An invitation to scientific computing with finite elements

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Chapter 1

Models for incompressible fluids

1.1 Navier-Stokes equations

1.1.1 A first glance at the equation

The motion of fluids like air or water can be described in some situations by the Navier-Stokes equations

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \mu\Delta\mathbf{u} - \nabla p + \mathbf{f}, \tag{1.1}$$

$$\text{div } \mathbf{u} = 0, \tag{1.2}$$

where $\rho > 0$ is the (constant) density of the fluid, $\mu > 0$ its viscosity, and $\mathbf{f}$ the force per unit volume applied to the fluid. The unknowns are $\mathbf{u} = (u_1, u_2, u_3)$ for the velocity and p for the pressure of the fluid.

Equation (1.1) represents the balance law (i.e. $m\mathbf{a} = \mathbf{F}$), while (1.2) reflects the incompressibility of the fluid.

1.1.2 Forces

Only two types of forces are allowed, namely the body forces and the contact forces. Body forces are described by a density. the typical example is the gravity in which

$$\mathbf{f} = -\rho g \mathbf{e}_3, \tag{1.3}$$

where g is the acceleration of gravity and $\mathbf{e}_3$ the vertical unit vector pointing upward.

Contact forces, on the other hand, represent the forces experienced by the fluid from the exterior world at its boundary, and are given by a surface density s. The same description is used for forces exerted on (virtual) surfaces inside the fluid. More precisely, let $\mathcal{S}$ be an oriented smooth surface inside the fluid, $\mathbf{x} \in \mathcal{S}$ and $\mathbf{n}$ be the positive normal to $\mathcal{S}$ at $\mathbf{x}$. Then $\mathbf{s}(\mathbf{x}, \mathbf{n})$ represents the force per unit area at $\mathbf{x}$ exerted by the material on the positive side of $\mathcal{S}$ on the material on the negative side. That $\mathbf{s}$ depends only on $\mathcal{S}$ through its normal - and not, say, on its curvature - is an assumption called Cauchy stress postulate. It is a working hypothesis defining a class of materials on which the theory applies. It turns out to be a sufficiently large class to make the theory widely applicable.

From this, we can write the total force $\mathbf{F}$ acting on a region Ω occupied by the fluid as

$$\mathbf{F} = \int_\Omega \mathbf{f}(\mathbf{x})\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{s}(\mathbf{x}, \mathbf{n}(\mathbf{x}))\, d\sigma(\mathbf{x}). \tag{1.4}$$

1.1.3 Kinematics

The field $\mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid particle occupying position $\mathbf{x}$ at time t. We are interested in the acceleration of that particle in order to write Newton's second law. We accomplish this by labelling particles by their positions at time $t = 0$. Let Ω_0 (resp. Ω_t) be the region occupied by the fluid at time 0 (resp. time t). Usually, Ω_0 is called the "reference configuration" whereas Ω_t stands for the "current configuration". The map $\boldsymbol{\chi}(\mathbf{X}, t)$ which gives the position at time t of the particle that was in $\mathbf{X} \in \Omega_0$ at time 0, is called "the motion" and satisfies the ODE

$$\frac{d\boldsymbol{\chi}}{dt}(\mathbf{X}, t) = \mathbf{u}(\boldsymbol{\chi}(\mathbf{X}, t), t), \tag{1.5}$$

$$\boldsymbol{\chi}(\mathbf{X}, 0) = \mathbf{X}. \tag{1.6}$$

Its inverse, defined on Ω_t will be denoted by $\boldsymbol{\xi}$. Now, the acceleration of the particle at time t and position $\boldsymbol{\chi}(\mathbf{X}, t)$ is given by

$$\begin{aligned} a(\boldsymbol{\chi}(\mathbf{X}, t), t) &= \frac{d}{dt}\mathbf{u}(\boldsymbol{\chi}(\mathbf{X}, t), t), \\ &= \frac{\partial \mathbf{u}}{\partial t}(\boldsymbol{\chi}(\mathbf{X}, t), t) + \sum_i \frac{\partial \mathbf{u}}{\partial x_i}(\boldsymbol{\chi}(\mathbf{X}, t), t)\frac{d\chi_i}{dt}(\mathbf{X}, t) \end{aligned}$$

which from (1.5) reduces to

$$\begin{aligned} \mathbf{a}(\boldsymbol{\chi}(\mathbf{X}, t), t) &= \frac{\partial \mathbf{u}}{\partial t}(\boldsymbol{\chi}(\mathbf{X}, t), t) + \sum_i \frac{\partial \mathbf{u}}{\partial x_i}(\boldsymbol{\chi}(\mathbf{X}, t), t) u_i(\boldsymbol{\chi}(\mathbf{X}, t), t), \\ &= \frac{\partial \mathbf{u}}{\partial t}(\boldsymbol{\chi}(\mathbf{X}, t), t) + (\mathbf{u} \cdot \nabla)\mathbf{u}(\boldsymbol{\chi}(\mathbf{X}, t), t). \end{aligned}$$

For $\mathbf{x} \in \Omega_t$, using $\mathbf{X} = \boldsymbol{\xi}(\mathbf{x}, t)$ in the preceding formula leads to

$$\mathbf{a}(\mathbf{x}, t) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) + (\mathbf{u} \cdot \nabla)\mathbf{u}(\mathbf{x}, t).$$

The same computation for an arbitrary function $\phi(\mathbf{x}, t)$ leads to

$$\dot{\phi}(\mathbf{x}, t) := \frac{\partial}{\partial t}\phi(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla)\phi(\mathbf{x}, t), \tag{1.7}$$

which is sometimes called the "total time derivative", the "material derivative" or the time derivative of $\phi(\mathbf{x}, t)$ at $\mathbf{X}$ fixed. With this notation, $\mathbf{a}(\mathbf{x}, t) = \dot{\mathbf{u}}(\mathbf{x}, t)$.

1.1.4 Balance laws (Newtonian mechanics)

For any subset E of Ω_0, let us call $E_t = \boldsymbol{\chi}(E,t)$ the region occupied by the particles of fluid which were in E at time 0. The balance laws that control the evolution of E_t, according to Newtonian mechanics are:

- Conservation of mass,
- Balance of linear momentum,
- Balance of angular momentum.

Conservation of mass

Let us call

$$M(E_t) = \int_{E_t} \rho(\mathbf{x},t)\, d\mathbf{x}, \tag{1.8}$$

the total mass of fluid contained in E_t. Conservation of mass states that $M(E_t)$ stays constant in time. Making the change of variable $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$

$$\int_{E_t} \rho(\mathbf{x},t)\, dx = \int_E \rho(\boldsymbol{\chi}(\mathbf{X},t),t) J(\mathbf{X},t)\, d\mathbf{X}\, dt, \tag{1.9}$$

where $J(\mathbf{X},t) > 0$ is the jacobian determinant of $\boldsymbol{\chi}$ at $(\mathbf{X},t)$. Then using Euler's lemma (see Exercise 1)

$$\frac{\partial J}{\partial t}(\mathbf{X},t) = J(\mathbf{X},t)\mathrm{div}_{\mathbf{x}}\mathbf{u}(\mathbf{x},t)_{|\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)}, \tag{1.10}$$

and (1.9), conservation of mass becomes

$$\begin{aligned} 0 &= \frac{d}{dt} M(E_t) \\ &= \int_E \dot{\rho} J + \rho \frac{\partial J}{\partial t}\, d\mathbf{X} \\ &= \int_E (\dot{\rho} + \rho \mathrm{div}_{\mathbf{x}}\mathbf{u}) J\, d\mathbf{X} \\ &= \int_{E_t} (\dot{\rho} + \rho \mathrm{div}_{\mathbf{x}}\mathbf{u})\, d\mathbf{x} \end{aligned} \tag{1.11}$$

for every $E \subset \Omega_0$ (and thus for every subset E_t of Ω_t). As a consequence, we get

$$\dot{\rho} + \rho \mathrm{div}_{\mathbf{x}}\mathbf{u} = 0 \text{ in } \Omega_t. \tag{1.12}$$

The incompressibility constraint can be stated and handled in the same way. Namely,

by saying that the volume of E_t stays constant in time. This becomes

$$\begin{aligned} 0 &= \frac{d}{dt}\mathrm{vol}(E_t) \\ &= \frac{d}{dt}\int_{E_t} 1\, d\mathbf{x} \\ &= \frac{d}{dt}\int_{E} J(\mathbf{X},t)\, d\mathbf{X} \\ &= \int_{E} \frac{\partial J}{\partial t}(\mathbf{x},t)\, d\mathbf{x} \end{aligned}$$

from which one infers $\dfrac{\partial J}{\partial t} = 0$. But since at $t = 0$ one has $J = 1$ ($\boldsymbol{\chi}(\mathbf{X}, 0) = \mathbf{X}$), it stays constant in time and space. Using this in (1.10) leads to $\mathrm{div}_{\mathbf{x}}\mathbf{u}(\mathbf{x},t) = 0$ from which one deduces (from (1.12)) that $\dot{\rho} = 0$. Therefore provided the density is constant at $t = 0$, it stays constant in time (and space).

At this point we have justified that incompressibility is expressed by (1.2), and that ρ in (1.1) is a positive constant.

Balance of linear momentum

Again, we set as before

$$\mathbf{L}(E_t) = \int_{E_t} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\, d\mathbf{x}, \tag{1.13}$$

the total linear momentum contained in E_t. Balance of linear momentum is simply the transcription of Newton's law ($\mathbf{F} = m\mathbf{a}$) which states that the rate of change of $L(E_t)$ equals the total force on E_t. In view of (1.4), this is

$$\frac{d}{dt}\mathbf{L}(E_t) = \int_{E_t} \mathbf{f}(\mathbf{x},t)\, d\mathbf{x} + \int_{\partial E_t} \mathbf{s}(\mathbf{x},t,\mathbf{n})\, d\sigma(\mathbf{x}), \tag{1.14}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x},t)$ stands for the unit outward normal to ∂E_t. Using Exercise 2, we get $\dfrac{d}{dt}\mathbf{L}(E_t) = \displaystyle\int_{E_t} \rho(\mathbf{x},t)\dot{\mathbf{u}}(\mathbf{x},t)\, d\mathbf{x}$.

Thanks to Cauchy's theorem (see Exercise 3), $\mathbf{s}(\mathbf{x},t,\mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}$, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. Therefore

$$\int_{\partial E_t} \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}\, d\sigma(\mathbf{x}) = \int_{E_t} \mathrm{div}\boldsymbol{\sigma}(\mathbf{x},t). \tag{1.15}$$

Now, collecting terms, and using that E_t is arbitrary, we get

$$\rho\dot{\mathbf{u}} = \mathrm{div}\boldsymbol{\sigma} + \mathbf{f}. \tag{1.16}$$

Balance of angular momentum

As we shall see, the balance law for angular momentum gives the new feature that the Cauchy stress tensor has to be symmetric.

Let x_0 be any point. We define

$$\mathbf{A}(E_t) = \int_{E_t} (\mathbf{x} - \mathbf{x}_0) \wedge \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)\, d\mathbf{x} \tag{1.17}$$

the total angular momentum with respect to $\mathbf{x}_0$ of the fluid contained in E_t. Balance of angular momentum states that the rate of change of $\mathbf{A}(E_t)$ equals the total momentum with respect to $\mathbf{x}_0$ of the forces acting on E_t. This means

$$\frac{d}{dt}\mathbf{A}(E_t) = \int_{E_t} (\mathbf{x} - \mathbf{x}_0) \wedge \mathbf{f}(\mathbf{x}, t)\, d\mathbf{x} + \int_{\partial E_t} (\mathbf{x} - \mathbf{x}_0) \wedge \mathbf{s}(\mathbf{x}, t, \mathbf{n})\, d\sigma(\mathbf{x}). \tag{1.18}$$

Again, $\dfrac{d}{dt}\mathbf{A}(E_t) = \displaystyle\int_{E_t} \rho(\mathbf{x}, t)(\mathbf{x}-\mathbf{x}_0)\wedge\dot{\mathbf{u}}(\mathbf{x}, t)\, d\mathbf{x}$ (see Exercise 2), and thus (1.18) rewrites

$$\int_{E_t} (\mathbf{x}-\mathbf{x}_0)\wedge(\rho\dot{\mathbf{u}}-\mathbf{f}-\mathrm{div}\boldsymbol{\sigma})\, d\mathbf{x} = -\int_{E_t} (\mathbf{x}-\mathbf{x}_0)\wedge\mathrm{div}\boldsymbol{\sigma}\, d\mathbf{x} + \int_{\partial E_t} (\mathbf{x}-\mathbf{x}_0)\wedge\boldsymbol{\sigma}\mathbf{n}\, d\mathbf{x}. \tag{1.19}$$

From the balance of linear momentum, the left-hand side vanishes, which means that $\boldsymbol{\sigma}$ is symmetric (see Exercise 4).

