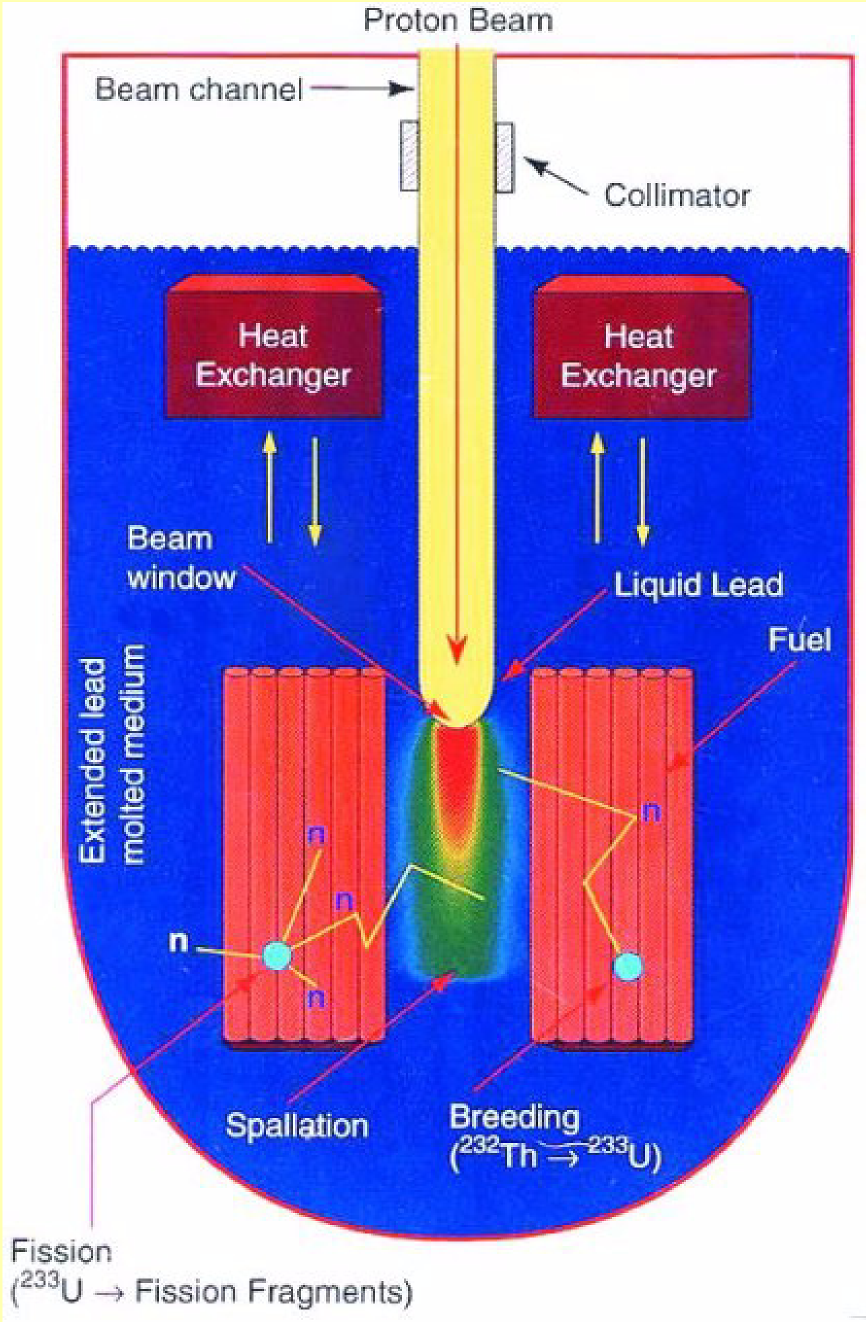
- Accelerator driven subcritical reactors for transmutation of nuclear waste
- Neutron fluctuations in a randomly varying medium
- Nuclear safeguards

<u>Waste disposal</u>: transmutation in accelerator driven systems (ADS)

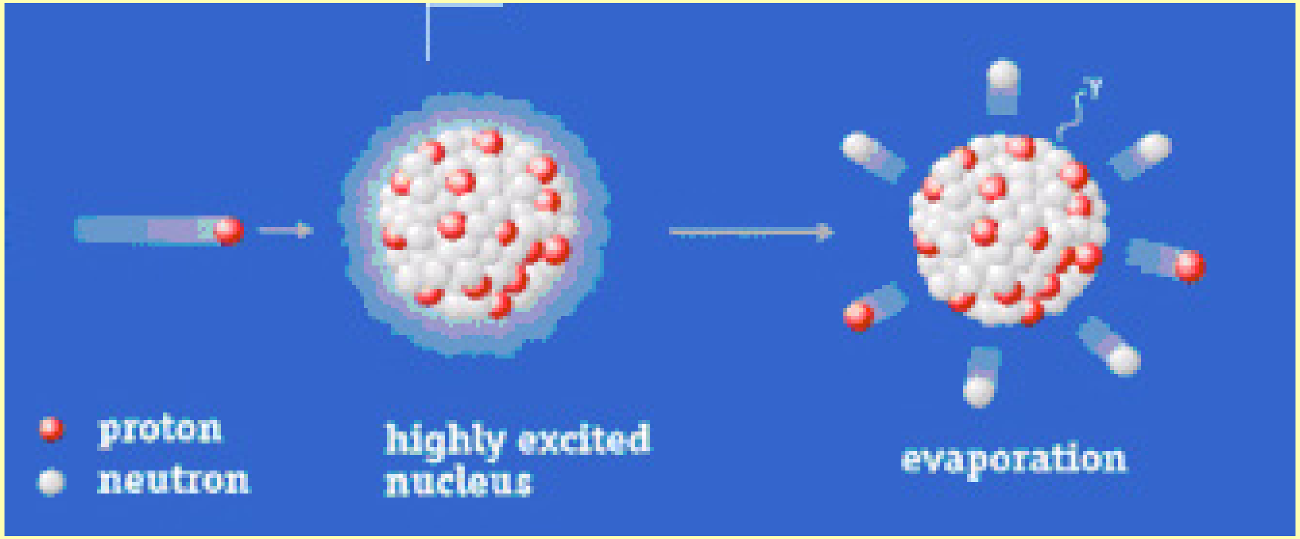
It consists of a subcritical core with a strong neutron source, based on spallation.

Advantages:

- Better operational safety (sub-critical)
- Less high level waste (can even incinerate existing waste)
- Better utilisation of the fuel resources (breeding): can also use thorium.



The spallation process



Spallation sources are usually driven in <u>pulsed</u> mode.

So the source statisctics is very different from the traditional case - theory must be developed further.

Another area: nuclear safeguards

Branching processes

Particles in the system undergo *reactions* with an intensity Q, with the possible outcomes

- die
- renew
- multiply

External source:

- immigration

Markov process:

Time interval τ between reactions has exponential distribution

$$\mathcal{P}\{\tau \geq t\} = T(t) = e^{-Qt} \tag{1}$$

Distribution of the number of particles generated in one reaction, ν:

$$\mathcal{P}\{\nu = k\} = f_k, \qquad k = 0, 1, \ldots, \tag{2}$$

- f_0: probability of annihilation
- f_1: probability of renewal
- $f_k,\ \ k > 1$: probability of multiplication

The state of the constant multiplying medium is defined by the parameters $\{Q, f_k,\ k \in \mathcal{Z}^+\}$.

Introduce the generating function $q(z)$ of the random variable ν as

$$q(z) = \mathbf{E}\{z^{\nu}\} = \sum_{k=0}^{\infty} f_k z^k \tag{3}$$

Let $< \nu >$ and $< \nu(\nu - 1) >$ denote the first and second factorial moments of f:

$$\left[\frac{dq(z)}{dz}\right]_{z=1} = q'(1) =< \nu >, \tag{4}$$

and

$$\left[\frac{d^2q(z)}{dz^2}\right]_{z=1} = q''(1) =< \nu(\nu - 1) > . \tag{5}$$

The parameter

$$\boxed{\alpha \equiv Q[q'(1) - 1] = Q(< \nu > -1) \equiv \frac{\rho}{\Lambda}}$$

will determine the behaviour of the system.

Constant medium

Let $\mathbf{n}(t)$ denote the number of particles at time $t \geq t_0$. Then

$$\mathcal{P}\{\mathbf{n}(t) = n | \mathbf{n}(0) = 1\} = p(t, n). \tag{6}$$

is the distribution of particles at time t, initiated by one particle at $t = 0$.

