CONDUCTIVITY INTERFACE PROBLEMS. PART I: SMALL PERTURBATIONS OF AN INTERFACE

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Abstract. We derive high-order terms in the asymptotic expansions of the boundary perturbations of steady-state voltage potentials resulting from small perturbations of the shape of a conductivity inclusion with \(C^2\)-boundary. Our derivation is rigorous and based on layer potential techniques. The asymptotic expansion in this paper is valid for \(C^1\)-perturbations and inclusions with extreme conductivities. It extends those already derived for small volume conductivity inclusions and is expected to lead to very effective algorithms, aimed at determining certain properties of the shape of a conductivity inclusion based on boundary measurements.

1. Introduction

The main objective is to present a schematic way based on layer potential techniques to derive high-order terms in the asymptotic expansions of the boundary perturbations of steady-state voltage potentials resulting from small perturbations of the shape of a conductivity inclusion with \(C^2\)-boundary.

More precisely, consider a homogeneous conducting object occupying a bounded domain \(\Omega \subset \mathbb{R}^2\), with a connected \(C^2\)-boundary \(\partial \Omega\). We assume, for the sake of simplicity, that its conductivity is equal to 1. Let \(D\) with \(C^2\)-boundary be a conductivity inclusion inside \(\Omega\) of conductivity equal to some positive constant \(k \neq 1\). We assume that \(\text{dist}(D, \partial \Omega) \geq C > 0\).

The voltage potential in the presence of the inclusion \(D\) is denoted by \(u\). It is the solution to

\[
\begin{align*}
\nabla \cdot \left( 1 + (k - 1)\chi_D \right) \nabla u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n}\bigg|_{\partial \Omega} &= g, \quad \int_{\partial \Omega} u = 0,
\end{align*}
\]

where \(\chi_D\) is the indicator function of \(D\). Here \(\nu\) denotes the unit outward normal to the domain \(\Omega\) and \(g\) represents the applied boundary current; it belongs to the set \(L^2_0(\partial \Omega) = \{f \in L^2(\partial \Omega), \int_{\partial \Omega} f = 0\}\).

Let \(D\) be an \(\epsilon\)-perturbation of \(D\), \(i.e., \) let \(h \in C^1(\partial D)\) and \(\partial D\) be given by

\(\partial D_\epsilon = \{ \tilde{x} : \tilde{x} = x + \epsilon h(x)\nu(x), \ x \in \partial D \}\).

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Let $u_\epsilon$ be the solution to

\[
\begin{cases}
\nabla \cdot \left( 1 + (k - 1)\chi_D \right) \nabla u_\epsilon = 0 \quad \text{in } \Omega, \\
\partial u_\epsilon / \partial \nu \big|_{\partial \Omega} = g, \int_{\partial \Omega} u_\epsilon = 0.
\end{cases}
\]

The main achievement of this paper is a rigorous derivation, based on layer potential techniques, of high-order terms in the asymptotic expansion of $(u_\epsilon - u)|_{\partial \Omega}$ as $\epsilon \to 0$.

The solution $u_\epsilon$ to (1.2) can be represented using integral operators (see formula (3.1)), and hence derivation of asymptotic formula for $u_\epsilon$ is reduced to that of the integral operator $K^*_D\epsilon$ defined by

\[
K^*_D\epsilon \varphi(\tilde{x}) = \frac{1}{2\pi} \mathrm{p.v.} \int_{\partial D_\epsilon} \frac{\langle \tilde{x} - \tilde{y}, \tilde{\nu}(\tilde{x}) \rangle}{|\tilde{x} - \tilde{y}|^2} \varphi(\tilde{y}) d\sigma(\tilde{y}),
\]

where p.v. stands for the Cauchy principal value. The operator $K^*_D\epsilon$ is a singular integral operator and known to be bounded on $L^2(\partial D_\epsilon)$ [8, 7]. It was proved in [9] that $K^*_D\epsilon$ converges to the operator $K^*_D\varphi$ on the non-perturbed domain $D$, defined for a density $\varphi \in L^2(\partial D)$, by

\[
K^*_D\varphi(x) = \frac{1}{2\pi} \int_{\partial D} \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2} \varphi(y) d\sigma(y).
\]