1.1.5 Stresses in a Newtonian fluid

The validity of the balance laws of previous section is not restricted to fluids, and actually we will use them for the modelling of solids. The type of material is specified by a constitutive equation which relates the Cauchy stress tensor to the motion. A fluid reacts to attempt to change its volume (with a pressure, and incompressibility is an extreme case) and to imposed shear rates, typically with a shear force. For an incompressible Newtonian fluid, the considered equation is

$$\boldsymbol{\sigma} = -p\mathbf{Id} + 2\mu\mathbf{D}(\mathbf{u}), \tag{1.20}$$

where p is the pressure of the fluid, $\mathbf{D}(\mathbf{u}) = \dfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the symmetric part of the velocity gradient (here $\nabla\mathbf{u}^T$ stands for the transpose of $\nabla\mathbf{u}$), and $\mu > 0$ is a scalar called the viscosity. The sign of μ is a consequence of positivity of dissipation as we will see later.

On the right-hand side, the first term would describe a fluid at rest. As we shall see later, the pressure $p = -\frac{1}{3}tr(\boldsymbol{\sigma})$ can be viewed as a Lagrange multiplier associated with the incompressibility constraint $\mathrm{div}\,\mathbf{u} = 0$. The other term is symmetric, linear in the gradient of the velocity, isotropic and Galilean invariant (i.e. invariant with respect to superimposed translations and rigid rotations).

1.1.6 Navier Stokes equations

Substituting $\boldsymbol{\sigma}$ in (1.16) by the formula given in (1.20), and remarking that

$$\mathrm{div}\nabla\mathbf{u}^T = \nabla\mathrm{div}\mathbf{u} = 0, \tag{1.21}$$

gives Navier-Stokes equations

$$\begin{cases} \rho\left[\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \mu\Delta\mathbf{u} - \nabla p + \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0. \end{cases} \tag{1.22}$$

Navier-Stokes equations are a very important model for many reasons. First, they are very fundamental. The only non-generic aspect is the linearity of the constitutive equation (1.20), but this is at least the most generic when the velocity has only small gradients. Then, it turns out that despite this simple constitutive equation, they model accurately a wide range of physical phenomena (from airplane wings, to wind on suspension bridges and skyscrapers, to ships and sail boats, to flow of water in a pipe, to swimming of fishes and bacteria, etc.) including natural instabilities (turbulence), which poses a lot of trouble for their numerical simulations. Eventually, they pose a severe challenge to mathematicians so that they have been chosen as one of the seven millenium problems selected by the Clay Mathematical Institute.

The physically relevant boundary conditions are typically of two kinds

- Prescribed velocity (in particular the no-slip condition $\mathbf{u} = 0$ is applied where the boundary is a solid at rest) ;
- Prescribed boundary forces. Calling $\mathbf{g}$ a known force per unit area applied to the fluid from the environment, it writes

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g}. \tag{1.23}$$

1.1.7 Energy and dissipation in an incompressible Newtonian fluid

By definition, the dissipation rate is equal to the difference between power expended on a system by external forces and the rate of change of stored energy. From thermodynamics, it is always non-negative. Here, the fluid only has kinetic energy and therefore we can write (E_t being any subset of Ω_t as usual)

$$\int_{E_t} \mathbf{f}\cdot\mathbf{u}\,dx + \int_{\partial E_t} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u}\,d\sigma - \frac{d}{dt}\left(\int_{E_t} \rho\frac{u^2}{2}\,d\mathbf{x}\right) \geq 0. \tag{1.24}$$

Again, for the last term, one has

$$\frac{d}{dt}\left(\int_{E_t} \rho\frac{u^2}{2}\,d\mathbf{x}\right) = \int_{E_t} \rho\mathbf{u}\cdot\dot{\mathbf{u}}\,d\mathbf{x}. \tag{1.25}$$

The second term is handled as

$$\int_{\partial E_t} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u}\,d\sigma = \int_{\partial E_t} \boldsymbol{\sigma}^T\mathbf{u}\cdot\mathbf{n}\,d\sigma = \int_{E_t} \mathrm{div}(\boldsymbol{\sigma}^T\mathbf{u})\,d\mathbf{x}, \tag{1.26}$$

which using the identity

$$\mathrm{div}(\boldsymbol{\sigma}^T\mathbf{u}) = \mathrm{div}\boldsymbol{\sigma}\cdot\mathbf{u} + \boldsymbol{\sigma} : \nabla\mathbf{u}, \tag{1.27}$$

and the fact that $\boldsymbol{\sigma}$ is actually symmetric, reduces to

$$\int_{\partial E_t} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u}\,d\sigma = \int_{E_t} \mathrm{div}\,\boldsymbol{\sigma}\cdot\mathbf{u}\,d\mathbf{x} + \int_{E_t} \boldsymbol{\sigma} : \mathbf{D}(\mathbf{u})\,d\mathbf{x}. \tag{1.28}$$

Collecting all terms, one arrives to

$$\int_{E_t} (\mathbf{f} + \mathrm{div}\boldsymbol{\sigma} - \rho\dot{\mathbf{u}})\cdot\mathbf{u}\,d\mathbf{x} + \int_{E_t} \boldsymbol{\sigma} : \mathbf{D}(\mathbf{u})\,d\mathbf{x} \geq 0. \tag{1.29}$$

From, the balance of linear momentum (1.16), the first term vanishes, whereas from (1.20), one gets

$$\int_{E_t} \boldsymbol{\sigma} : \mathbf{D}(\mathbf{u})\,d\mathbf{x} = \int_{E_t} (-p\mathbf{Id} + 2\mu\mathbf{D}(\mathbf{u})) : \mathbf{D}(\mathbf{u})\,d\mathbf{x} = 2\mu\int_{E_t} |\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x}. \tag{1.30}$$

The dissipation being non-negative one must have $\mu \geq 0$.

Remark 1 *In the case where there is no body force ($\mathbf{f} = 0$) and for example $\mathbf{u} = 0$ at the boundary, then the kinetic energy decreases in time providing a kind of stability to the equations. Of course, this is formal because the reasonning relies on the smoothness of u and the existence of sufficiently regular solutions is not known in general.*

1.2 The Reynolds[1] number - Stokes equations

Stokes equations are obtained as formal limits of Navier-Stokes equations (1.1,1.2) when some terms are neglected. To make this statement more precise, we make the change of variables and rescale the unknowns by considering

$$\mathbf{x}^* = \frac{1}{L}\mathbf{x}, \quad \mathbf{u}^* = \frac{1}{U}\mathbf{u}\,,\ t^* = \frac{U}{L}t, \tag{1.31}$$

where L, and U respectively stand for a typical length and velocity involved in the considered flow. The equation on dimensionless variables writes

$$\frac{\partial\mathbf{u}^*}{\partial t^*} + (\mathbf{u}^*\cdot\nabla_*)\,\mathbf{u}^* - \frac{\mu}{\rho U L}\Delta\mathbf{u} + \nabla p^* = \mathbf{f}^* \tag{1.32}$$

where $p^* = p/(\rho U^2)$ is the adimensioned pressure, and $\mathbf{f}^* = \mathbf{f}L/(\rho U^2)$ the adimensioned forcing term. The quantity

$$\mathrm{Re} = \frac{\rho U L}{\mu}$$

is called the "Reynolds number". The Reynolds number is a number without dimension which measures the intensity of the inertial forces (typically of order $\frac{\rho U^2}{L}$) over the viscous

[1] "Osborne Reynolds (23 August 1842 - 21 February 1912) was a prominent innovator in the understanding of fluid dynamics. Separately, his studies of heat transfer between solids and fluids brought improvements in boiler and condenser design." (Source Wikipedia).

forces inside the fluid (of order $\frac{\mu U}{L^2}$). When Re is much smaller than one, the flow is governed by the viscosity while on the contrary for Re >> 1, viscosity is negligible and inertia dominates.

Depending whether the coefficients in (1.32) are small or not the corresponding terms may be neglected or not. This gives several simplified equations which should formally be approximations of (1.1,1.2). Namely we distinguish

- The stationary Stokes equations, in the regime Re $\to 0$, all the other parameters being taken as constant.

$$-\Delta_* \mathbf{u}^* + \nabla_* p^* = 0, \tag{1.33}$$
$$\mathrm{div}_* \mathbf{u}^* = 0. \tag{1.34}$$

  In such flows, the viscosity dominates inertial effects, and the rate of change of loads (or boundary conditions) is small compared to intrinsic response times of the fluid. Whether or not gravity plays a role depends on the value of λ. Going back to the physical quantities and units, we get the equations

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.35}$$
$$\mathrm{div}\ \mathbf{u} = 0, \tag{1.36}$$

- The non-stationary Euler equations, in the regime Re $\to +\infty$. Formally the viscosity is negligible and we remain with

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] + \nabla p = \mathbf{f}, \tag{1.37}$$
$$\mathrm{div}\ \mathbf{u} = 0, \tag{1.38}$$

Let us emphasize that this limit is purely formal, and the actual behavior of flows governed by Navier-Stokes equations can be very different from those governed by Euler equation. In particular, the standard condition at a wall for NS equations is a no-slip condition, whereas natural conditions for Euler affect the normal component only. As a consequence, the behavior of both models near walls can be drastically different, even at very high Reynolds numbers.

Remark 2 *The description of phenomena associated to flows at high Reynolds number goes far beyond the scope of this course, and the understanding of* turbulence *(which refers to situations where complex motions occur over a large large a space scales) still raises many open questions. Let us simply say here that high Reynolds flows can be pictured as containing* eddies *over a large range of sizes, starting to the global size of the observed phenomenon, down to much smaller scales. It is usually considered that dissipation occurs at some space scale η (smaller size of the eddies). According to Kolmogorov's theory, L/η is of the order $Re^{3/4}$. This formula gives a precious indication in the context of numerical simulations. If one aims at discretizing the space in order to "capture" (i.e. represent on the mesh) the smallest eddies, the number of mesh vertices in each direction scales*

at $Re^{3/4}$, so that in $3d$, the total number of vertices is $Re^{9/4}$. This remark concerns Direct Navier-Stokes *simulations (called DNS). Other methods, like Large Eddy Simulation (LES) method, or methods based on the so-called $k-\varepsilon$ model, have been introduced to limit the cost of numerical simulation by discretizing at a scale larger than η. Those approaches rely on assumptions regarding what happens at scales smaller than the mesh size, and involve extra unknown pertaining to those phenomena (like the kinetic energy k associated to smaller scales in the $k-\varepsilon$ model).*

Oscillatory flows. In the case of oscillatory flows, which plays an important role in many situations (e.g. in the swimming motion of living organisms), it is natural to follow a slightly different approach: denoting by ω the frequency of the phenomenon which is observed, the time is made dimensionless according to $t^* = \omega t$, which leads to

$$\omega \frac{L}{U} \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla_*)\, \mathbf{u}^* - \frac{1}{\mathrm{Re}} \Delta \mathbf{u} + \nabla p^* = \mathbf{f}^*.$$

This approach decouples the time derivative and its advective counterpart, which allows to consider situations where Re goes to 0 together with $U/(L\omega)$, which leads to the non-stationary Stokes equations, which writes

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f}.$$

As typical numerical applications, we consider the case of a bacterium or a dolphin swimming into water at room temperature. In these cases, we get $\dfrac{\rho}{\mu} = \dfrac{10^3 \mathrm{kg\ s^{-1}}}{10^{-3}\ \mathrm{Pa\ s}} = 10^6 \mathrm{m^{-2}s}$ and

- For the dolphin, $\omega \sim 1\ \mathrm{s}^{-1}$, $L \sim 1$ m, $U \sim 10$ m s^{-1}, and Re $\sim 10^7$, $\dfrac{\omega L}{U} \sim 1$, and, using $\mathbf{f} = -\rho g \mathbf{e}_3$ (the force due to gravity), we get that the force coefficient is of order $\dfrac{Lg}{U^2} \sim 10^{-1}$. This means that inertia plays a proeminent role compared to viscosity. The dolphin moves mainly by accelerating the water and Stokes equations are not a valid approximation to Navier-Stokes equations in that case. One should rather use in this situation the Euler equations (1.37,1.38).
- For the bacterium, $\omega \sim 1\ \mathrm{s}^{-1}$, $L \sim 1\ \mu$m, $U \sim 10\ \mu$m s^{-1}, and Re $\sim 10^{-5}$, Re$\frac{\omega L}{U} \sim 10^{-6}$, and Re$\dfrac{Lg}{U^2} \sim 1$. In that case, the Stokes equations is a valid model, however, the gravity cannot be neglected. A bacterium can only move by exploiting viscous forces.