With m starting particles at $t = 0$:

$$p(t, n|m)$$

Then, due to the independent evolution of the individual particles:

$$\boxed{p(t, n|m) = \sum_{n_1 + \cdots + n_m = n} \prod_{\ell=1}^{m} p(t, n_\ell)} \tag{7}$$

holds.

Introducing the *generating functions*

$$g(t,z) = \sum_{n=0}^{\infty} p(t,n) z^n, \qquad |z| \le 1$$

and

$$g(t,z|m) = \sum_{n=0}^{\infty} p(t,n|m) z^n, \qquad |z| \le 1$$

eqn (7) becomes

$$\boxed{g(t,z|m) = g^m(t,z)} \tag{8}$$

The generating function $g(t,z)$ fulfils the *backward master equation*

$$\frac{\partial g(t,z)}{\partial t} = Q \sum_{k=0}^{\infty} f_k \, g^k(t,z) - Qg(t,z) = Q \, q[g(t,z)] - Qg(t,z) \tag{9}$$

and the *forward master equation*

$$\frac{\partial g(t,z)}{\partial t} = Q[q(z) - z] \frac{\partial g(t,z)}{\partial z}. \tag{10}$$

Moments, variances

Define the expectation

$$\mathsf{E}\{\mathbf{n}(t)[\mathbf{n}(t)-1]\cdots[\mathbf{n}(t)-k+1]\} =$$

$$= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,p(t,n) = m_k(t), \qquad \forall\ t \in [0,\infty) \tag{11}$$

as the kth factorial moment of the random variable $\mathbf{n}(t)$.
It can be shown that the equation

$$\mathsf{E}\{\mathbf{n}(t)[\mathbf{n}(t)-1]\cdots[\mathbf{n}(t)-k+1]\} = m_k(t) = \lim_{z\uparrow 1} \frac{\partial^k g(t,z)}{\partial z^k}, \tag{12}$$

holds for all positive integers k.

In what follows we shall need mostly the first two lowest order moments.
The variance $\mathbf{D}^2\{\mathbf{n}(t)\}$ of the random variable $\mathbf{n}(t)$ is defined by the equation

$$\mathbf{D}^2\{\mathbf{n}(t)\} = m_2(t) + m_1(t) - [m_1(t)]^2\,. \tag{13}$$

The following results are easily obtained for the first and second factorial moments:

$$m_1(t) = e^{\alpha t}, \qquad \text{where} \quad \alpha = Q[q'(1) - 1] = Q[< \nu > -1] = \frac{\rho}{\Lambda}, \tag{14}$$

and

$$m_2(t) = \begin{cases} Q\frac{q''(1)}{\alpha}\, e^{\alpha t}\left(e^{\alpha t} - 1\right), & \text{if } \alpha \neq 0, \\ Q\, q''(1)t, & \text{if } \alpha = 0, \end{cases} \tag{15}$$

where $q''(1) =< \nu(\nu - 1) >$.

For the variance of $\mathbf{n}(t)$ one obtains

$$\mathbf{D}^2\left\{\mathbf{n}(t)\right\} = \begin{cases} \left[Q\,\frac{q''(1)}{\alpha} - 1\right]\, e^{\alpha t}\left(e^{\alpha t} - 1\right), & \text{if } \alpha \neq 0, \\ Q\, q''(1)t, & \text{if } \alpha = 0. \end{cases} \tag{16}$$

The variance diverges *linearly* in a critical medium (when $\alpha = 0$)

In the derivation of the above (traditional) formula, the source is assumed to have simple Poisson statistics.

Reactivity measurements have become interesting again in the so-called accelerator-driven systems (ADS).

In such systems, the source does not have simple Poisson statistics. The source is a spallation source which can be either continuous, in which case it has

- compound Poisson statistics (several source neutrons emitted simultaneously, the emissions follow Poisson statistics) (talk at ICTT-16, Atlanta);

  Or it can be pulsed, and further

- a pulsed source with finite pulse width (simple Poisson but with time dependent intensity);
- a pulsed periodic source with narrow (Dirac-δ) pulses, i.e. a non-Poisson source (this talk).

Derivation of the backward master equation

The derivation of the traditional case is usually made by the backward Kolmogorov or master equations.
A backward (adjoint) equation cannot handle an extended source, only a point source, hence the case of a continuous source is handled in two steps.
First, the generating function

$$g_d(z, t, T) = \sum_{n=0}^{\infty} z^n p_d(n, t, T)$$

of the probability $p_d(n, t, T)$ that there will be n detections during a measurement time T, induced by a *single* neutron emitted to the system at time t is determined from a backward Kolmogorov equation. This quantity has long been known.

Derivation of the backward master equation (cont)

Then the generating function

$$G_d(z,T) = \sum_{n=0}^{\infty} z^Z P_d(Z,T)$$

of the probability that there will be Z detections during the detection time T due to an extraneous source with intensity $S(t)$ can be expressed with the single-neutron generating function as

$$G_d(z,T) = \lim_{t\to\infty} \exp\left\{\int_0^{\infty} \left[S(t')g_d(z,t-t',T) - 1\right] dt'\right\} \tag{4}$$

This is the generalization of the expression of the first moment relationship giving the expectation of the detector counts as an integral over the adjoint function (neutron importance) and the source function.

The case of narrow pulses

For narrow pulses, with a pulse width much shorter than the characteristic times of the system (e.g. neutron lifetime, pulse repetition time), it would appear practical to express the source as a train of Dirac-delta pulses:

$$S(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0) \tag{5}$$

However, the intensity function of the source in a stochastic theory cannot be an irregular (unbounded) function, hence the traditional method is not applicable.

A quantitative example

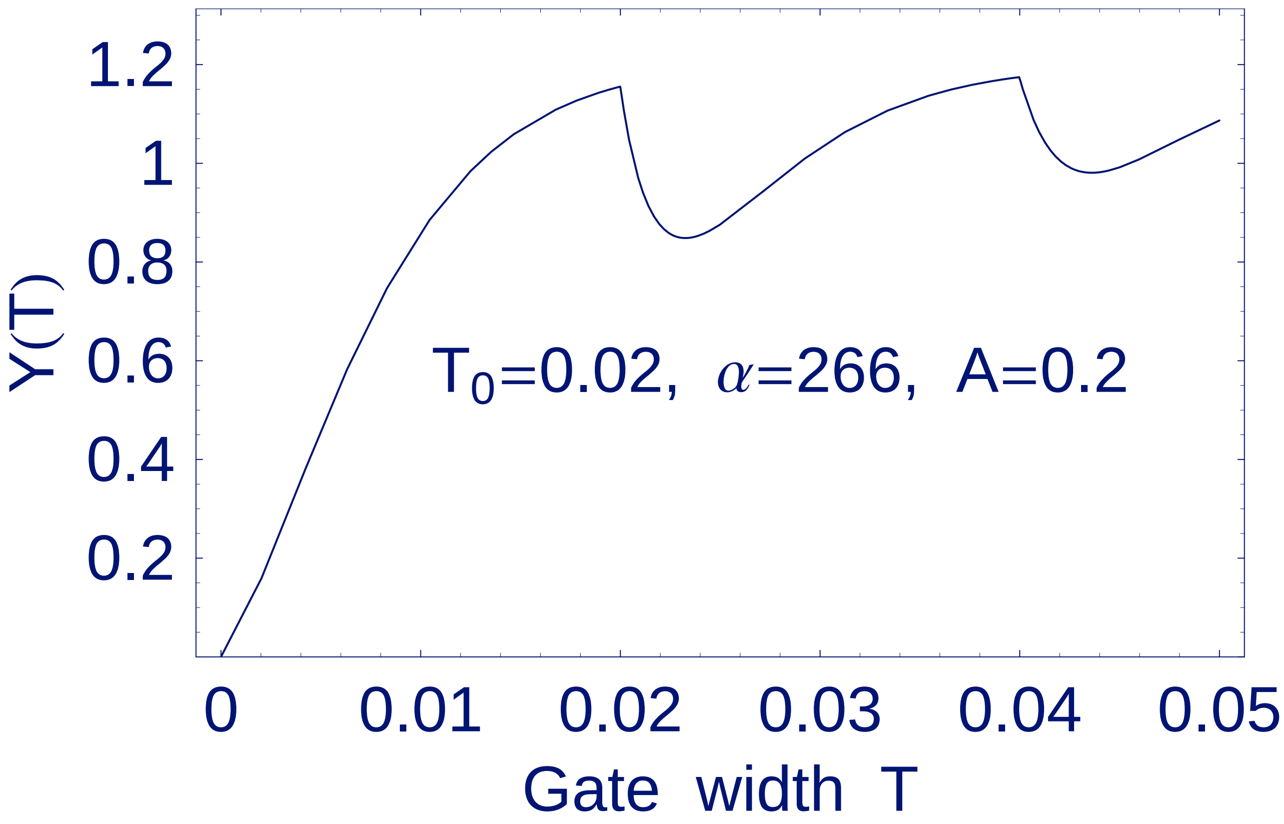


Figure 2: Feynman-alpha curve for periodic instantaneous pulses with $T_0 = 0.02$ sec, $\alpha = 266s^-1$ and $\delta^* = 0.2$, for the case of deterministic pulsing.