In this paper we will derive a complete asymptotic expansion of the singular integral operator $K^*_D\epsilon$ on $L^2(\partial D_\epsilon)$ in terms of $\epsilon$. This asymptotic expansion yields an expansion of $u_\epsilon - u$ which extends those already derived for small volume inclusions [1, 2, 3, 4, 5, 10, 16, 19]. Our formula is of significant interest from an “imaging point of view”. For instance, if one has a very detailed knowledge of the “boundary signatures” of conductivity inclusions, then it becomes possible to design very effective algorithms to identify certain properties of their shapes. Since it carries very precise information on the shape of the inclusion, we will show how it can be efficiently exploited for designing significantly better algorithms. In connection with this, we refer to [6, 13, 17].

2. High-order terms in the expansion of $K^*_D\epsilon$

Let $a, b \in \mathbb{R}$, with $a < b$, and let $X(t) : [a, b] \to \mathbb{R}^2$ be the arclength parametrization of $\partial D$, namely, $X$ is a $C^2$-function satisfying $|X'(t)| = 1$ for all $t \in [a, b]$ and

\[
\partial D := \{ x = X(t), t \in [a, b] \}.
\]

Then the outward unit normal to $\partial D$, $\nu(x)$, is given by $\nu(x) = R_{-\pi/2} X'(t)$, where $R_{-\pi/2}$ is the rotation by $-\pi/2$, the tangential vector at $x$, $T(x) = X'(t)$, and $X'(t) \perp X''(t)$. Set the curvature $\tau(x)$ to be defined by

\[
X''(t) = \tau(x)\nu(x).
\]

We will sometimes use $h(t)$ for $h(X(t))$ and $h'(t)$ for the tangential derivative of $h(x)$. 


Then, \( \tilde{X}(t) = X(t) + \epsilon h(t) \nu(x) = X(t) + \epsilon h(t) R_{-\frac{\pi}{2}} X'(t) \) is a parametrization of \( \partial D_\epsilon \). By \( \tilde{\nu}(\tilde{x}) \), we denote the outward unit normal to \( \partial D_\epsilon \) at \( \tilde{x} \). Then, we have

\[
\tilde{\nu}(\tilde{x}) = \frac{R_{-\frac{\pi}{2}} \tilde{X}'(t)}{|\tilde{X}'(t)|} = \frac{(1 - \epsilon h(t) \tau(x)) \nu(x) - \epsilon h'(t) X'(t)}{\sqrt{\epsilon^2 h'(t)^2 + (1 - \epsilon h(t) \tau(x))^2}} \cdot (1)
\]

and hence \( \tilde{\nu}(\tilde{x}) \) can be expanded uniformly as

\[
\tilde{\nu}(\tilde{x}) = \sum_{n=0}^{\infty} \epsilon^n \nu^{(n)}(x), \quad x \in \partial D,
\]

where the vector-valued functions \( \nu^{(n)} \) are bounded. In particular, the first two terms are given by

\[
\nu^{(0)}(x) = \nu(x), \quad \nu^{(1)}(x) = -h'(t) T(x).
\]

Likewise, we get a uniformly convergent expansion for the length element \( d\sigma_\epsilon(\tilde{y}) \):

\[
d\sigma_\epsilon(\tilde{y}) = |\tilde{X}'(s)| ds = \sqrt{(1 - \epsilon \tau(s) h(s))^2 + \epsilon^2 h'^2(s)} ds = \sum_{n=0}^{\infty} \epsilon^n \sigma^{(n)}(y) d\sigma(y),
\]

where \( \sigma^{(n)} \) are bounded functions and

\[
\sigma^{(0)}(y) = 1, \quad \sigma^{(1)}(y) = -\tau(y) h(y).
\]

Set

\[
x = X(t), \quad \tilde{x} = \tilde{X}(t) = x + \epsilon h(t) R_{-\frac{\pi}{2}} X'(t),
\]

\[
y = X(s), \quad \tilde{y} = \tilde{X}(s) = y + \epsilon h(s) R_{-\frac{\pi}{2}} X'(s).
\]