The Reynolds number distinguishes between the so-called *laminar* flows (Re $<<$ 1) and *turbulent* flows (Re $>>$ 1). Examples of laminar flows are given by very viscous flows, or flows inside a glacier (see Fig. 1.2) while in turbulent flows one usually sees vortices and non stationary regimes (see Fig. 1.2)

Remark 3 *In Stokes equations, taking the divergence of (1.33) leads to*

$$\Delta_* p^* = 0. \tag{1.39}$$

Figure 1.1: In a glacier the flow seems very smooth. Nice laminar structures appear in the direction of the flow.

Figure 1.2: Flow around a wing or in a river. In both cases complex structures appear and the flow is non stationnary, and turbulent.

1.3 Equivalent writings of Navier-Stokes equations

It is very important to understand that the pressure in Navier-Stokes equations is one of the unknowns (together with the three components of the velocity $\mathbf{u}$). In particular, using the identity (known as Bernoulli's identity in this context)

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(|\mathbf{u}|^2) - \mathbf{u}\times \text{curl } \mathbf{u}$$

one can transform the original equation (1.1,1.2) into

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times \text{curl } \mathbf{u}\right] = \mu\Delta\mathbf{u} - \nabla\tilde{p} + \mathbf{f}, \tag{1.40}$$

$$\text{div } \mathbf{u} = 0, \tag{1.41}$$

where the new pressure $\tilde{p} = p + \frac{\rho}{2}|\mathbf{u}|^2$ is usually called the dynamical pressure.

Another writing of the non linear term is the following

$$\begin{aligned}(\mathbf{u}\cdot\nabla)\mathbf{u} &= \sum_{i=1}^{3} u_i \partial_{x_i}\mathbf{u} \\ &= \sum_{i=1}^{3} \partial_{x_i}(u_i\mathbf{u})\end{aligned}$$

in view of div $\mathbf{u} = 0$. This latter expression is often denoted by div $(\mathbf{u}\otimes\mathbf{u})$. With this notation, the Navier-Stokes equations become

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \text{div } (\mathbf{u}\otimes\mathbf{u})\right] = \mu\Delta\mathbf{u} - \nabla p + \mathbf{f}, \tag{1.42}$$

$$\text{div } \mathbf{u} = 0, \tag{1.43}$$

1.4 Some typical problems

We describe here some situations for which the exact solution cannot be given analytically.

1.4.1 Flow around an obstacle

A great amount of computational strategies to approximate solutions to the Navier-Stokes equation have been developed during the last decades to address the following problem (represented in the two dimensional setting by Fig. 1.3): consider a flow of a viscous fluid around a fixed obstacle B, one is interested in estimating the forces exerted by the fluid on the obstacle. The problem consists in solving incompressible Navier-Stokes equations in the fluid part Ω, with no-slip conditions on the boundary γ of the obstacle. The question of boundary conditions on the outer boundary raises some interesting issues. A first model consists in assuming that the outer boundary is far enough from ω, so that the flow there

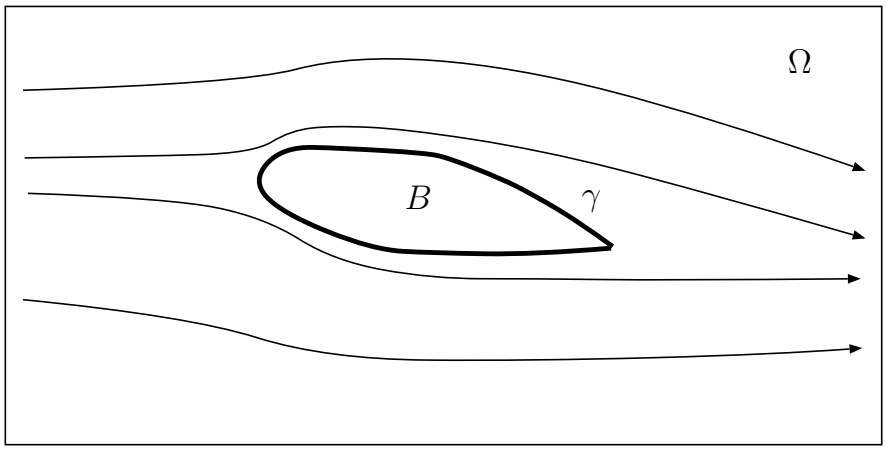

Figure 1.3: Flow around an obstacle

is not affected by the presence of the obstacle. In this case, it is reasonable to assume the velocity to be fixed on $\Gamma = \partial\Omega$, equal to a "velocity at infinity" $\mathbf{U}_\infty$. The approach may lead to a large exterior domain, and thereby high computational costs. A great amount of investigations has been undertaken to propose alternative strategies to overcome this problem, by elaborating appropriate boundary conditions on the outer boundary.

In the case where ω represent an airfoil, quantities of interested are the lift force (*portance* in french) $\mathbf{F}_y$, and the drag (*traînée*) $\mathbf{F}_x$

$$\mathbf{F}_y = -\mathbf{e}_y \cdot \int_\gamma \boldsymbol{\sigma}\mathbf{n}\ , \quad \mathbf{F}_x = -\mathbf{e}_x \cdot \int_\gamma \boldsymbol{\sigma}\mathbf{n}.$$

The lift force maintains the aircraft in the air (it balances the total weight of the flying plane), whereas drag is opposed to motion, thereby conditioning the price to pay to make the plane flying.

1.4.2 Some simple fluid-structure interaction problems

In real life application, one is commonly interested in modeling the way a fluid interacts with another medium, like an elastic structure. This problem raises issues which are still the object of active research, both on the theoretical and numerical aspects. We present here here two situations where the number of degrees of freedom for the structure is finite.

Spring-mass-fluid system. We consider the situation represented in Fig. 1.4 (left). The domain Ω is bounded on the right side by a piston with mass m, attached to a spring with stiffness $k > 0$. Despite its formal simplicity, this example calls for a special care of boundary conditions : as the motion of the piston is not compatible with the no-slip condition on the lateral boundary, we shall assume that the fluid may slip freely along this part of the boundary. Assuming the fluid sticks to the piston, the global problem

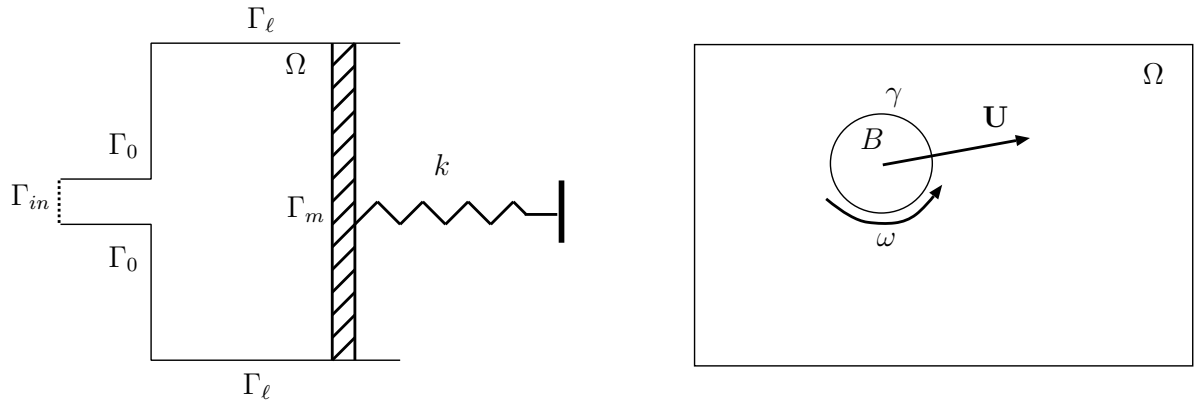

Figure 1.4: Interaction with a rigid structure

may be written

$$\begin{aligned}
\rho\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) - \mu\Delta\mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega(t)\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega(t)\\
\mathbf{u} &= 0 \quad \text{on } \Gamma_0\\
\boldsymbol{\sigma}\mathbf{n} &= 0 \quad \text{on } \Gamma_{in}\\
\mathbf{u}\cdot\mathbf{n} = 0\,,\ \ \mathbf{t}\cdot\boldsymbol{\sigma}\mathbf{n} &= 0 \quad \text{on } \Gamma_\ell\\
\mathbf{u} = \dot{x}\,\mathbf{e}_x &= 0 \quad \text{on } \Gamma_m
\end{aligned}$$

supplemented by Newton's law for the piston:

$$m\frac{d^2x}{dt^2} = -k(x-x_0) - \int_{\Gamma_m}\boldsymbol{\sigma}\mathbf{n}.$$

Fluid particle flow. In this second example (see Fig. 1.4, right), we consider a rigid disc flowing freely in a viscous fluid. Denoting by ρ_s the density of the solid, and by m its mass, the fluid problem can be written in the form of incompressible Navier-Stokes equations in the moving domain $\Omega(t)$ (with right-hand side $\rho\mathbf{g}$, supplemented by two types of coupling conditions:

1. Kinematic coupling. No-slip condition on the boundary of the body writes
$$\mathbf{u} = \mathbf{U} + \omega\wedge\mathbf{r} \quad \text{on } \gamma.$$
2. Dynamic coupling. Newton's law for the particle translational and angular velocity write
$$m\frac{d\mathbf{U}}{dt} = -\int_\gamma \boldsymbol{\sigma}\mathbf{n} + m\mathbf{g}\,,\ \ J\frac{d\omega}{dt} = 0,$$
where J is the moment of inertia of the disc.

The problem is written here for a single particle, but it can be extended straightforwardly to the many-body situation (with 3 degrees of freedom per particle, together with a three dimensional coupling between each particle and the fluid).

1.4.3 Flow in complex geometries: example of the bronchial tree

The bronchial tract can be seen as a network of interconnected pipes following a tree-like structure, with 23 levels of bifurcations. It is a typical example of a situation where a full computation of the fluid flow in the overall domain is out of reach. A careful look at the Reynolds number throughout the tree, during normal breathing, reveals that inertial effects are predominant in the first generations only, whereas Reynolds number decreases to very small values at the end of the tree. As a consequence, it is natural to consider Navier-Stokes for the upper part of the tree, for the N first generations, couple the 2^N outlets to equivalent resistances (which account for the truncated subtrees), and finally plug those resistances onto a spring mass system (which accounts for the lung - thoracic cage system). The obtained multi-compartment system is represented in Fig. 1.5.