A quantitative example

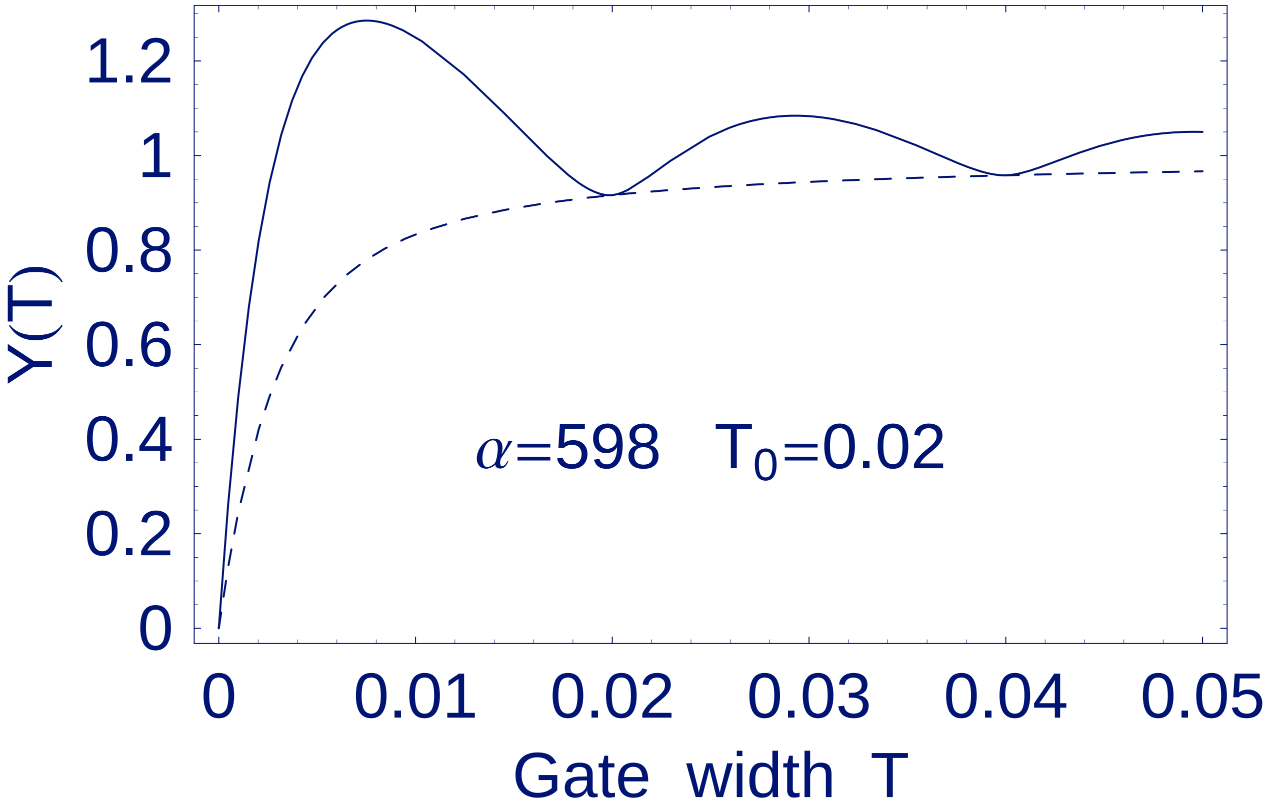


Figure 3: Feynman-alpha curve for periodic instantaneous pulses with $T_0 = 0.02$ sec, $\alpha = 266s^{-}1$ and $\delta^* = 0.2$, for the case of stochastic pulsing.

Comparison with experiments (Kyoto Univ. Res. React.)

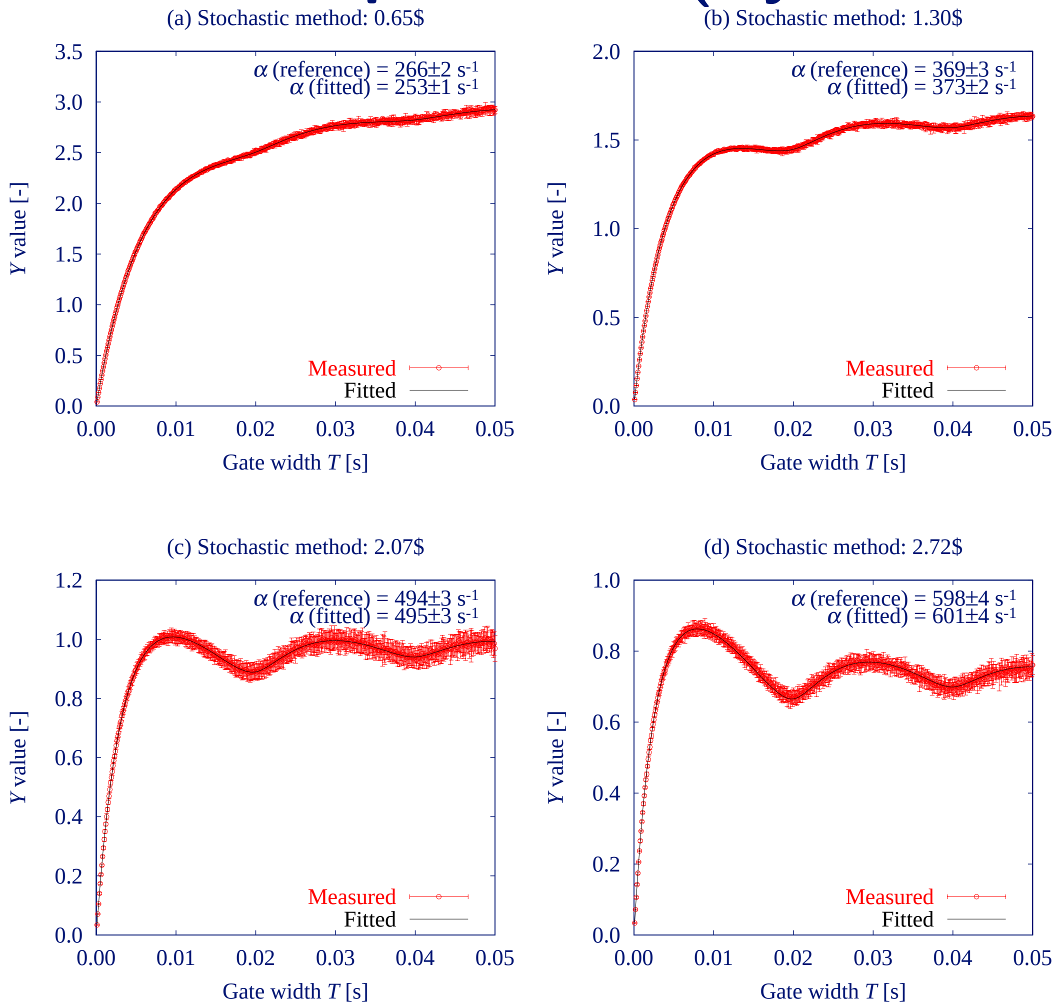


Figure 4: Measured and fitted results for the stochastic pulsing.