Since

\[
|\tilde{x} - \tilde{y}|^2 = |x - y|^2 + 2 \epsilon \langle x - y, h(t) \nu(x) - h(s) \nu(y) \rangle + \epsilon^2 |h(t) \nu(x) - h(s) \nu(y)|^2,
\]

we get

\[
\frac{1}{|\tilde{x} - \tilde{y}|^2} = \frac{1}{|x - y|^2} \frac{1}{1 + 2 \epsilon F(x, y) + \epsilon^2 G(x, y)},
\]

where

\[
F(x, y) = \frac{\langle x - y, h(t) \nu(x) - h(s) \nu(y) \rangle}{|x - y|^2},
\]

and

\[
G(x, y) = \frac{|h(t) \nu(x) - h(s) \nu(y)|^2}{|x - y|^2}.
\]
One can easily see that
\[ |F(x,y)| + |G(x,y)| \leq C\|x\|c_2\|h\|c_1. \]

It follows from (2.1), (2.3), (2.5), and (2.6) that
\[
\frac{\langle \hat{x} - \hat{y}, \nu(\hat{x}) \rangle}{|\hat{x} - \hat{y}|^2} d\sigma_\epsilon(\hat{y}) = \left( \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2} + \epsilon \left[ \frac{\langle h(t)\nu(x) - h(s)\nu(y), \nu(x) \rangle}{|x - y|^2} - \frac{\langle x - y, \tau(x)h(t)\nu(x) + h'(t)T(x) \rangle}{|x - y|^2} \right] \right.
\]
\[ - \epsilon^2 \left( \frac{h(t)\nu(x) - h(s)\nu(y), \tau(x)h(t)\nu(x) + h'(t)T(x) \rangle}{|x - y|^2} \right) \times \frac{1}{1 + 2\epsilon F(x,y) + \epsilon^2 G(x,y)} \frac{\sqrt{(1 - \epsilon\tau(y)h(s))^2 + \epsilon^2 h^2(s)}}{\sqrt{(1 - \epsilon\tau(x)h(t))^2 + \epsilon^2 h^2(t)}} d\sigma(y)
\]
\[ = \left( K_0(x,y) + \epsilon K_1(x,y) + \epsilon^2 K_2(x,y) \right) \times \frac{1}{1 + 2\epsilon F(x,y) + \epsilon^2 G(x,y)} \frac{\sqrt{(1 - \epsilon\tau(y)h(s))^2 + \epsilon^2 h^2(s)}}{\sqrt{(1 - \epsilon\tau(x)h(t))^2 + \epsilon^2 h^2(t)}} d\sigma(y). \]

Let
\[
\frac{1}{1 + 2\epsilon F(x,y) + \epsilon^2 G(x,y)} \frac{\sqrt{(1 - \epsilon\tau(y)h(s))^2 + \epsilon^2 h^2(s)}}{\sqrt{(1 - \epsilon\tau(x)h(t))^2 + \epsilon^2 h^2(t)}} \sum_{n=0}^{\infty} \epsilon^n F_n(x,y),
\]
where the series converges absolutely and uniformly. In particular, we can easily see that
\[ F_0(x,y) = 1, \quad F_1(x,y) = -2F(x,y) + \tau(x)h(x) - \tau(y)h(y). \]

Then we now have
\[
\frac{\langle \hat{x} - \hat{y}, \nu(\hat{x}) \rangle}{|\hat{x} - \hat{y}|^2} d\sigma_\epsilon(\hat{y}) = \left( \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2} d\sigma(y) + \epsilon \left( K_0(x,y)F_1(x,y) + K_1(x,y) \right) d\sigma(y) \right.
\]
\[ + \epsilon^2 \sum_{n=0}^{\infty} \epsilon^n \left( F_{n+2}(x,y)K_0(x,y) + F_{n+1}(x,y)K_1(x,y) + F_n(x,y)K_2(x,y) \right) d\sigma(y). \]