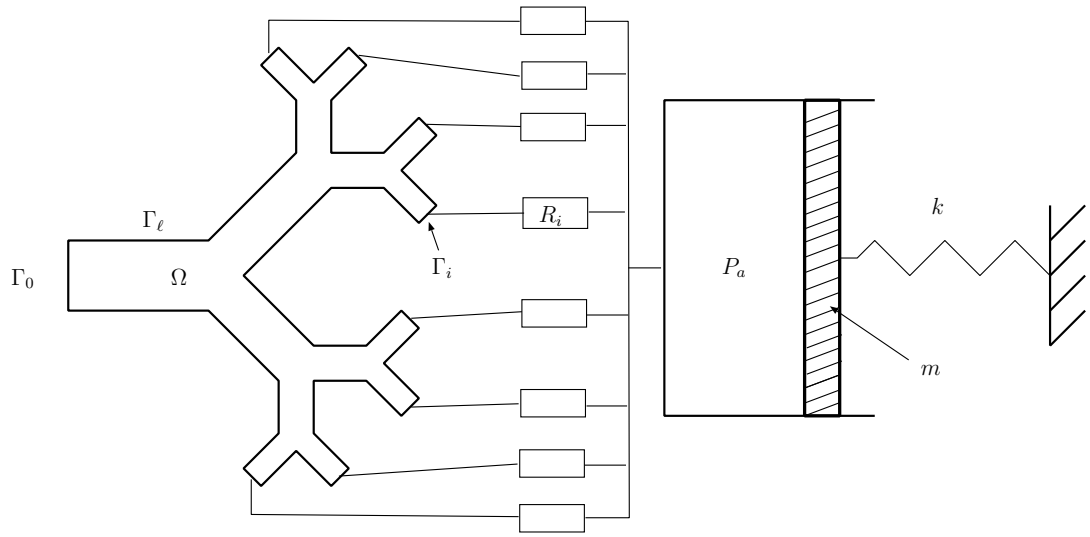

Figure 1.5: Three compartment lung

We introduce a pressure Π_i for each outlet i (this pressure is assumed to be constant over each section), and one write Poiseuille's law for each subtree, which expresses the linear dependence between flux and pressure drop:

$$\Pi_i - P_a = R_i \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n}, \;\; R_i \geq 0 \, .$$

The global system is obtained by eliminating auxiliary pressures Π_i.

$$\left\{\begin{array}{rcll} \rho\dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\triangle\mathbf{u} + \nabla p &=& 0\,, & \text{dans } \Omega\,, \\ \nabla\cdot\mathbf{u} &=& 0\,, & \text{dans } \Omega\,, \\ \mathbf{u} &=& 0\,, & \text{sur } \Gamma_\ell\,, \\ \mu\nabla\mathbf{u}\mathbf{n} - p\mathbf{n} &=& -P_0\mathbf{n} & \text{sur } \Gamma_0\,, \\ \mu\nabla\mathbf{u}\mathbf{n} - p\mathbf{n} &=& -P_a\mathbf{n} - R_i\left(\displaystyle\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}\,, & \text{sur } \Gamma_i\,, \\ m\ddot{x} + kx &=& f_{ext} + SP_a\,, & \\ S\dot{x} &=& \displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\mathbf{u}\cdot\mathbf{n} = -\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{n}. & \end{array}\right. \tag{1.44}$$

1.5 Exercises

Exercise 1 *Prove Euler's lemma*

Exercise 2 *Prove that conservation of mass implies*

$$\frac{d}{dt}\int_{E_t}\rho(\mathbf{x},t)\phi(\mathbf{x},t)\,dx = \int_{E_t}\rho(\mathbf{x},t)\dot{\phi}(\mathbf{x},t)\,d\mathbf{x}. \tag{1.45}$$

Hint: Use formula (1.11)

Exercise 3 *Prove Cauchy's theorem*

Exercise 4 *By expanding the computation, prove that* $-\int_E(\mathbf{x}-\mathbf{x}_0)\wedge div\,\boldsymbol{\sigma}\,d\mathbf{x} + \int_{\partial E}(\mathbf{x}-\mathbf{x}_0)\wedge\boldsymbol{\sigma}\mathbf{n}\,d\mathbf{x} = 0$ *for all* E *if and only if* $\boldsymbol{\sigma}$ *is symmetric.*

Chapter 2

Analysis of the Stokes equations

2.1 The Poisson problem, minimization

We first consider the so-called Dirichlet problem posed on a bounded regular domain Ω, for a right-hand side $f \in L^2(\Omega)$

$$\left[\begin{array}{rcl} -\Delta u & = & f \text{ in } \Omega, \\ u & = & 0 \text{ on } \partial\Omega. \end{array} \right. \tag{2.1}$$

The variational equation is obtained by taking a test function v that vanishes over $\Gamma = \partial\Omega$. One obtains

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v.$$

The problem in this form fits into the following abstract framework

$$\left| \begin{array}{l} V \text{ is a Hilbert space} \\ a(\cdot,\cdot) \text{ bilinear form on } V \times V\,,\ \varphi \in V' \\ \text{Find } u \in V \text{ such that }, \\ a(u,v) = \langle \varphi\,, v \rangle \quad \forall v \in V. \end{array} \right. \tag{2.2}$$

As we shall see, the Lax Milgram theorem asserts well-posedness of this problem, under some assumptions on the bilinear form $a(\cdot,\cdot)$. In the present case, $a(\cdot,\cdot)$ is symmetric, and the problem takes a simpler form: it consists in representing a linear functional ($\varphi \in V'$) by an element of V, with respect to $a(\cdot,\cdot)$. As soon as this bilinear form can be taken as a scalar product on V, the Riesz theorem (Th. **??**) gives existence and uniqueness of such a element. To fit the problem within this representation theorem, the Hilbert space in which the problem will be set has to be chosen in accordance with the bilinear form $\int \nabla u \cdot \nabla v$.

Let us introduce the Sobolev space

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \ \text{ such that } \nabla u \in L^2(\Omega) \right\}, \tag{2.3}$$

where the gradient ∇u is taken in the sense of distributions on Ω. When equipped with the scalar product $(\cdot,\cdot)_{H^1(\Omega)}$ and the associated norm $||\cdot||_{H^1(\Omega)}$ defined by

$$\begin{aligned} ||u||^2_{H^1(\Omega)} &= \int_\Omega |u(x)|^2\,dx + \int_\Omega |\nabla u|^2\,dx \\ (u,v)_{H^1(\Omega)} &= \int_\Omega u(x)v(x)\,dx + \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx \end{aligned}$$

the space $H^1(\Omega)$ is a Hilbert space. The boundary condition on the Dirichlet problem (2.9) is taken into account by considering the subspace $H^1_0(\Omega)$ of $H^1(\Omega)$ defined as the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ (with respect to the H^1 norm). The Poincaré inequality

$$\exists C\,(=C(\Omega)),\ \text{ s.t. } \forall u\in H^1_0(\Omega),\ ||u||_{L^2(\Omega)} \le C\int_\Omega |\nabla u(x)|^2\,dx \tag{2.4}$$

permits to show that

$$|u|_1^2 := \int_\Omega |\nabla u(x)|^2\,dx \tag{2.5}$$

is a norm on $H^1_0(\Omega)$ which is equivalent to the usual H^1 norm. As a consequence, we have:

Proposition 1 *Let Ω be a bounded domain, and $f\in L^2(\Omega)$. The problem*

$$\textit{Find } u\in H^1(\Omega)\,,\ \int_\Omega \nabla u\cdot\nabla v = \int_\Omega fv \quad \forall u\in H^1_0, \tag{2.6}$$

admits a unique solution u, which is also the unique minimizer of

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv.$$

Proof : It is a straightforward application of Riesz theorem in the Hilbert space $V = H^1_0(\Omega)$.

A solution to (2.6) is called a *weak solution* to Problem (2.9). The link between the original equation and the variational formulation is clear, but not yet formalized (the path followed to establish the variational formulation is informal). It is necessary to complete the process by establishing in what sense the original equation stands. This step can be very delicate, as it raises regularity issues (H^1 regularity is not sufficient to give a classic sense to the Laplace operator). If we admit here that the solution provided by Proposition 1 is H^2, which is the case if one assumes that the domain is bounded and regular (twice differentiable), or if it is a convex polyhedron, we may proceed as follows: For any $v\in\mathcal{D}(\Omega)$,

$$\int_\Omega \nabla u\cdot\nabla v = \int_\Omega fv.$$

As u is twice differentiable in the Sobolev sense (its second derivatives are L^2 functions), we may use Green's formula to obtain

$$\int_\Omega (-\Delta u - f)\,v = 0,$$

so that $-\Delta u = f$ almost everywhere. The Dirichlet boundary condition, which was part of the initial problem, is "included" in the space where the well-posedness theorem was obtained. For this reason, such a condition is called *essential*.

Non homogeneous Dirichlet conditions.

Before considering such conditions, let us say a few words on traces. The notion of trace of a function (more precisely of a class of functions, as we consider that two functions in the classical sense identify as soon as they are equal almost everywhere) extends the notion of restriction for regular functions (in a proper sense). If a function is continuous in $\mathbf{R}^2$, its restriction to a regular curve Γ is well-defined. If one considers now a function in L^2, this "restriction" does not make sense as Γ has zero measure. Yet, under some additional regularity assumptions, even if the considered functions are not continuous (so that pointwise values might not be well-defined), it is possible to extend this notion of restriction. In particular for functions in $H^1(\Omega)$, where Ω is a bounded domain with smooth boundary Γ, one can define a trace $v \mapsto \gamma(v)$ which maps H^1 onto $L^2(\Gamma)$. Actually trace functions are more regular than L^2, the range of the trace operator is denoted by $H^{1/2}(\Gamma)$, which means informally that half a derivative is in L^2 (which can be expressed straightforwardly in the Fourier setting). In what follows, we shall denote by $v_{|\Gamma}$, or even v when no confusion is possible, what should be denoted by $\gamma(v)$.

When the value at which u is set on the boundary, the problem takes the following form

$$\left[\begin{array}{rcl} -\Delta u & = & f \text{ in } \Omega, \\ u & = & g \text{ on } \partial\Omega, \end{array} \right. \tag{2.7}$$

where g belongs is (the trace on the boundary of) a H^1 function $\tilde{g}$. Indeed, writing $u = \tilde{g} + \tilde{u}$ we are looking for $\tilde{v} \in H_0^1(\Omega)$ solution of the variational formulation associated with the problem

$$\text{Find } \tilde{u} \in H_0^1(\Omega), \text{ s.t. } \forall v \in H_0^1(\Omega), \int_\Omega \nabla\tilde{u} \cdot \nabla v = \int_\Omega fv - \int_\Omega \nabla\tilde{g} \cdot \nabla v. \tag{2.8}$$

This variational formulation is then solved thanks to Lax-Milgram theorem since

$$v \longmapsto \int_\Omega \nabla\tilde{g} \cdot \nabla v$$

is a continuous linear functional on H_0^1. Note that it is sufficient to assume that g is the trace of a function in H^1 (i.e. that g is in $H^{1/2}(\Gamma)$) to obtain existence and uniqueness of a weak solution in $H^1(\Omega)$, but a better regularity (like the H^2 regularity, which will be needed to obtain optimal convergence for first order finite element approximations) necessitates stronger assumptions on g.

Remark 4 *The problem can also be written as follows*[1]*: Find u in the affine space*

$$\{w \in H^1(\Omega)\,,\ w = g \text{ on } \Gamma\},$$

such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega).$$

[1]The discrete variational formulation will be written in this spirit in the FreeFem++ framework.

Neuman boundary conditions. Other types of boundary conditions, namely Neuman boundary conditions, can be considered. In the context of the heat equation, they consist in prescribing the flux (whereas for Dirichlet conditions the value of the temperature itself is prescribed). We consider the following problem

$$\left[\begin{array}{rcl} u - \Delta u &=& f \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} &=& 0 \text{ on } \partial\Omega. \end{array}\right. \tag{2.9}$$

The variational formulation is

$$\int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v = \int fv \quad \forall v \in H^1(\Omega).$$

If one applies Lax-Milgram theorem (or Riesz theorem) on $H = H^1(\Omega)$ (and not $H_0^1(\Omega)$) with $a(u,v) = (u,v)_{H^1(\Omega)}$, and $\langle \varphi\,,v\rangle = \displaystyle\int_\Omega fv$ as usual, we obtain that there exists a unique $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv. \tag{2.10}$$

The unique solution u can be shown to verify the initial boundary value problem. Admitting again that the solution is in $H^2(\Omega)$ (which is indeed the case as soon as the domain is bounded and regular), we can use the variational formulation with test functions in $\mathcal{D}(\Omega)$, and integrate by part to obtain

$$\int_\Omega (u - \Delta u - f)v = 0 \quad \forall v \in \mathcal{D}(\Omega),$$

so that $u - \Delta u = f$ a.e. (as a function of $L^2(\Omega)$). Considering now test functions which do not vanish on the boundary, integrating by parts, and using the equation which has been established in Ω, one obtains

$$\int_\Gamma \frac{\partial u}{\partial n} v = 0,$$

so that $\partial u/\partial n$ (which is well defined as $u \in H^2$) vanishes almost everywhere on Γ.