MEDIUM WITH RANDOMLY VARYING PARAMETERS

Simplest possible example of a randomly varying medium: jumping randomly between two states:

- two discrete states $\mathcal{S}_1 = \{Q_1, f_k^{(1)}\}$ and $\mathcal{S}_2 = \{Q_2, f_k^{(2)}\}$
  These will lead to two different values of α, i.e.
  $\alpha_1 = Q_1(<\nu>_1 - 1)$ and $\alpha_2 = Q_2(<\nu>_2 - 1)$

- transition probability λ:
  $\lambda\,\Delta t + o(\Delta t)$ is the probability that during time Δt the transition $\mathcal{S}_1 \rightarrow \mathcal{S}_2$ or the transition $\mathcal{S}_2 \rightarrow \mathcal{S}_1$ takes place.

  Main quantity of interest:

$$p_{i,j}(t, n|m), \tag{17}$$

the probability that at time $t \geq 0$ the system contains n particles and is in the state $\mathcal{S}_j$, $j = 1, 2$, provided that at time $t = 0$ it contained m particles and was in state $\mathcal{S}_i$, $i = 1, 2$.

Time dependence of the system changes

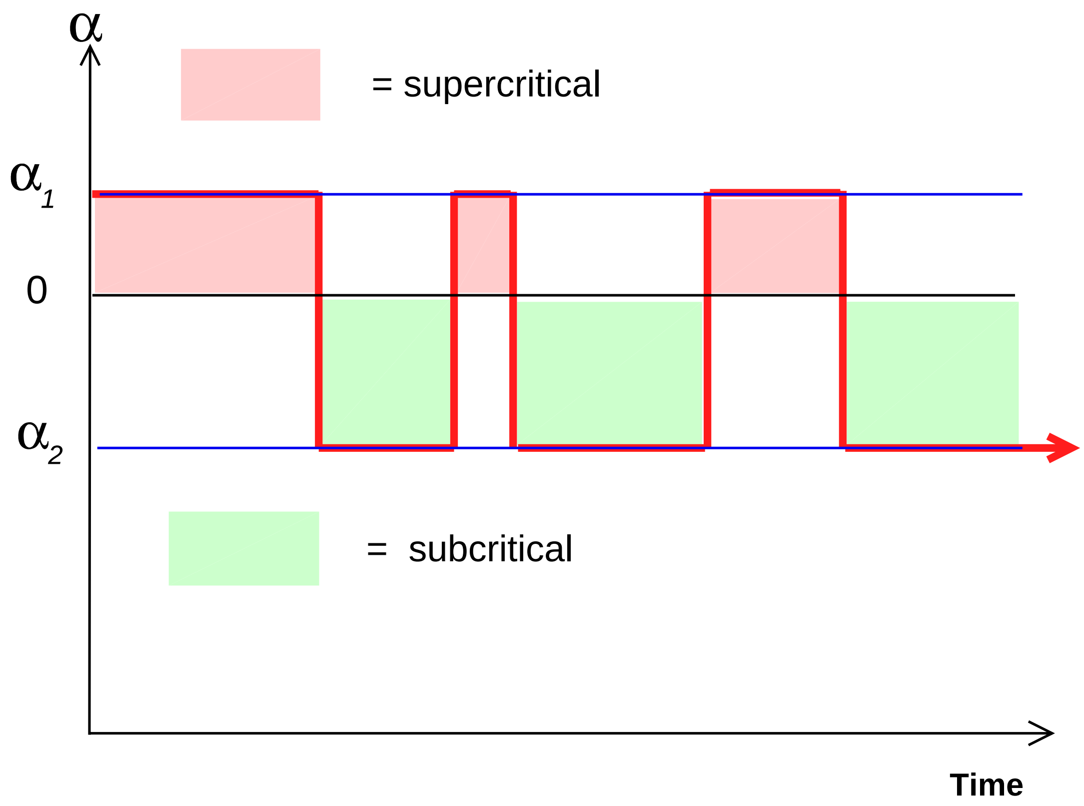


Figure 1: Time dependence of the system parameter $\alpha(t)$ between the two states α_1 and α_2 in a randomly varying system.

When we are not interested in which final state the medium is:

$$p_i(t,n|m) = p_{i,1}(t,n|m) + p_{i,2}(t,n|m), \qquad i = 1,2. \tag{18}$$

Again, we will use the generating function

$$g_{i,j}(t,z|m) = \sum_{n=0}^{\infty} p_{i,j}(t,n|m)\, z^n, \tag{19}$$

and similarly for $g_i(t,z|m)$.

Backward equation approach is now not applicable because

$$\boxed{g_{i,j}(t,z|m) \neq g_{i,j}^m(t,z|1), \quad i,j = 1,2}$$

The processes started by particles that are present in the system at a given time are *not independent in a randomly varying medium.*

Forward master equations

In the forward approach one can neglect notations on the initial particle number; it is accounted for in the initial conditions. Hence we shall use the probability distributions

$$p_{i,j}(t,n), \quad i,j = 1,2.$$

and their generating functions.

Forward master equations:

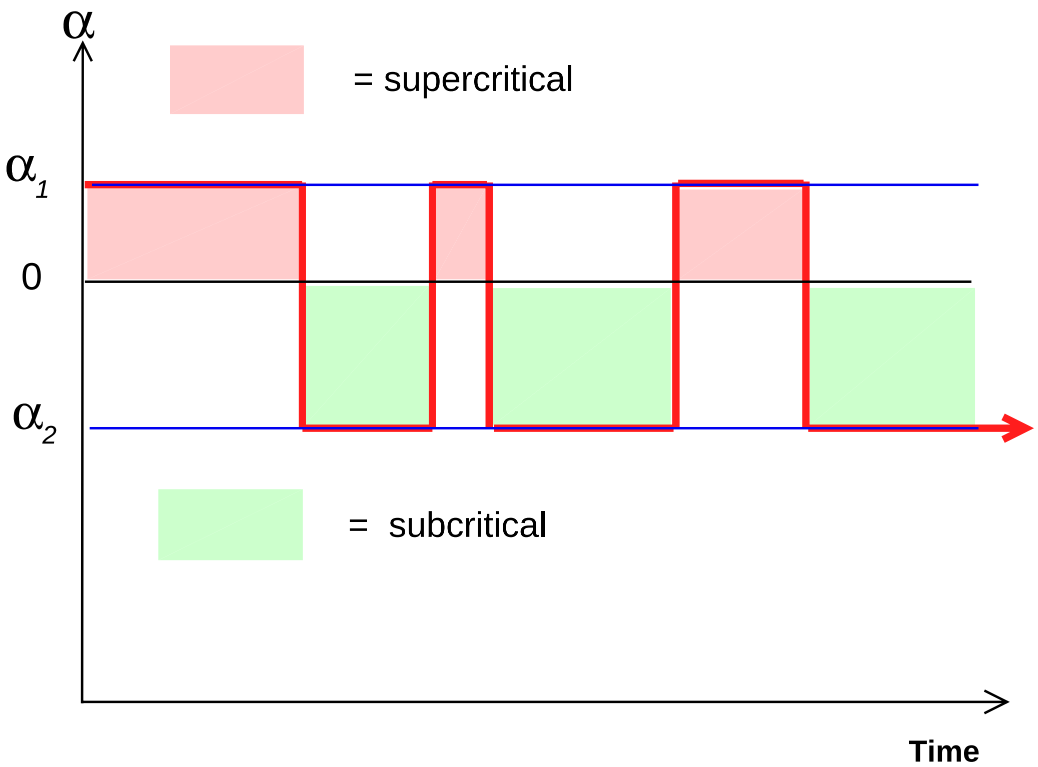
$$\frac{dp_{i,1}(t,n)}{dt} = -(nQ_1+\lambda)\, p_{i,1}(t,n) + \lambda\, p_{i,2}(t,n) + Q_1 \sum_{k=0}^{\infty} (n-k+1)\, f_k^{(1)}\, p_{i,1}(t,n-k+1),$$

$$\frac{dp_{i,2}(t,n)}{dt} = -(nQ_2+\lambda)\, p_{i,2}(t,n) + \lambda\, p_{i,1}(t,n) + Q_2 \sum_{k=0}^{\infty} (n-k+1)\, f_k^{(2)}\, p_{i,2}(t,n-k+1).$$

Properties of the first moments (expectations)

Constant medium: criticality if $\alpha = 0$

Random medium: α is a random variable. $<\alpha>= \frac{\alpha_1+\alpha_2}{2} = 0 \rightarrow$ supercritical!