Therefore, we obtain that
\[
\frac{\langle \hat{x} - \hat{y}, \nu(\hat{x}) \rangle}{|\hat{x} - \hat{y}|^2} d\sigma_\epsilon(\hat{y}) = \sum_{n=0}^{\infty} \epsilon^n k_n(x,y) d\sigma(y),
\]
where
\[ k_0(x,y) = \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2}, \quad k_1(x,y) = K_0(x,y)F_1(x,y) + K_1(x,y), \]
and for any \( n \geq 2, \)
\[ k_n(x,y) = F_n(x,y)K_0(x,y) + F_{n-1}(x,y)K_1(x,y) + F_{n-2}(x,y)K_2(x,y). \]

Introduce a sequence of integral operators \( K_D^{(n)} \) defined for any \( \phi \in L^2(\partial D) \) by:
\[ K_D^{(n)} \phi(x) = \int_{\partial D} k_n(x,y) \phi(y) d\sigma(y) \quad \text{for } n \geq 0. \]

Note that \( K_D^{(0)} = K_D \). Observe that the same operator with the kernel \( k_n(x,y) \) replaced with \( K_j(x,y), j = 0, 1, 2, \) is bounded on \( L^2(\partial D) \). In fact, it is an immediate consequence of
the celebrated theorem of Coifman-McIntosh-Meyer [7]. Therefore each $K_D^{(n)}$ is bounded on $L^2(\partial D)$.

Let $\Psi_\epsilon$ be the diffeomorphism from $\partial D$ onto $\partial D_\epsilon$ given by $\Psi_\epsilon(x) = x + \epsilon h(t)\nu(x)$, where $x = X(t)$. The following theorem holds.

**Theorem 2.1.** Let $N \in \mathbb{N}$. There exists $C$ depending only on $N$, $\|X\|_{C^1}$, and $\|h\|_{C^1}$ such that for any $\tilde{\phi} \in L^2(\partial D_\epsilon)$,

\[
\|(K_{D,\epsilon}^* \tilde{\phi}) \circ \Psi_\epsilon - K_D^* \phi - \sum_{n=1}^N \epsilon^n K_D^{(n)} \phi\|_{L^2(\partial D)} \leq C\epsilon^{N+1} \|\phi\|_{L^2(\partial D)},
\]

where $\phi := \tilde{\phi} \circ \Psi_\epsilon$.

### 3. Derivation of the Full Asymptotic Formula for the Steady-State Voltage Potentials

In this section we derive high-order terms in the asymptotic expansion of $(u_\epsilon - u)|_{\partial \Omega}$ as $\epsilon \to 0$.

Suppose that the conductivity of $D$ is $k$. Let $\lambda := \frac{k+1}{2(k-1)}$. Define the background voltage potential, $U$, to be the unique solution to

\[
\begin{cases}
\Delta U = 0 & \text{in } \Omega, \\
\frac{\partial U}{\partial \nu}|_{\partial \Omega} = g, \int_{\partial \Omega} U = 0.
\end{cases}
\]

Let $N(x, y)$ be the Neumann function for $\Delta$ in $\Omega$ corresponding to a Dirac mass at $y$, that is, $N$ is the solution to

\[
\begin{cases}
\Delta_x N(x, y) = -\delta_y & \text{in } \Omega, \\
\frac{\partial N}{\partial \nu}|_{\partial \Omega} = -\frac{1}{|\partial \Omega|}, \\
\int_{\partial \Omega} N(x, y) d\sigma(x) = 0 & \text{for } y \in \Omega.
\end{cases}
\]

Let $N_D$ be defined by

\[
N_D \varphi(x) := \int_{\partial D} N(x, y) \varphi(y) d\sigma(y), \quad x \in \partial \Omega,
\]

for $\varphi \in L^2_0(\partial D)$.