A variation between Dirichlet and Neumann boundary conditions is also possible. Namely, we consider Γ_D a part of the boundary on which we apply a Dirichlet boundary condition, and $\Gamma_N = \partial\Omega \setminus \Gamma_D$:

$$\left[\begin{array}{rcl} -\Delta u &=& f \text{ in } \Omega, \\ u &=& g \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} &=& h \text{ on } \Gamma_N. \end{array}\right. \tag{2.11}$$

In that case the variational formulation will be

$$\int_\Omega \nabla \tilde{u} \cdot \nabla v - \int_{\Gamma_N} hv = \int_\Omega fv - \int_\Omega \nabla \tilde{g} \cdot \nabla v.$$

2.2 Stokes equations

In the case of Stokes equations, the unknown is the velocity-pressure couple $(\mathbf{u}, p)$. As we shall see in this section, the pressure field p can be interpreted as a Lagrange multiplier associated to the constraint $\nabla \cdot \mathbf{u} = 0$. We consider the Stokes problem in a bounded, regular domain Ω, with homogeneous Dirichlet boundary conditions (we take $\mu = 1$ for simplicity):

$$\left\{ \begin{array}{rcl} -\Delta \mathbf{u} + \nabla p & = & \mathbf{f} \text{ in } \Omega\,, \\ \nabla \cdot \mathbf{u} & = & 0 \text{ in } \Omega\,, \\ \mathbf{u} & = & 0 \text{ on } \Gamma = \partial\Omega\,. \end{array} \right. \tag{2.12}$$

As far as the velocity is concerned, this problem can be studied straightforwardly by writing the variational formulation with divergence free test functions (this approach is followed in MAP431 [1]). It writes

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int \mathbf{f} \cdot \mathbf{v},$$

which fits into the framework of Lax-Milgram (or Riesz) theorem, as the set

$$K = \{\mathbf{v} \in H^1(\Omega)^d\,,\ \nabla \cdot \mathbf{v} = 0\}$$

is a Hilbert space when endowed with the H^1 norm (and scalar product). It provides existence and uniqueness of a velocity field $\mathbf{u}$, which is a solution of the original problem in some sense. The pressure field can be recovered by use of the De Rham theorem (see again [1]), which asserts existence of a unique pressure field $p \in L^2(\Omega)$ with zero mean value such that

$$\int_\Omega p \nabla \cdot \mathbf{v} = \langle \psi\,, \mathbf{v} \rangle,$$

as soon as $\psi \in V'$ vanishes against any divergence free field. Taking

$$\langle \psi\,, \mathbf{v} \rangle = \int \mathbf{f} \cdot \mathbf{v} - \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}$$

in the De Rham theorem allows to obtain well-posedness in terms of pressure. We shall follow another approach here, based on a *saddle point* formulation of the problem, which will constitute the basis of mixed finite element method (see Chapter 3).

Firstly, note that uniqueness cannot be expected for the Stokes problem with homogeneous Dirichlet boundary conditions, as the pressure p is obviously defined up to a constant. We shall handle this indeterminacy by prescribing an extra constraint on the pressure, namely zero mean over the domain. The variational formulation of this problem is obtained as follows. Consider test functions $\mathbf{v}$ and q for velocity and pressure, respectively, with $\mathbf{v} = 0$ on Γ, we multiply the first equation by $\mathbf{v}$, the second one by q, and we integrate over the domain to obtain

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} & = & \displaystyle\int_\Omega \mathbf{f} \cdot \mathbf{v}\,, \\ \displaystyle\int_\Omega q \nabla \cdot \mathbf{u} & = & 0\,. \end{array} \right. \tag{2.13}$$

This leads to the following abstract formulation

$$\left\{ \begin{array}{rcll} a(u,v) + b(v,p) & = & \langle \varphi \, , v \rangle & \forall v \in V \, , \\ \\ b(u,q) & = & 0 & \forall q \in X \, . \end{array} \right. \tag{2.14}$$

Here $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear forms over $V \times V$ and $V \times X$, with $V = H^1(\Omega)^d$ and

$$X = L^2_0 = \left\{ q \in L^2(\Omega) \, , \ \int_\Omega q = 0 \right\} .$$

This abstract form can also be written in terms of operators. We introduce $A \in \mathcal{L}(V, V')$ and $B \in \mathcal{L}(V, X)$ defined by

$$\langle Av, w \rangle = a(v,w) \quad \forall w \in V \, , \ (Bv, q) = b(v,q) \quad \forall q \in X.$$

Eq.(2.14) can then be written

$$\left\{ \begin{array}{rcl} Au + B^* p & = & \varphi \, , \\ \\ Bu & = & 0 \, . \end{array} \right. \tag{2.15}$$

The well-posedness theorem for Stokes equations relies on a so called *saddle-point*[2] formulation of the following problem :

$$\left\{ \begin{array}{l} u \in K = \ker B, \\ J(u) = \inf\limits_{v \in K} J(v), \end{array} \right. \tag{2.16}$$

where J is defined by

$$J(v) = \frac{1}{2} a(v,v) - \langle \varphi \, , v \rangle.$$

Before detailing the abstract framework, let us give a overview of the situation: the minimization problem above is obviously well-posed under reasonable assumptions (in particular $a(\cdot,\cdot)$ is supposed to be coercive), it admits a unique minimizer u. Expressing the fact that u minimizes J over K, i.e. $J(u+v) \leq J(u)$ for any $v \in K$, one obtains that ∇J is in $K^\perp = \ker B^\perp$. In the finite dimensional setting (B can be seen as a matrix), $\ker B^\perp = \mathrm{R}(B^\star)$ implies existence of $p \in X$ such that $\nabla J + B^\star p = 0$, i.e. $Au + B^\star p = \varphi$, which provides existence of a solution to Problem (2.15). Uniqueness for the Lagrange multiplier p is obtained as soon as $B^\star$ is injective, i.e. B is surjective.

[2]This name is due to the fact that Problem (2.15) is equivalent to: $(\mathbf{u}, p)$ is a saddle-point for the function (called the *Lagrangian*)

$$L(v,q) = J(v) + (Bv, q) \, ,$$

which means that

$$L(u,q) \leq L(u,p) \leq L(v,p) \quad \forall v \in V \, , \ \forall q \in X.$$

When the dimension is infinite (which is the case for the Stokes problem), $\ker B^\perp = \mathrm{R}(B^\star)$ is replaced by the weaker property

$$(\ker B)^\perp = \overline{\mathrm{R}(B^\star)}.$$

As a consequence, if one assume that $B^\star$ has a closed range (or equivalently, B has a closed range), one obtains existence of a couple (u,p) solution to the saddle-point formulation (2.16), and uniqueness as soon as B is surjective.

In the following we shall consider the following set of assumptions and notations.

$$\left|\begin{array}{l} V \text{ and } X \text{ Hilbert spaces,} \\ a(\cdot,\cdot) \text{ bilinear continuous symmetric coercive on } V\times V\,,\ \varphi\in V' \\ B\in\mathcal{L}(V,X), \\ K=\ker B=\{v\in V\,,\ Bv=0\} \\ J(v)=\dfrac{1}{2}a(v,v)-\langle\varphi,v\rangle\,,\quad u=\arg\min\limits_K J, \end{array}\right\} \tag{2.17}$$

Let us express rigorously the informal considerations above.

Proposition 2 *Let (u,p) verify (2.15). Then u is a solution to the minimization problem (2.16)*

Proof : As $Au+B^\star p=\varphi$, we have $Au-\varphi\in\ker B^\perp = K^\perp$, i.e. $(\nabla J, v)=0$ for any $v\in K$. As J is a quadratic functional, it implies that u minimizes J over K.

Proposition 3 *If the range of B is closed, then Problem (2.15) admits a solution (u,p).*

Proof : This is a direct consequence of the fact that, as B has closed range, so does $B^\star$ (see Theorem **??**, page **??**). By Theorem **??**, $\ker B^\perp = \mathrm{R}(B^\star)$, which gives the existence of $p\in X$ such that $B^\star p=\varphi-Au$.

Proposition 4 *If B is surjective, then Problem (2.15) admits a unique solution (u,p).*

Proof : As $\ker B^\star = \mathrm{R}(B)^\perp=\{0\}$, the Lagrange multiplier of the previous proposition is unique.

Surjectivity of B can expressed with the so called inf-sup (or LBB) condition (see Prop. **??**, page **??**):

$$\inf_{q\in X}\sup_{v\in V}\frac{|(q,Bv)|}{|v|\,\|q\|}\geq\beta>0.$$

Back to Stokes equation. Stokes problem fits into the framework defined by (2.17), with

$$V=H^1(\Omega)^d\,,\ X=L^2_0=\left\{q\in L^2(\Omega)\,,\ \int_\Omega q=0\right\}\,,\ a(u,v)=\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}$$

and B is defined by $\mathbf{v} \longmapsto B\mathbf{v} = -\nabla \cdot \mathbf{v}$, $K = \ker B$. Note that well-posedness of the constrained minimization problem is straightforward, so that existence and uniqueness of $\mathbf{u}$, solution to the Stokes problem in some way, is not an issue. To obtain well-posedness for the mixed formulation, one needs to verify the main assumption of Prop. 4, i.e. that B maps $H_0^1(\Omega)^d$ onto $L_0^2(\Omega)$ in a surjective way.

Theorem 1 *Let Ω be a bounded, regular domain. The variational form of the Stokes problem 2.13 admits a unique solution $(\mathbf{u}, p)$.*

Proof : As already mentioned, the well-posedness of the constrained minimization problem is straightforward: there exists a unique velocity field $\mathbf{u}$ which minimizes

$$J(\mathbf{v}) = \frac{1}{2}\int_\Omega |\nabla \mathbf{v}|^2 - \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

To establish existence and uniqueness of a pressure field p, we use Prop. 4. To that purpose, we need to establish that B is surjective from $V = H_0^1(\Omega)^d$ onto $X = L_0^2(\Omega)$. Consider $q \in L_0^2(\Omega)$, and the Poisson problem:

$$-\Delta\psi = q,$$

with Neuman boundary conditions. As q has 0 mean value, it admits a solution ψ. As the domain is regular, this solution belongs to $H_0^1 \cap H^2$. The field $\mathbf{v} = \nabla\psi$ is such that $B\mathbf{v} = -\nabla \cdot \mathbf{v} = -\Delta\psi = q$, $\mathbf{v} \cdot \mathbf{n} = 0$, and $|\mathbf{v}|_1 \leq C\,|q|_0$. The proof is not completed, as $\mathbf{v}$ is not in H_0^1 (its tangential component on the boundary does not vanish). The last step of the construction, which we do not detail here, consists in introducing a divergence free field $\mathbf{w}$ wich identifies to $-\mathbf{v}$ on $\partial\Omega$ (which is purely tangential). We refer to [10] or [16] for details on this construction, and the fact that the H^1 semi-norm of $\mathbf{w}$ controlled by that of $\mathbf{v}$.

To recover Stokes equations in a strong sense (i.e. as identity between functions) , one needs to have some regularity.

Theorem 2 *Let Ω be a bounded domain. We assume that the boundary of Ω is C^2. Then the solution $(\mathbf{u}, p)$ to the variational Stokes problem belongs to $H^2(\Omega) \times H^1(\Omega)$.*

Proof : The regularity of solutions to the Stokes problem is a delicate issue, which we shall not address here in details (references). If we admit $H^2 \times H^1$ regularity for the velocity-pressure couple, Stokes equations in the form (2.12) can be recovered by a simple integration by parts.

2.3 Exercises

Exercise 5 *Prove that if H is a separable Hilbert space, then H possesses a hilbertian basis.*

Exercise 6 *Show that Lemma* **??** *is a consequence of Rellich's lemma.*

Chapter 3

Finite Element Approximation of Stokes equations

We now turn to the numerical approximation of the solutions of Stokes equations by the finite element method. This chapter is divided into four parts. We first recall the fundamental principles of the finite element method for the model problem, and then focus on the peculiarities of the method for Stokes equations. This leads to the so-called *mixed* finite element formulation for which we provide the reader with the principal results. We then apply the method to some standard flows, using the software FreeFem++, and end this chapter with computational considerations, like numerical methods for solving the underlying linear system.

3.1 The classical finite element method

The finite element method is a numerical method intended to solve partial differential equations. Among its main features are the fact that it has the flexibility of being easily adaptable on quite general domains, and that the unknowns are really functions defined on Ω_h the approximation of the domain Ω (in contrast for instance with the finite differences method where the solution only exists at discretization points).