Condition of criticality in the mean:

Criticality condition: $s_1 = 0 \to$

$$2\lambda - \alpha_1 - \alpha_2 = \sqrt{(\alpha_1 - \alpha_2)^2 + 4\lambda^2}$$

which is fulfilled if $\lambda = \lambda_{cr}$, where

$$\lambda_{cr} = \frac{\alpha_1\, \alpha_2}{\alpha_1 + \alpha_2} > 0 \tag{30}$$

This is *necessary, but not sufficient condition* – it can be fulfilled also when $\alpha_1 > 0$ and $\alpha_2 > 0$, in which case the system is evidently supercritical.
Necessary and sufficient condition of the criticality:

$$\alpha_1 + \alpha_2 < 0; \qquad \alpha_1\, \alpha_2 < 0$$

and

$$\lambda = \lambda_{cr} > 0.$$

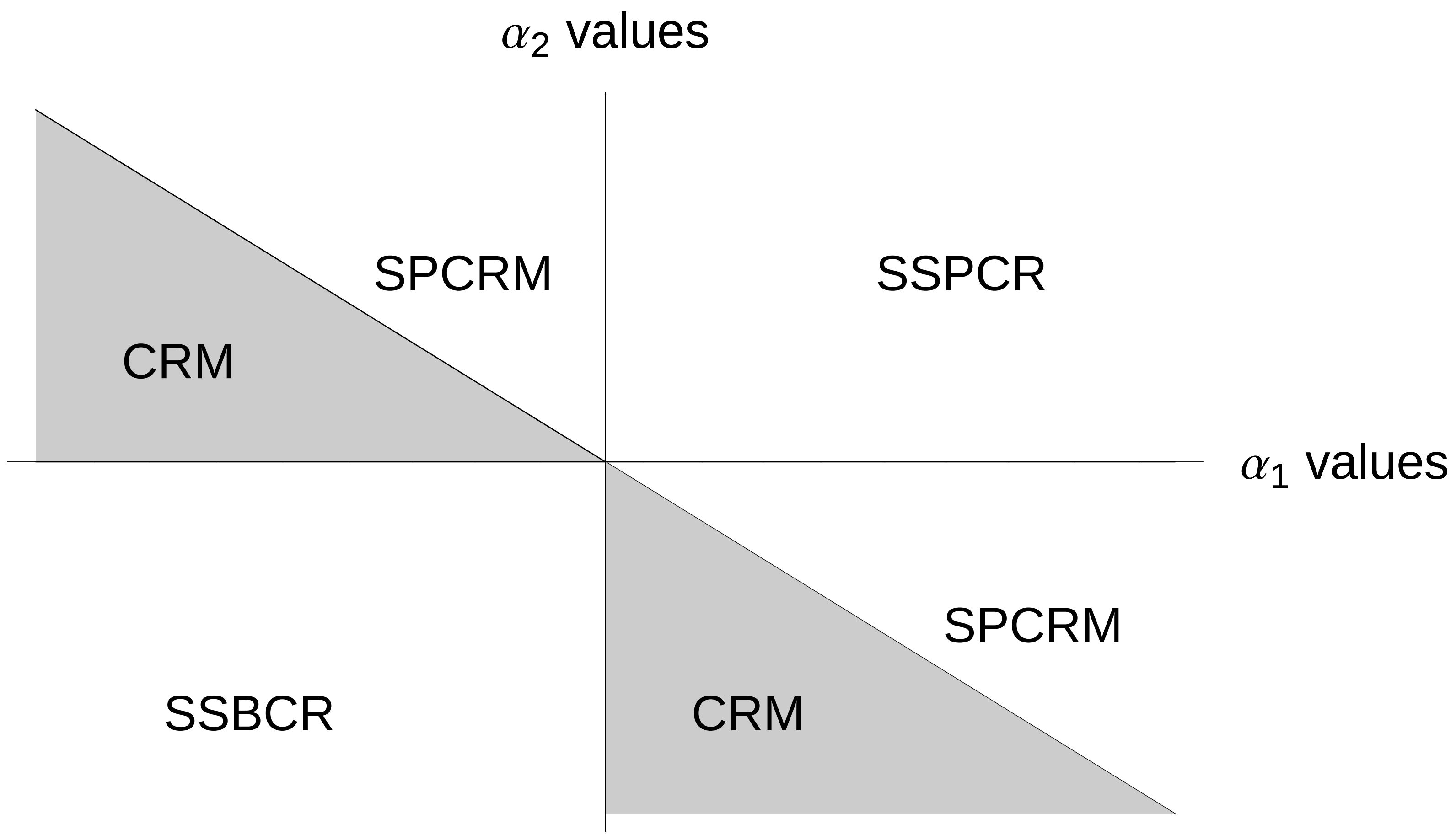


Figure 2: The parameter plane $(\alpha_1,\ \alpha_2)$ defined on the base of the time dependence of $m_i^{(1)}(t),\ i=1,2$. The value of λ plays a role only in the domain CRM

Time behaviour in a system critical in the mean

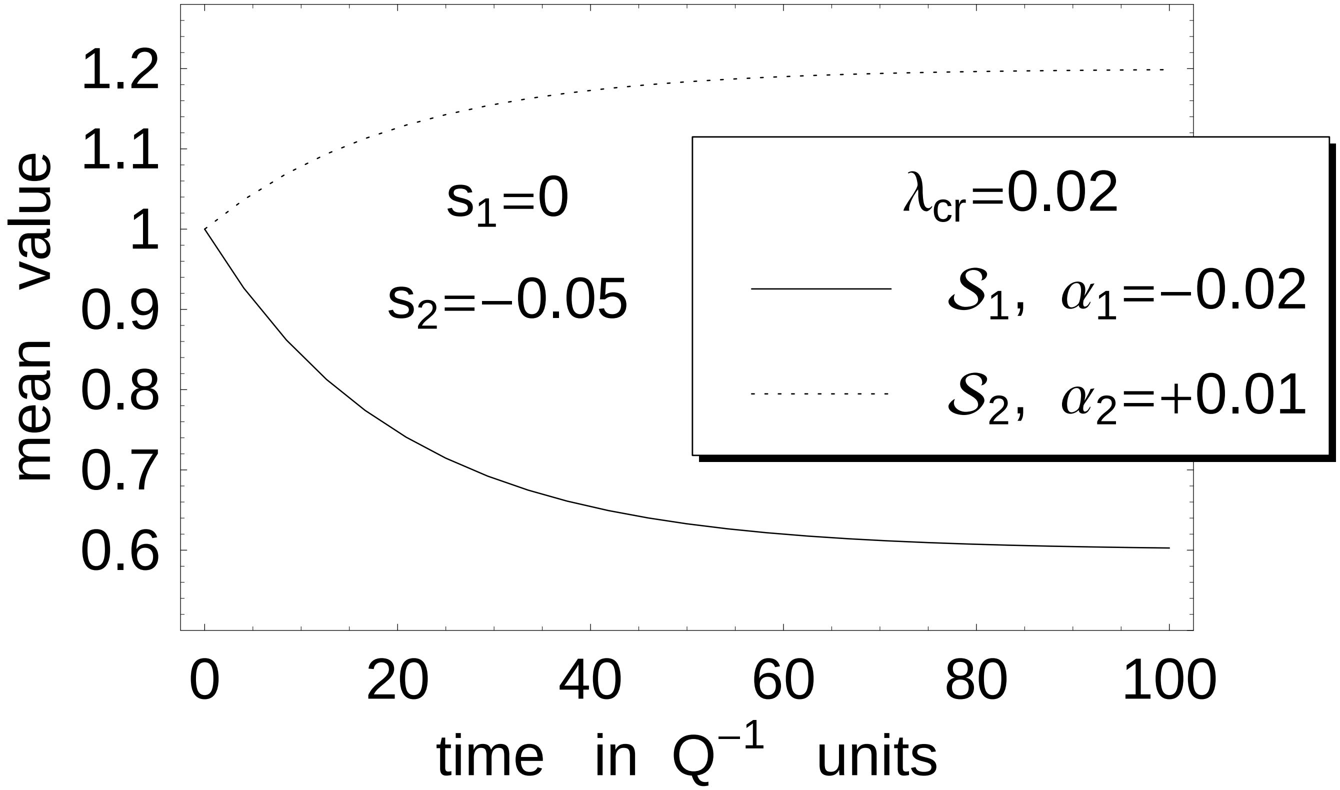


Figure 3: Time dependence of the expectation of the particle number in a system starting from a subcritical $\{\alpha_2 = +0.01\}$ and a supercritical $\{\alpha_1 = -0.02\}$ state, respectively.