Let $u_\epsilon$ be the solution to (1.2). Then the following representation formula holds [1]:

\[
(3.1) \quad u_\epsilon(x) = U(x) - N_D \tilde{\phi}_\epsilon(x), \quad x \in \partial \Omega
\]

where $\tilde{\phi}_\epsilon \in L^2_0(\partial D_\epsilon)$ is given by

\[
(3.2) \quad \tilde{H}_\epsilon(x) := D_{\Omega}(u_\epsilon|_{\partial \Omega})(x) - S_{\Omega} g(x), \quad x \in \Omega,
\]

and

\[
(\lambda - K_{D,\epsilon}^*) \tilde{\phi}_\epsilon(x) = \frac{\partial H_\epsilon}{\partial \nu}(x), \quad x \in \partial D_\epsilon.
\]
Here and throughout this paper $\mathcal{S}_{\Omega}$ and $\mathcal{D}_{\Omega}$ denote the single and double layer potential on $\partial \Omega$:

\[
\mathcal{D}_{\Omega} \varphi(x) = \frac{1}{2\pi} \int_{\partial \Omega} \frac{\langle y - x, \nu(y) \rangle}{|x - y|^2} \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial \Omega,
\]

\[
\mathcal{S}_{\Omega} \varphi(x) = \frac{1}{2\pi} \int_{\partial \Omega} \log |x - y| \varphi(y) d\sigma(y), \quad x \in \mathbb{R}^2.
\]

Likewise the solution $u$ to (1.1) has the representation

\[
u(x) = \mathcal{D}_{\Omega}(u|_{\partial \Omega})(x) - \mathcal{S}_{\Omega} g(x), \quad x \in \Omega,
\]

and

\[
(\lambda I - \mathcal{K}_D) \psi(x) = \frac{\partial H}{\partial \nu}(x), \quad x \in \partial \Omega.
\]

We then get

\[
u(x) - u(x) = -\mathcal{N}_D \tilde{\psi}_\epsilon(x) + \mathcal{N}_D \psi(x), \quad x \in \partial \Omega.
\]

We now investigate the asymptotic behavior of $\mathcal{N}_D \tilde{\psi}_\epsilon$ as $\epsilon \to 0$. After the change of variables $\tilde{y} = \Psi(x, y)$, we get from (2.3) and the Taylor expansion of $N(x, y)$ that

\[
\mathcal{N}_D \tilde{\psi}_\epsilon(x) = \int_{\partial D} \left( \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{e^n}{\alpha!} (h(y) \nu(y))^\alpha \partial^\alpha_x N(x, y) \right) \tilde{\psi}_\epsilon(x) \left( \sum_{n=0}^{\infty} e^n \sigma^{(n)}(y) \right) d\sigma(y),
\]

where $\phi_\epsilon = \tilde{\psi}_\epsilon \circ \Psi$. One can see from Theorem 2.1 that, for each integer $N$, $\phi_\epsilon$ satisfies

\[
\left( \lambda I - \mathcal{K}_D - \sum_{n=1}^{N} e^n \mathcal{K}^{(n)}_D \right) \phi_\epsilon + O(\epsilon^{N+1}) = (\nabla H_\epsilon)(\Psi_\epsilon) \cdot \nu(\Psi_\epsilon) \quad \text{on } \partial D.
\]

We obtain from (2.2) that

\[
(\nabla H_\epsilon)(\Psi_\epsilon)(y) \cdot \nu(\Psi_\epsilon)(y) = \left( \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{e^n}{\alpha!} (\nabla \partial^\alpha_x H_\epsilon(y) (h(y) \nu(y))^\alpha \right) \cdot \left( \sum_{n=0}^{\infty} e^n \nu^{(n)}(y) \right)
\]

\[
\sum_{n=0}^{\infty} e^n G_n(y).
\]

Note that

\[
G_0(y) = \frac{\partial H_\epsilon}{\partial \nu}(y), \quad G_1(y) = h(y) (D^2 H_\epsilon(y) \nu(y), \nu(y)) - h'(y) (\nabla H_\epsilon, T(y)),
\]

where $D^2 H_\epsilon$ is the Hessian of $H_\epsilon$. Therefore, we obtain the following integral equation to solve:

\[
\left( \lambda I - \mathcal{K}_D - \sum_{n=1}^{N} e^n \mathcal{K}^{(n)}_D \right) \phi_\epsilon + O(\epsilon^{N+1}) = \sum_{n=0}^{\infty} e^n G_n \quad \text{on } \partial D.
\]