To start with, let us consider the model problem

$$\left[\begin{array}{l} -\Delta u = f \text{ in } \Omega\,, \\ u = 0 \text{ on } \partial\Omega\,. \end{array} \right. \tag{3.1}$$

As we have recalled in Section 2.1, the variational formulation that is used to solve (3.1) reads

$$\text{Find } u \in V, \text{ such that } a(u,v) = \langle \varphi\,, v \rangle, \tag{3.2}$$

where the Hilbert space $V = H_0^1(\Omega)$, a is the continuous bilinear form on V

$$\begin{array}{rcl} a : V \times V & \to & \mathbb{R} \\ (u,v) & \mapsto & \displaystyle\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx \end{array}$$

and l is the linear continuous form defined by

$$\begin{array}{rcl} l : V & \to & \mathbb{R} \\ v & \mapsto & \displaystyle\int_\Omega f(x)v(x)\,dx \end{array}$$

A finite element formulation of the problem is obtained by finding a *finite dimensional* subspace $V_h \subset V$ and solve the *approximate* variational formulation

$$\text{Find } u_h \in V_h, \quad \text{such that } a(u_h, v_h) = \langle \varphi\,, v\rangle\,. \tag{3.3}$$

3.1.1 Céa's Lemma

Without further precision, one can estimate the error between u solution of (3.2) and u_h solution of (3.3) through the Céa's lemma

Lemma 1 *(Céa's lemma) There exists a constant C such that*

$$\|u - u_h\|_V \le C \inf_{v_h \in V_h} \|u - v_h\|_V\,. \tag{3.4}$$

Proof : Since a is V coercive, one has

$$\alpha\|u - u_h\|_V^2 \le a(u - u_h, u - u_h)\,.$$

Moreover, since u and u_h solve (3.2) and (3.3) respectively, one deduces

$$\forall v_h \in V_h,\ a(u - u_h, u_h - v_h) = \langle \varphi\,, u_h - v_h\rangle - \langle \varphi\,, u_h - v_h\rangle = 0\,. \tag{3.5}$$

Summing with the preceding equation, one obtains $\forall v_h \in V_h$

$$\begin{array}{rcl} \alpha\|u - u_h\|_V^2 & \le & a(u - u_h, u - u_h) + a(u - u_h, u_h - v_h) \\ & = & a(u - u_h, u - v_h) \\ & \le & M\|u - u_h\|_V\,\|u - v_h\|_V \end{array}$$

from which we deduce $\|u - u_h\|_V \le \dfrac{M}{\alpha}\|u - v_h\|_V$ for all $v_h \in V_h$ which gives (3.4).

What is remarkable in Céa's lemma is that it relates the *approximation* error (*i.e.* the error made by the method, the left hand side of (3.4)) to the *interpolation* error, *i.e.* the error made by the fact that V_h does not a priori contain the exact solution u. In this latter error, the problem that we aim at solving is not important, only the fact that V_h contains a map which is close to the exact solution.

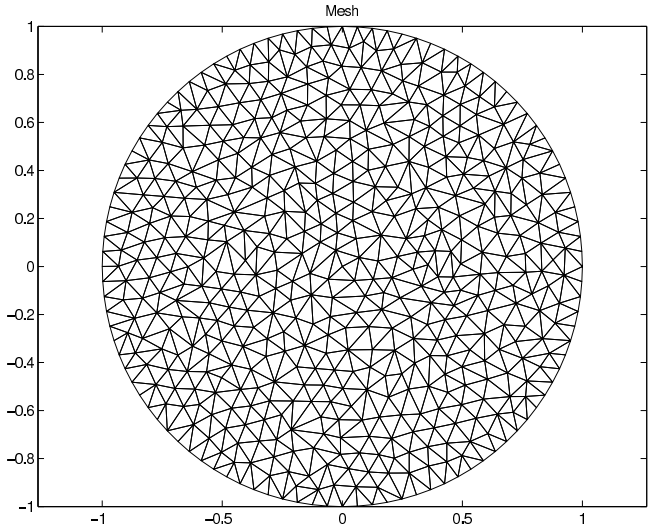

Figure 3.1: A conformal mesh of the unit disk in 2D.

3.1.2 The Lagrange finite element P^k

The finite element method is a canonical way of finding convergent approximations, which are furthermore practically tractable (*i.e.* with the help of a computer, one can compute the approximation u_h). This is classically done by considering a family of conformal meshes $\{\Omega_h\}_{h>0}$ of Ω, and building a family of finite dimensional spaces $\{V_h\}_{h>0}$. Although there are other possibilities[1], typical meshes are built with simplices, which are nothing but triangles in 2D or tetrahedra in 3D. Therefore the meshes we consider in this book are all triangular in 2D and tetrahedral in 3D. Both the families $\{\Omega_h\}_{h>0}$ and $\{V_h\}_{h>0}$ are parameterized by h which usually measures the coarseness of the mesh (h is typically the biggest edge-length in the mesh Ω_h. For all $k \geq 0$, the P^k approximation consists in taking

$$V_h = \left\{ u_h \in V \ \text{ such that } \forall K \in \Omega_h,\ u_h|_K \in P^k(K) \right\} \tag{3.6}$$

where P^k is the space of polynomials of degree less than or equal to k.

Calling N_h the dimension of V_h, it is very common to describe V_h by giving one of its basis $(\phi_k)_{1 \leq k \leq N_h}$. Although there are infinitely many ways to do this, there exists a canonical one which is given hereafter.

3.1.3 Degrees of freedom and basis functions

On a simplex K functions of $P^k(K)$ are obtained as interpolations of their values on a standard regular lattice of points defined on K. This lattice depends on the chosen degree k used for the P^k approximation, and is such that for any set of values, one for each point of the lattice, there exists a unique function $f \in P^k(K)$ which interpolates these values in K. As a consequence, the number of points of the lattice has to be equal to the dimension of the corresponding $P^k(K)$. The lattices used for the P^k, $k = 1, 2, 3$ approximation are

[1] See [1] for instance.

shown in Fig. 3.1.3 while their corresponding 3D versions are given in Fig. 3.1.3. We easily check that in 2D, we have

$$\begin{aligned} P^1(K) &= \text{Span}\{1, x, y\}, \\ P^2(K) &= \text{Span}\{1, x, y, x^2, xy, y^2\}, \\ P^3(K) &= \text{Span}\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}, \end{aligned}$$

which are of dimension 3, 6 and 10 respectively, exactly as the number of points of the corresponding lattice. Similarly, in 3D, one has

$$\begin{aligned} &P^1(K) = \text{Span}\{1, x, y, z\}, \\ &P^2(K) = \text{Span}\{1, x, y, z, x^2, y^2, z^2, xy, yz, xz\}, \\ &P^3(K) = \text{Span}\{1, x, y, z, x^2, y^2, z^2, xy, yz, xz, x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz\}, \end{aligned}$$

of dimensions 4, 10, 20 respectively.

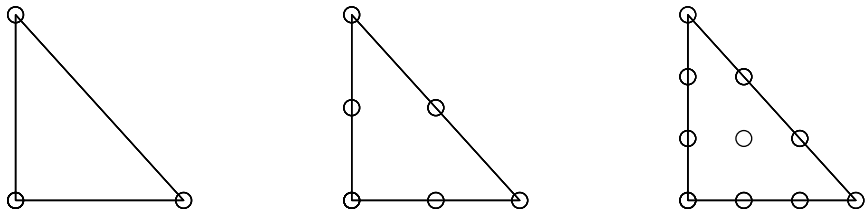

Figure 3.2: The lattices in 2D, respectively for the P^1, P^2, and P^3 approximation.

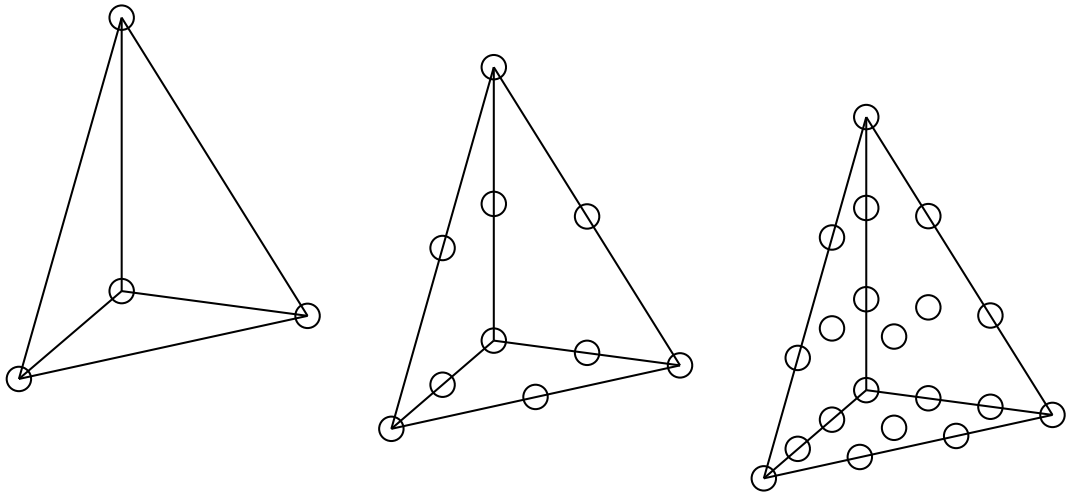

Figure 3.3: The lattices in 3D, respectively for the P^1, P^2, and P^3 approximation. Notice that the trace of the 3D $k-$lattice on each face of the tetrahedron is a 2D $k-$lattice.

Merging together all the lattice points of all the simplices of $\mathcal{T}_h$ gives a set of points called $(\mathbf{x}_k)_{1\leq k\leq N_h}$ in the sequel. Such typical sets, for P^1 and P^2 approximation are shown

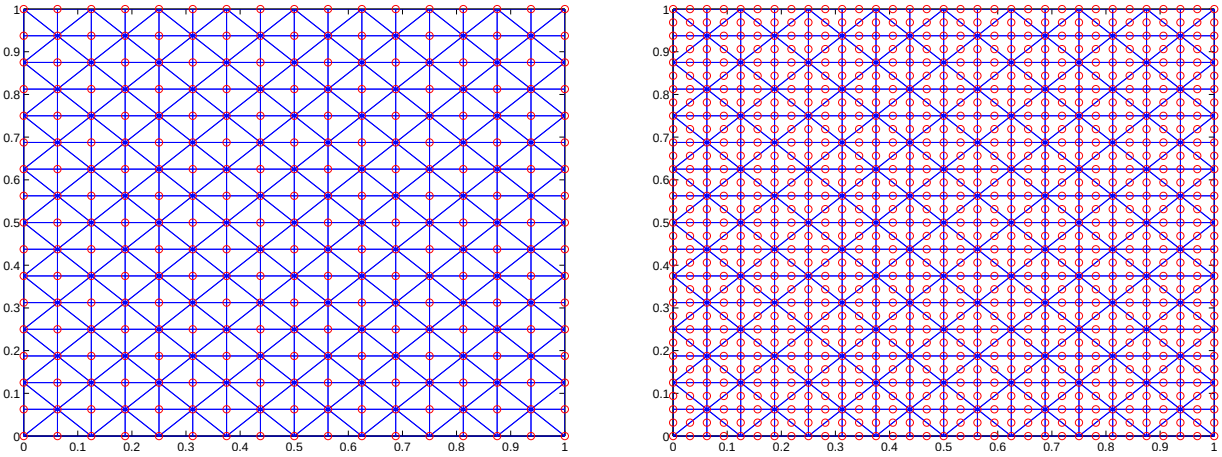

Figure 3.4: A triangular mesh of the unit square (in blue) with the P^1 (left) and p^2 (right) degrees of freedom (in red). There are many more degrees of freedom in P^2 than in P^1.

in Fig. 3.1.3. The value that a function of V_h takes at the points $(\mathbf{x}_k)_{1\leq k\leq N_h}$ are called the *degrees of freedom* meaning that for any possible values $(u_1, \cdots, u_{N_h}) \in \mathbb{R}^{N_h}$, there exists a unique function $u_h \in V_h$ such that

$$\forall 1 \leq k \leq N_h,\ u_h(\mathbf{x}_k) = u_k\,.$$

In order to solve in practice the approximate problem (3.3) within the P^k approximation defined by (3.6), a natural choice consists in taking for the basis functions $(\phi_k)_{1\leq k\leq N_h}$ the function of V_h which satisfies

$$\phi_k(x_l) = \delta_{kl}\,, \forall (k,l) \in \{1, \cdots, N_h\}^2\,,$$

where δ_{kl} stands for the Kronecker symbol. A typical basis function (also called *hat function* in this context) is shown for the P^1 approximation in Fig. 3.1.3.

3.1.4 Estimation of the approximation error

Cea's lemma 1 relates the error between computed and exact solutions to the approximation error

$$\inf_{v_h \in V_h} |v_h - u|\,.$$

We describe in this section how this approximation error can be estimated. This estimation is based on the explicit definition of an operator which maps linearly a smooth field onto an element of the finite element space, in a way that the difference between both can be evaluated according to various norms.