Time behaviour in a system subcritical in the mean

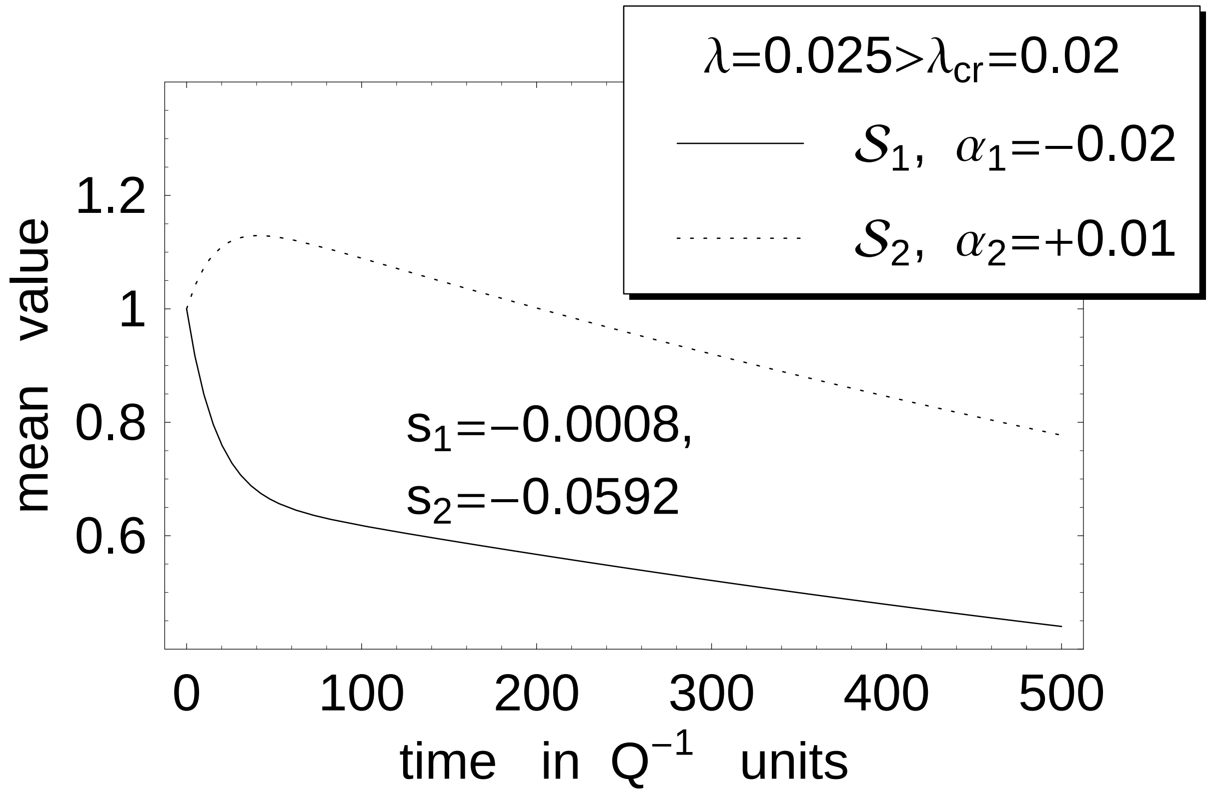


Figure 4: Time dependencies of the expectations in a subcritical system with $\lambda > \lambda_{cr}$. Upper curve: start from $\{\alpha_2 = +0.01\}$, lower curve: start from $\{\alpha_1 = -0.02\}$.

Time behaviour in a system supercritical in the mean

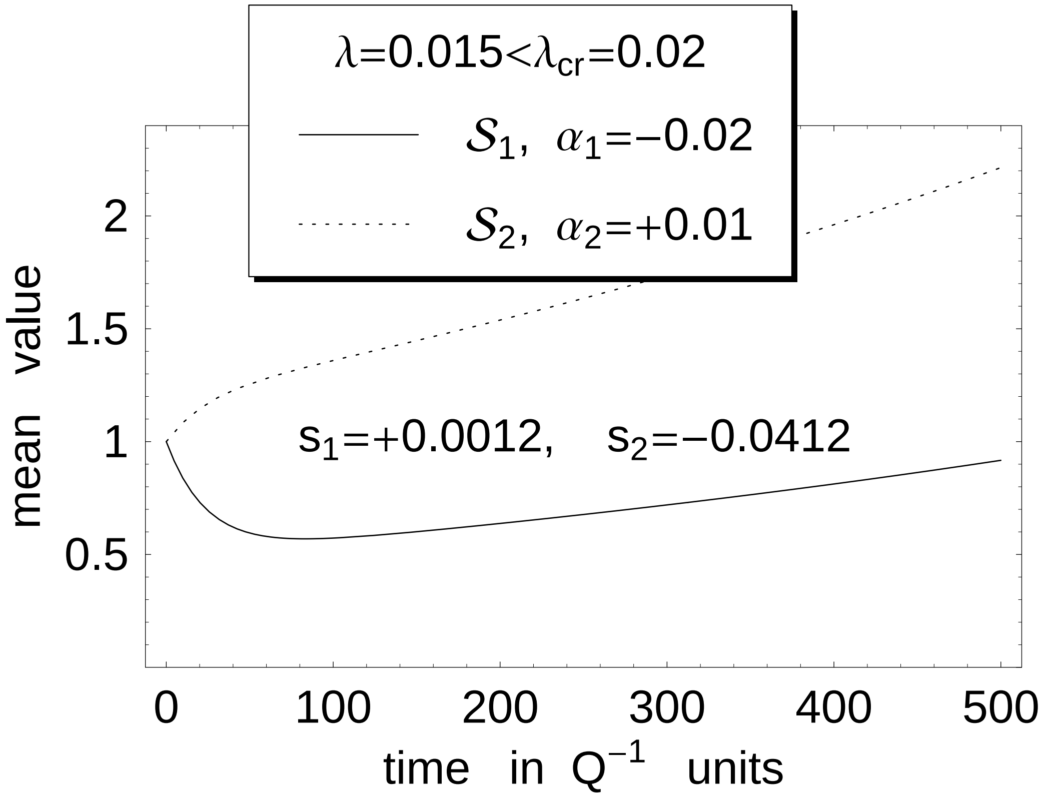


Figure 5: Time dependence of the expectation in a supercritical system with $\lambda > \lambda_{cr}$. Upper curve: start from $\{\alpha_2 = +0.01\}$, lower curve: start from $\{\alpha_1 = -0.02\}$.

Multiplicities and Number Distribution of Neutrons and Gammas

Characteristics of emission from a fissile sample

Andreas Enqvist

Introduction

- Fissile materials.
- Spontaneous fission.
- Induced fission – multiplying sample.
- Multiplicities change – increasing with sample mass.
- Measuring multiplicities can be used to quantify the sample, as well as determining the isotopes involved in the fission.

Neutrons

- Neutron life in sample is short
  - May induce fission but all chains are shortlived. The induced neutrons can be seen as a correction to the original multiplicity.
  - The correction depends on the probability to induce fission - p, which depends on sample size.
  - Compare with the fast fission factor in 4-factor formula. ”the mean number of primary neutrons increase”

Analytical tools

- Factorial moments - Multiplicities: the relative frequencies of singlets, doublets, triplets etc. (of neutrons or gammas) in one single fission.
  - Induced fission events will change the multiplicities.
  - Up to about 3rd moment needed and measured.
- Probabilities P(n) for emitting n neutrons or gammas from 1 source event.
  - Needed for all n where it isn't negligible.
  - Number of P(n) needed increase with sample size.
  - P(n) can be found with recursive formulas.
- Matematical tool - Probability Generating Functions (PGF)

- The PGF *g(z)* of the probability distribution of *f(n)* is defined as

$$g(z) = \sum_{n=0}^{\infty} f(n) z^n$$

- The multiplicities (factorial moments) of *f* are obtained as derivatives of *g(z)* at *z=1:*

$$\langle n \rangle = \left. \frac{\partial g(z)}{\partial z} \right|_{z=1} = \sum_n n f(n)$$

$$\langle n(n-1) \rangle = \left. \frac{\partial^2 g(z)}{\partial z^2} \right|_{z=1} = \sum_n n(n-1) f(n)$$ etc...