The equation (3.8) can be solved recursively in the following way: Define

\[
\phi^{(0)} = (\lambda I - \mathcal{K}_D)^{-1} G_0 = (\lambda I - \mathcal{K}_D)^{-1} \left( \frac{\partial H_\epsilon}{\partial \nu} \right|_{\partial D},
\]
and for $1 \leq n \leq N$,

$$
(3.10) \quad \phi^{(n)} = (\lambda I - \mathcal{K}_D^+)^{-1} \left( G_n + \sum_{p=0}^{n-1} \mathcal{K}_D^{(n-p)} \phi^{(p)} \right).
$$

We obtain the following lemma.

**Lemma 3.1.** Let $N \in \mathbb{N}$. There exists $C$ depending only on $N$, the $C^2$-norm of $X$, and the $C^1$-norm of $h$ such that

$$
\| \phi_I - \sum_{n=0}^{N} \epsilon^n \phi^{(n)} \|_{L^2(\partial D)} \leq C \epsilon^{N+1},
$$

where $\phi^{(n)}$ are defined by the recursive relation (3.10).

Define, for $n \in \mathbb{N}$ and for $x \in \partial \Omega$,

$$
(3.11) \quad v_n(x) := \sum_{i+j+k=n} \int_{\partial D} \left( \sum_{|\alpha|=1} \frac{1}{\alpha!} (h(y)\nu(y))^\alpha \partial_y^\alpha \mathcal{N}(x, y) \right) \sigma^{(j)}(y) \phi^{(k)}(y) d\sigma(y).
$$

It then follows from (3.7) and (3.11) that

$$
\mathcal{N}_D \phi_I(x) = \mathcal{N}_D (\lambda I - \mathcal{K}_D^+)^{-1} \left( \frac{\partial H_I}{\partial \nu} |_{\partial D} \right) + \sum_{n=1}^{N} \epsilon^n v_n(x) + O(\epsilon^{N+1}).
$$

Hence we get from (3.5) that

$$
(3.12) \quad u_\epsilon(x) - u(x) = -\mathcal{N}_D (\lambda I - \mathcal{K}_D^+)^{-1} \left( \frac{\partial H_I}{\partial \nu} |_{\partial D} - \frac{\partial H}{\partial \nu} |_{\partial D} \right) - \sum_{n=1}^{N} \epsilon^n v_n(x) + O(\epsilon^{N+1}), \quad x \in \partial \Omega.
$$

Observe from (3.2) and (3.3) that

$$
(3.13) \quad (I + \mathcal{E})(u_\epsilon - u)(x) = -\sum_{n=1}^{N} \epsilon^n v_n(x) + O(\epsilon^{N+1}), \quad x \in \partial \Omega.
$$

We need the following lemma.

**Lemma 3.2.** The operator $I + \mathcal{E}$ is invertible on $L^2_0(\partial \Omega)$.

Let us continue derivation of asymptotic expansion of $(u_\epsilon - u)|_{\partial \Omega}$ leaving the proof of Lemma 3.2 at the end of this section.

We get from (3.13) that

$$
(3.14) \quad u_\epsilon(x) - u(x) = -\sum_{n=1}^{N} \epsilon^n (I + \mathcal{E})^{-1}(v_n)(x) + O(\epsilon^{N+1}), \quad x \in \partial \Omega.
$$