The approach we follow here can be decomposed onto three steps:

1. An general stability estimate for a general class of linear operators acting on spaces of functions defined over a simplex K. It can be seen as a generalization of the so called Poincaré-Wirtinger, which asserts that

$$|v - m(v)|_{0,K} \leq |v|_{1,K}\,.$$

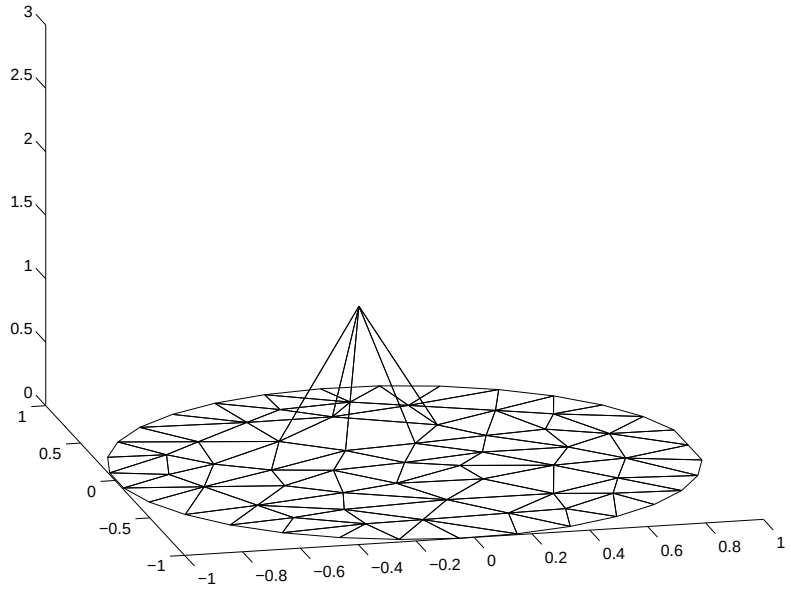

Figure 3.5: A basis function in P^1 approximation defined on the mesh of a unit disc.

The latter inequality expresses that, if one controls the norm of the gradient and *something* (here the mean value, but it could be anything) which handles the constant functions, one controls the L^2 norm. An extension can be expressed as follows: if one controls the H^2 semi norm, and *something* that controls the affine functions, one controls all norms weaker than H^2 (i.e. L^2 and H^1). The estimate (3.7) below generalizes this principle.

2. The second step is crucial, yet based on simple arguments : it consists in transposing the previous stability estimate onto simplices of any size and shape. In particular, we shall see that, if we consider simplices of size h, a factor h^β will appear in the estimate

3. The last step, which is elementary in the setting we consider, will consists in summing up all local errors (for simplices) to obtain an error overall the domain.

Step 1. (Stability estimate)

Proposition 5 *Let $K \subset \mathbf{R}^d$ be a non-degenerated simplex, $m \leq k \leq 1$, and I_K a linear operator from H^{k+1} onto H^m which preserves polynomials of degree $\leq k$.*

$$|v - I_K v|_{m,K} \leq C\,|v|_{k+1,K} \quad \forall v \in H^{k+1}(K) \tag{3.7}$$

Proof : We prove this property by contradiction: assume that there exists (v_n) such that

$$|v_n - I_K v_n|_{m,K} > nC\,|v_n|_{k+1,K}\,.$$

On choose v_n in $(P^k)^\perp$ (which is possible, as we can add any polynomial of degree k without changing the inequality), and such that the norm of v_n is 1 in H^{k+1}. This

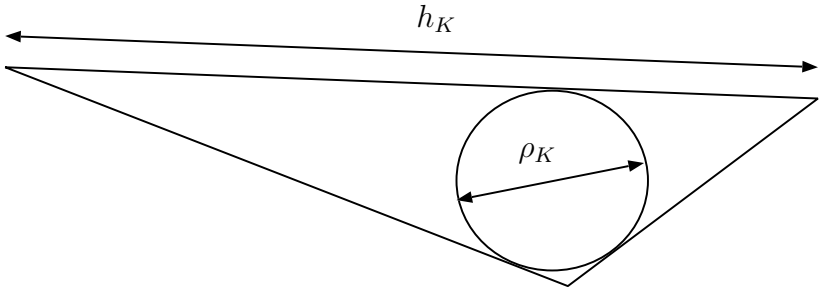

Figure 3.6: Definition of h and ρ for a triangle

sequence is bounded in H^{k+1}, so that one can extract a subsequence which converges weakly to $u \in H^{k+1}$. This subsequence (still denoted $(v_n()$ converges strongly in H^k (by compact injection of H^{k+1} onto H^k). As $|v_n|_{k+1,K}$ goes to 0, the Cauchy quantity $|v_p - v_q|_{k+1,K}$ also converges to 0, so that the sequence converges strongly in H^{k+1}. The limit u is orthogonal to P^k, and all its $k+1$ derivatives vanish: it is a polynomial of degree $\leq k$, so that $u = 0$, which is a contradiction, as u has norm 1 in H^{k+1}.

Step 2. (Change of variables) The constant C in the estimate (3.7) depends on simplex K. Let us consider this estimate applied to the reference simplex $\tilde{K}$, the constant is now universal (it only depends on the space dimension d and on parameter k and m). Considering now a general simplex $K \subset \mathbf{R}^d$, the idea is to map K onto $\tilde{K}$, use the stability estimate, and transport back the obtained estimate to K. Let us denote by Φ the affine mapping $\tilde{K} \longrightarrow K$. It can be checked that its jacobian (which is simply its linear part) verifies

$$\|\nabla\Phi\| \leq C h_K.$$

where h_K is the diameter of K (see Fig. 3.6). Indeed, $\|\nabla\Phi\|$ measures the maximal possible ratio between the length of a segment and its image by Φ. On the other way around, we have

$$\left\|\nabla\Phi^{-1}\right\| \leq C\frac{1}{\rho_K}.$$

Proposition 6 *Let $K \subset \mathbf{R}^d$ be a non-degenerated simplex, $m \leq k \leq 1$, and I_K a linear operator from H^{k+1} onto H^m which preserves polynomials of degree $\leq k$. There exists a constant (which depends on d, k, and m only) such that*

$$|v - I_K v|_{m,K} \leq C\frac{h_K^{k+1}}{\rho_K^m}|v|_{k+1,K} \quad \forall v \in H^{k+1}(K) \tag{3.8}$$

Proof : *We consider $v \in H^{k+1}(K)$, and we denote by $\tilde{v} \in H^{k+1}(\tilde{K})$ the transported function $v \circ \Phi$. As $|v - I_K v|_{m,K}$ involves m derivatives in space, we get*

$$|v - I_K v|_{m,K} \leq C\frac{1}{\rho_K^m}|\nabla\Phi|\,|\tilde{v} - I_{\tilde{K}}\tilde{v}|_{m,\tilde{K}}.$$

We apply then the estimate over $\tilde{K}$

$$|\tilde{v} - I_{\tilde{K}}\tilde{v}|_{m,\tilde{K}} \leq C\,|\tilde{v}|_{k+1,\tilde{K}}\,,$$

where C *is a universal constant, and we perform the reciprocal change of variable* $\tilde{x} \mapsto x = \Phi(\tilde{x})$*, which gives*

$$|\tilde{v}|_{k+1,\tilde{K}} \leq Ch_K^{k+1}\left|\nabla\Phi^{-1}\right|\,|v|_{k+1,K}$$

which yields (3.10). Note that tha constant C *might have changed in the process, but it still does not depend on* K*.*

Step 3. (Estimate over the whole domain) We consider now a polyhedral[2] domain, and a family (Ω_h) of triangulations. We shall assume that the family is *regular*, i.e. the aspect ratio of element is controlled: h_K/ρ_K is bounded uniformly with respect to h (i.e. the triangulation) and K (the individual simplex). We also consider that h denotes[3] the diameter of Ω_h We denote by I_h a global interpolation operator, i.e. I_h maps $H^{k+1}(\Omega)$ onto $H^m(\Omega)$, and its restriction to any element K preserves polynomials of degree $\leq k$.

Proposition 7 *Let* Ω *be a polyhedral domain, and* (Ω_h) *a regular family of triangulations, and* (I_h) *the corresponding family of interpolation operators. We have*

$$|v - I_h v|_{m,\Omega} \leq Ch^{k+1-m}\,|v|_{k+1,\Omega} \quad \forall v \in H^{k+1}(\Omega) \tag{3.9}$$

Proof : We simply decompose the squared quantity over elements of Ω_h:

$$\begin{aligned}|v - I_h v|^2_{m,\Omega} = \sum_K |v - I_h v|^2_{m,K} &\leq C\sum_K \frac{h_K^{2(k+1)}}{\rho_K^{2m}}\,|v|^2_{k+1,K} \leq C'\sum_K h_K^{2(k+1-m)}\,|v|^2_{k+1,K}\\ &\leq C'h^{2(k+1-m)}\sum_K |v|^2_{k+1,K} = C'h^{2(k+1-m)}\,|v|^2_{k+1,\Omega}\,.\end{aligned}$$

Application to first order (P^1) elements. The simplest (and most commonly used approach) to solve elliptic problems is based on piecewise affine functions. Given a polyhedral domain Ω and a mesh Ω_h, for any continuous function v over Ω, we denote by $I_h v$ the function which is affine on each triangle of Ω_h, and such that $v_h(x) = v(x)$ for any vertex x of Ω_h. As H^2 functions are continuous in the physical dimensions $d = 1$, 2 or 3, this operator maps continuously $H^2(\Omega)$ onto $H^1(\Omega)$, and the previous proposition takes the following form (with $k = m = 1$).

Proposition 8 *Let* Ω *be a polyhedral domain of* $\mathbf{R}^d$*,* $d = 1$*,* 2 *or* 3*,* (Ω_h) *a regular family of triangulations, and* I_h *the first order interpolation operator on* Ω_h*. We have*

$$|v - I_h v|_{1,\Omega} \leq Ch\,|v|_{2,\Omega} \quad \forall v \in H^2(\Omega) \tag{3.10}$$

[2]This approach can be extended to more general domain, but in the case of curved boundaries, the fact that Ω_h might be different from Ω calls for a special care of elements in contact with the boundary.

[3]This double role played by h (as *label* of the triangulation in the family, and as diameter of the corresponding triangulation) might seem confusing at first sight. Yet it is the standard convention in this context, and

3.2 Mixed finite element formulations

3.2.1 Finite element formulation of the Stokes problem

We consider the Stokes problem in a bounded domain Ω, with Dirichlet boundary conditions. The variational formulation writes

$$\left\{ \begin{array}{rcll} \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} & = & \displaystyle\int_\Omega \mathbf{f} \cdot \mathbf{v} & \forall \mathbf{v} \in V = H_0^1(\Omega)^d \\ \displaystyle\int_\Omega q \nabla \cdot \mathbf{u} & = & 0 & \forall q \in X = L_0^2(\Omega). \end{array} \right. \tag{3.11}$$

This variational formulation can be written in an abstract way:

$$\left\{ \begin{array}{rcrcll} a(u,v) & + & b(v,p) & = & \langle \varphi \, , v \rangle & \forall v \in V \\ b(u,\mu) & & & = & 0 & \forall \mu \in X. \end{array} \right. \tag{3.12}$$

One denotes by $B \in \mathcal{L}(V,X)$ the operator defined by $(Bv,\mu) = b(v,\mu)$. Note that $b(v,p)$ can also be written $\langle B^\star p \, , v \rangle$.

It is natural to introduce V_h and X_h, approximation spaces for V and X, respectively, and the associated discretized problem

$$\left\{ \begin{array}{rcrcll} a(u_h,v_h) & + & b(v_h,p_h) & = & \langle \varphi \, , v_h \rangle & \forall v_h \in V_h \\ b(u_h,\mu_h) & & & = & 0 & \forall \mu_h \in X_h. \end{array} \right. \tag{3.13}$$

Operator $B_h \in \mathcal{L}(V,X_h)$ is defined by

$$(B_h v, \mu_h) = (Bv, \mu_h) \quad \forall \mu_h \in X_h,$$

and the discrete constrained space is denoted by

$$K_h = V_h \cap \ker B_h = \{ v_h \, , \; b(v_h,\mu_h) = 0 \quad \forall \mu_h \in X \}$$

Problem 3.14 consists in minimizing

$$J(v) = \frac{1}{2} a(v,v) - \langle \varphi \, , v \rangle$$

over K_h. Considering finite dimensional approximation spaces V_h and X_h, K_h has finite dimension, therefore it is closed, so that Problem 3.14 is obviously well-posed, at least in terms of existence. Uniqueness holds for the primal variable u_h, but not necessarily for the Lagrange multiplier p_h.

The question is now: assuming that (V_h) and (X_h) are suitable sequences of approximation spaces, i.e. $\mathrm{dist}(v,V_h)$ and $\mathrm{dist}(q,X_h)$ both go to zero as the discretization parameter h goes to 0, can u_h be expected to converge to the solution u of the continuous constrained minimization problem ? As for the dual part p_h, in case it is uniquely defined, one may also wonder whether it converges to p.