Master equation of backward type

- Neutrons, PGF *h(z)* of probability *p(n)* that n neutrons escape the sample.
  - Leave system with prob. *(1-p).*
  - Induce fission, creating n neutrons with probability *p*, each of these neutrons are to be viewed independently.

$$p(n) = (1-p)\delta_{n,1} + p\sum_{k=1}^{\infty} p_f(k) \prod_{\substack{i=1 \\ \{n_1+n_2+\ldots+n_k=n\}}}^{k} p(n_i)$$

- Getting PGF – multiply by z^n and sum over n. We can then identify the PGF's

$$\sum_{n=0}^{\infty} p(n) z^n = \sum_{n=0}^{\infty} (1-p)\delta_{n,1} z^n +$$

$$p \sum_{k=0}^{\infty} p_f(k) \underbrace{\sum_{n_1, n_2, \ldots, n_k} \sum_{n=0}^{\infty} p(n_1) p(n_2) \ldots p(n_k)\, z^{n_1} z^{n_2} \ldots z^{n_k}}_{[h(z)]^k}$$

$$\{n_1 + n_2 + \ldots + n_k = n\}$$

$$\Leftrightarrow$$

$$h(z) = (1-p)z + p\, q_f[h(z)]$$

- For one source event we get the coupled equation

$$H(z) = q_s[h(z)]$$

- With

$$q_{f,s}(x) = \sum_{n=0}^{\infty} p_{f,s}(n)\, x^n$$

Probabilities P(n)

- We return to the coupled master equations of backward type for neutrons.

$$h(z) = (1 - p)z + p\, q_f[h(z)]$$

$$H(z) = q_s[h(z)]$$

- q_f, q_s are probability generating functions (PGF) of p_f, p_s (known)

- *H(z)*, *h(z)* PGFs of *P(n)*, *p(n)*

$$H(z) = \sum_{n=0}^{\infty} P(n)\, z^n$$

- *P(n)*, *p(n)* are then corresponding Taylor expansion coefficients of *H(z)*, *h(z)*

$$P(n) = \frac{1}{n!} \frac{\partial^n H(z)}{\partial z^n}\bigg|_{z=0}$$

- Calculating the derivatives means we also need derivatives of *h(z).*
- Each n-th derivative contains all lower derivatives.
- Starting value is given by ”0th” derivative giving *p(0)*

$$p(0) = p\sum_{n=0}^{N} p_f(n)[p(0)]^n$$

- First probability *P(1)*, *p(1)* needs the derivative of the implicitly defined *h(z)*

$$\frac{\partial h(z)}{\partial z} = (1-p) + \frac{\partial q_f(h)}{\partial h}\frac{\partial h(z)}{\partial z}$$

$$\Leftrightarrow \qquad \frac{\partial h(z)}{\partial z} = \frac{(1-p)}{\left(1 - p\,\frac{\partial q_f(h)}{\partial h}\right)}$$

- New quantities that reminds of the mean number of particles produced

$$\left.\frac{\partial q_f(h)}{\partial h}\right|_{z=0} \approx \sum_{n=0}^{\approx 8} n\, p_f(n)[p(0)]^{n-1} \equiv \overline{\nu_f}$$

Symbolic tools

- The derivatives grow rapidly

$$h^{(3)}(z) = \frac{3p\, h'(z)h''(z)\, q[h(z)] + p\, h'(z)^3 q^{(3)}[h(z)]}{1 - p\, q'[h(z)]}$$

- For high n - *p(n)*, *P(n)* can contain thousands of terms
- Mathematica can handle it all symbolically

Mathematica procedure

- Writing code to give rule-based results.
  - `f[z_]:=(1-p)z+p q[h[z]]`
  - `Solve[f(i)[z]==h(i)[z], h(i)[z]]`

$$h^{(3)}(z) \to \frac{3p\, h'(z)h''(z)\, q[h(z)] + p\, h'(z)^3 q^{(3)}[h(z)]}{1 - p\, q'[h(z)]}$$

  - Rules used recursively with substitute commands: /. //.
  - Same procedure continues for H(z), p(n), P(n).

- Still all symbolically, recursively described.
- Last step is to evaluate the quantities we search by using numerical data, and the fact that $z=0$ for $p(n)$, $P(n)$

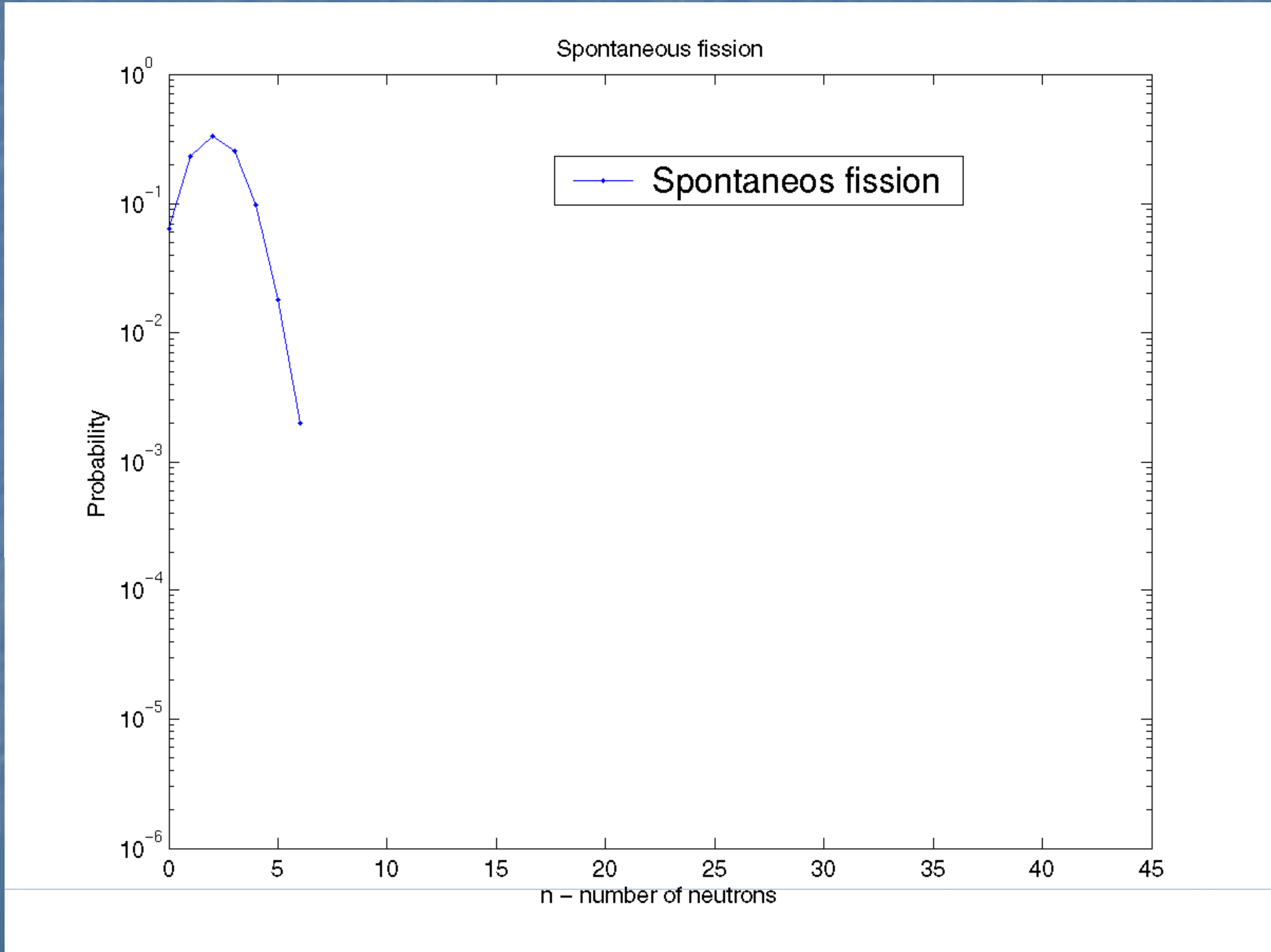
{P(1)→0.168749,P(2)→0.183567,
P(3)→0.119855, ... }

- The probabilities calculated, will very closely sum up to unity to confirm the general validity of the approach used.

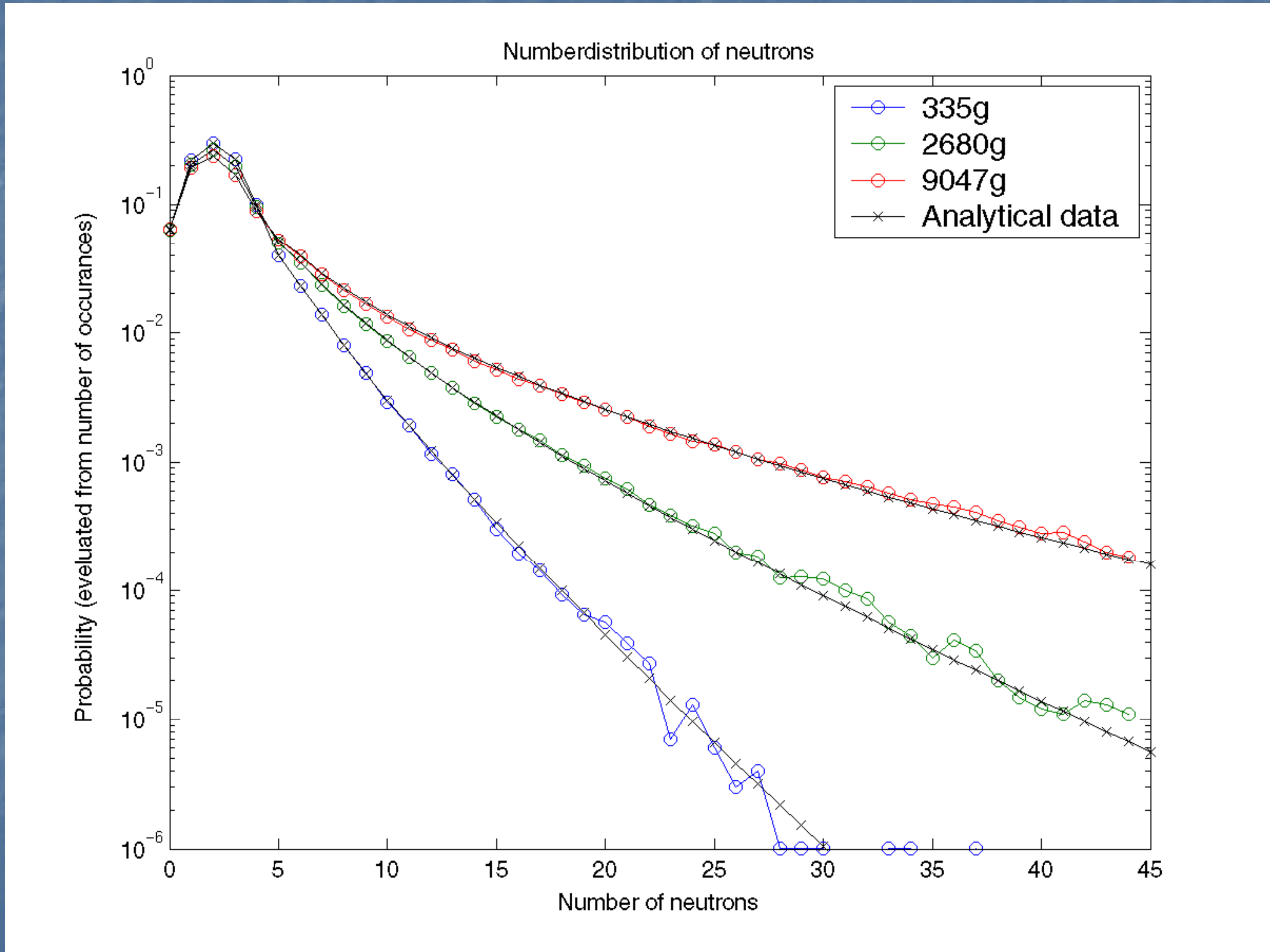
Results

- Comparison with results obtained by Monte Carlo simulations MCNP-PoliMi.
- Larger samples makes it possible to get larger neutron ”bursts”.
- The main distribution still follows the spontaneous distribution quite nicely

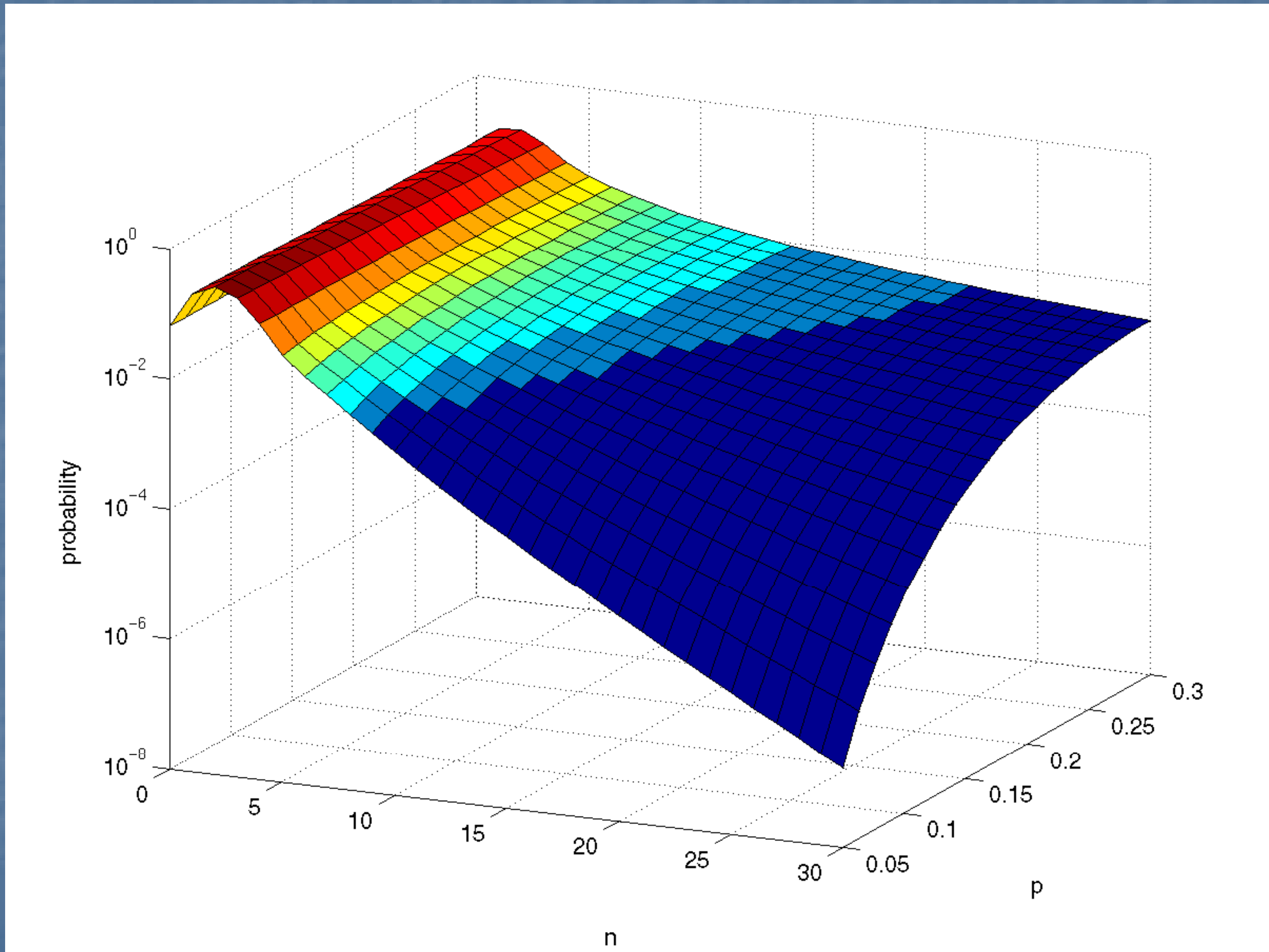
Spontaneous fission



Number distribution for three samples of different mass



Dependence on mass (the probability to induce fission)



Future work

- Probabilities for gammas.
  - As mentioned the multiplicities for gammas are higher and can be favourably measure with scintillator detectors.
- Examining closer the dependence on the fission probability – p.
- Developing the analytical description to account for absorption.
- Incorporation detector efficiencies.
- Response to initial events other than spontaneous fission.
- Inverse tasks – now the calculations gives us multiplicities by inserting data for sample mass.