Extension of the theory developed for standard (unconstrained) variational problems is in fact not straightforward. Informally, if X_h is much "richer" than V_h in terms of

approximation, the discrete problem might be overconstrained, so that K_h does not approximate V properly (it might even be reduced to $\{0\}$). On the other way, if X_h is too poor, the problem might be underconstrained, so that the solution u_h to the discrete constrained minimization problem might converge (if ever) outside K. Both approximations have obviously to be *balanced* in some way. This balance necessitates a fine tuning between the two discretization spaces, which will be expressed by the discrete inf-sup (or BBL) condition.

Remark 5 *In the case of Stokes problem, as the velocity is H^1 and the pressure L^2, it would be very natural to approximate velocities with continuous, piecewise P^1 functions (as for the poisson problem), while pressures would be approximated by P^0 functions. As we shall see, this straightforward approach does not work, for reasons that we shall detail below.*

3.2.2 A general estimate without inf-sup condition

To enlight how this inf-sup condition allows a proper error estimation of primal and dual quantities (i.e. $|u - u_h|$ and $|p - p_h|$), we shall start the numerical analysis by a very general estimation of the sole primal quantity, under very loose conditions (in particular without the inf-sup condition). Although this estimation will not be used as such in the context of divergence free constraint, it will prove powerful in other contexts (see Chapter **??**, dedicated to fictitious domain methods).

We assume here that $a(\cdot,\cdot)$ is a continuous, symmetric and coercive bilinear form over $V \times V$, and $B \in \mathcal{L}(V, X)$, so that $K = \ker B$ is closed. The problem which consists in minimizing

$$J(v) = \frac{1}{2}a(v,v) - \langle \varphi \, , v \rangle$$

over K is well-posed by Lax Milgram theorem, and we denote by u its solution. We do not make extra assumptions on B, so that the saddle-point problem may be ill-posed in terms of existence of a Lagrange multiplier. Concerning approximation spaces, we simply assume that $V_h \subset V$, but X_h is any finite dimensional space, and $B_h \in \mathcal{L}(V, X_h)$. Of course, it will be necessary to make extra assumptions on X_h and B_h to obtain useful estimates, but it is not necessary for the time being.

We consider the (abstract) discretized saddle-point problem:

$$\left\{ \begin{array}{rcccll} a(u_h, v_h) & + & (B_h v_h, p_h) & = & \langle \varphi \, , v_h \rangle & \forall v_h \in V_h \\ (B_h u_h, q_h) & & & = & 0 & \forall q_h \in X_h. \end{array} \right. \tag{3.14}$$

This problems consists in minimizing

$$J(v) = \frac{1}{2}a(v,v) - \langle \varphi \, , v \rangle$$

over $K_h = \ker B_h$. As X_h is finite dimensional, the saddle-point problem is well-posed in the following sense: it admits a solution (u_h, p_h), and u_h is uniquely defined as the minimizer of J over K_h.

Let us start by a very simple remark on the continuous problem: there exists a unique $\xi \in V'$ (more precisely in $K^\perp$) such that

$$Au + \xi = \varphi.$$

As u is uniquely defined, ξ is uniquely defined as $\varphi - Au$. The basis of the following estimate will be to prove that, even if ξ cannot be written $B^\star p$, some convergence of u_h toward u can be expected as soon as ξ can be approximated by $B_h^\star \mu_h$.

Proposition 9 *Let $a(\cdot,\cdot)$ be a continuous, symmetric, coercive, bilinear form over $V \times V$, $\varphi \in V'$, $B \in \mathcal{L}(V,X)$. We denote by u the unique minimizer of J over $K = \ker B$. Let $V_h \subset V$ and X_h be finite-dimensional spaces, $B_h \in \mathcal{L}(V,X_h)$, and $K_h = \ker B_h \cap V_h$. Let (u_h, p_h) be a solution of the saddle point-problem (3.14). Then*

$$|u - u_h| \leq C \left(\inf_{w_h \in K_h} |w_h - u| + \inf_{q_h \in X_h} \|\xi - B_h^\star q_h\|_{V_h'} \right) \tag{3.15}$$

Proof : For any $v_h \in K_h$, $(B_h v_h, p_h) = 0$, so that

$$a(u_h, v_h) = \langle \varphi \, , \, v_h \rangle.$$

The core of the proof consists in taking v_h in the form $u_h - w_h$, where w_h is in K_h. It comes

$$a(v_h, v_h) = \langle \varphi \, , \, v_h \rangle - a(w_h, v_h).$$

The exact solution u verifies

$$a(u, v_h) + \langle \xi \, , \, v_h \rangle = \langle \varphi \, , \, v_h \rangle,$$

so that

$$a(v_h, v_h) = a(u - w_h, v_h) + \langle \xi \, , \, v_h \rangle.$$

As v_h is in K_h, $\langle B_h^\star q_h \, , \, v_h \rangle = (B_h v_h, \mu_h) = 0$ for any $\mu_h \in X_h$, so that this quantity can be substracted to the right-hand side.

$$a(v_h, v_h) = a(u - w_h, v_h) + \langle \xi - B_h^\star \mu_h \, , \, v_h \rangle.$$

One obtains

$$\alpha \, |v_h|^2 \leq \|a\| \, |u - w_h| \, |v_h| + \|B_h^\star q_h - \xi\| \, |v_h| \Rightarrow \alpha \, |u_h - w_h| \leq \|a\| \, |u - w_h| + \|B_h^\star q_h - \xi\|$$

and therefore

$$|u_h - u| \leq C \left(|u - w_h| + \|B_h^\star q_h - \xi\|_{V_h'} \right).$$

for any $w_h \in K_h$, any $q_h \in X_h$, which ends the proof.

Estimate (3.15) expresses the required balance between both approximation spaces. Firstly, X_h has to be rich enough (and B_h has to approximate B in some sense) for the second term to be small. But if X_h is too rich, it may constrain excessively the problem, so that K_h may not be a good approximation space for K, preventing the first term to go to 0.

3.2.3 Estimates with the inf-sup condition

We may now introduce the framework which we will favor for Stokes equation. We consider here a saddle point formulation which is well-posed at the continuous level, i.e. B is surjective.

Proposition 10 *Notations and assumptions are those of Proposition 9. We furthermore assume here that $B \in \mathcal{L}(V, X)$ is surjective, so that the continuous saddle point problem admits a unique solution (u, p). We also assume that both approximations are conforming, i.e. $V_h \subset V$ and $X_h \subset X$, that B_h is defined by*

$$(B_h v, q_h) = (Bv, q_h) = b(v, q_h),$$

and that it verifies the discrete inf-sup condition:

$$\inf_{q_h \in X_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{|v_h|\,|q_h|} \geq \beta > 0,$$

where β is independent[4] *of h. Then we have the following error estimate*

$$|u - u_h| + |p - p_h| \leq C \left(\inf_{w_h \in V_h} |w_h - u| + \inf_{q_h \in X_h} |q_h - p| \right) \tag{3.16}$$

Proof : The proof is based on estimate (3.15). As preliminary step, let us first note that the infimum over X_h of $\|B_h^\star q_h - \xi\|_{V_h'}$ takes the form (as $\xi = B^\star p$)

$$\inf_{q_h \in X_h} |B^\star(q_h - p)| \leq \inf_{q_h \in X_h} \|B^\star\|\,|q_h - p|\,.$$

The first step of the proof, which is essential to use approximation properties of V_h, consists in showing that the first infimum in (3.15) can be replaced by an infimum over the unsconstrained approximation space V_h.

In a second step, we will show that the estimate on $|u - u_h|$ induces an estimate on $|p - p_h|$.

Step 1. As for the first term in estimate (3.15), we proceed as follows: consider an approximation $v_h \in V_h$ of u, we build an approximation $w_h = v_h + z_h \in K_h$ with the same approximation properties (up to a multiplicative constant which does not depend on h).

Let $v_h \in V_h$ be given. We denote by z_h the element of V_h which verifies

$$B_h z_h = -B_h v_h$$

and which minimizes the norm (i.e. $w_h = v_h + z_h$ is the projection of v_h onto K_h). This z_h is the primal part of the solution to the saddle point problem

$$\begin{aligned} (z_h, y_h) + (\eta_h, B_h y_h) &= 0 \quad \forall y_h \in V_h \\ (\mu_h, B_h z_h) &= -(\mu_h, B_h v_h) \quad \forall \mu_h \in X_h. \end{aligned}$$

[4] As always in such contexts, we implicitly consider sequences of approximation spaces (V_h) and (X_h), indexed by a parameter h (which will represent the mesh diameter in actual Finite Element discretization, which goes to 0 as the dimension of the spaces goes to infinity.

Thanks to the inf-sup condition, we have

$$|\eta_h| \leq \frac{1}{\beta} \sup_{y_h} \frac{\langle B_h^\star \eta_h \, , y_h \rangle}{|y_h|} = |z_h| \, .$$

On the other hand, taking $y_h = z_h$, it comes

$$|z_h|^2 \leq |(\eta_h, B_h z_h)| \leq |\eta_h| \, |B_h v_h| = |\eta_h| \, |B_h(u - v_h)| \leq C \, |z_h| \, |B_h(u - v_h)| \, .$$

The new approximant $w_h = v_h + z_h$ is then such that

$$|u - w_h| \leq |u - v_h| + |z_h| \leq C \, |u - v_h| \, .$$

Step 2. To estimate $|p - p_h|$, we substract the discrete variational formulation from the continuous one. We get

$$(p_h, Bv_h) = a(u - u_h, v_h) + (p, Bv_h),$$

so that, for any $q_h \in X_h$

$$(p_h - q_h, Bv_h) = a(u - u_h, v_h) + (p - q_h, Bv_h) \quad \forall q_h \in X_h.$$

By the inf-sup condition, we have

$$|p_h - q_h| \leq \frac{1}{\beta} \sup_{v_h \in V_h} \frac{|a(u - u_h, v_h) + (\lambda - \mu_h, Bv_h)|}{|v_h|}$$

$$\leq \frac{1}{\beta} \left(\|a\| \, |u - u_h| + \|B\| \, |p - q_h| \right) ,$$

so that, finally,

$$|p - p_h| \leq \frac{\|a\|}{\beta} \, |u - u_h| + \left(1 + \frac{\|B\|}{\beta} \right) \inf_{q_h \in X_h} |p - q_h| \, .$$

The discrete inf-sup condition, which is not easily verified in its native form, is equivalent to the existence of an operator from V to V_h which preserves the discrete constraint, with a bounded norm[5]. More precisely, we have

Proposition 11 *(Fortin's criterium)*
We assume that $B \in \mathcal{L}(V, X)$ verifies the inf-sup condition. Then the sequence (V_h, X_h) verifies the discrete inf-sup condition iff there exists a constant $C > 0$ and a family $(\Pi_h)_h$, with $\Pi_h \in \mathcal{L}(V, V_h)$, such that

$$b \left(\Pi_h v - v, q_h \right) = 0 \quad \forall (v, q_h) \in V \times X_h,$$

with

$$|\Pi_h v| \leq C \, |v| \, .$$

[5]As always, this assertion refers to a family (Π_h) of operators which is uniformly bounded with respect to h.

Proof : Assume that $b(\cdot,\cdot)$ verifies the inf-sup condition. For any $v \in V$, we build v_h which minimizes the norm among all those w_h that verify $Bw_h = Bv$ (as in the beginning of Proposition 10). The saddle point formulation of the problem writes

$$\begin{aligned}(v_h, w_h) + (p_h, Bw_h) &= 0 \quad \forall w_h \in V_h \\ (q_h, Bv_h) &= (q_h, Bv) \quad \forall q_h \in X_h.\end{aligned}$$

By the inf-sup condition, one has $|p_h| \leq |v_h| / \beta$, and by taking $w_h = v_h$ in the first line, we obtain $|v_h| \leq C\,|v|$. We define $\Pi_h v$ as v_h (orthogonal projection onto $B^\star(X_h)$.

On the other way, consider $q_h \in X_h \subset X$, denote by v its reciprocal image by B which minimizes the norm. This norm is controlled by that of q_h because the range of B is closed. We have

$$\sup_{v_h} \frac{b(v_h, q_h)}{|v_h|} \geq \frac{b(\Pi_h v, q_h)}{|\Pi_h v|} \geq \frac{1}{C}\,|q_h|\,,$$

which ends the proof.

3.3 Exercises

Exercise 7 *Show that on a conformal mesh Ω_h of a two-dimensional domain Ω the space*

$$P^1(\Omega_h) = \left\{ u_h \in H_0^1(\Omega, \mathbb{R}^2) \quad \textit{such that div } \mathbf{u}_h = 0 \right\}$$

is reduced to $\{0\}$.

FiXme: *Verifier et corriger les references* **FiXme** !

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