Observe that the function $v_n$ is still depending on $\epsilon$ since $G_n$ in (3.6) is defined by $H_I$ and hence $\phi^{(n)}$ depends on $\epsilon$. We can remove this dependence on $\epsilon$ from the asymptotic formula in an iterative way.
Observe from (3.14) that
\[(u_\epsilon - u)|_{\partial \Omega} = O(\epsilon),\]
and hence, by (3.12),
\[H_\epsilon(x) - H(x) = O(\epsilon).\]
Thus if we define \(G_{\epsilon}^n, n \in \mathbb{N},\) by (3.6) with \(H_\epsilon\) replaced with \(H,\) and define \(\phi_1^{(n)}\) and \(v_1^n\) by (3.9), (3.10), and (3.11), then \(v_n - v_1^n = O(\epsilon)\). Therefore we get
\[(u_\epsilon - u)|_{\partial \Omega} = \epsilon^1(I + E)^{-1}(v_1^1)(x) + O(\epsilon^2), \quad x \in \partial \Omega.\]
Repeat the same procedure with \(H - \epsilon D\Omega(I + E)^{-1}(v_1^n)\) instead of \(H\) to get \(v_2^n\). Then \(v_n - v_2^n = O(\epsilon^2)\) and hence
\[u_\epsilon(x) - u(x) = -\sum_{n=1}^{\infty} \epsilon^n(I + E)^{-1}(v_2^n)(x) + O(\epsilon^{N+1}), \quad x \in \partial \Omega.\]
Repeating the same procedure until we get \(v_N^n\), and we obtain the following theorem.

**Theorem 3.3.** Let \(v_N^n, n = 1, \ldots, N,\) be the functions obtained by the above procedure. Then the following formula holds uniformly for \(x \in \partial \Omega:\)
\[u_\epsilon(x) - u(x) = -\sum_{n=1}^{N} \epsilon^n(I + E)^{-1}(v_N^n)(x) + O(\epsilon^{N+1}).\]
The remainder \(O(\epsilon^{N+1})\) depends only on \(N, \Omega,\) the \(C^2\)-norm of \(X,\) the \(C^1\)-norm of \(h,\) and \(\text{dist}(D, \partial \Omega).\)

Let us compute the first order approximation of \((u_\epsilon - u)|_{\partial \Omega}\) explicitly. Note that \(\phi_1^{(0)} = \phi\) where \(\phi\) is defined by (3.4), and
\[\phi_1^{(1)} = (\lambda I - K_D)\left(h\langle D^2 H \nu, \nu \rangle - h'\langle \nabla H, T \rangle + K^{(1)}_D \phi \right).\]
Therefore, by (2.4) and (3.11), \(v_1^1\) takes the form
\[v_1^1(x) = \int_{\partial D} \nabla_x N(x, y) \cdot \nu(y)h(y)\phi(y)d\sigma(y) - \int_{\partial D} N(x, y)\tau(y)h(y)\phi(y)d\sigma(y) + \int_{\partial D} N(x, y)\phi_1^{(1)}(y)d\sigma(y).\]
Using this formula and (3.15) we find the first-order term in the asymptotic expansion of \(u_\epsilon - u\) on \(\partial \Omega.\)

The first term in the asymptotic expansions is exactly the domain derivative of the solution derived in [12, Theorem 1]. To see that, it suffices to prove that
\[(I + E)w = -\frac{1}{k-1}v_1^1 \quad \text{on } \partial \Omega.\]
where \(w\) is the solution of
\[
\begin{aligned}
\Delta w &= 0 \quad \text{in } (\Omega \setminus \mathcal{D}) \cup D, \\
(w|_+ - w|_- &= -h \frac{\partial u}{\partial \nu} |_-) \quad \text{on } \partial D, \\
\frac{\partial w}{\partial \nu} |_+ - k \frac{\partial w}{\partial \nu} |_- &= - \frac{\partial}{\partial T} (h \frac{\partial u}{\partial T}) \quad \text{on } \partial D, \\
\frac{\partial w}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

It is easy to see that
\[
w = \mathcal{D}_\Omega (w|_\partial \Omega) + \mathcal{D}_D (h \frac{\partial u}{\partial \nu} |_-) + S_D \theta, \quad x \in \Omega,
\]
where the density \(\theta\) on \(\partial D\) is given by
\[
\theta = (\lambda I - \mathcal{K}_D^*)^{-1} \left[ - \frac{1}{k-1} \left( \frac{\partial}{\partial T} h \frac{\partial u}{\partial T} + \frac{\partial}{\partial \nu} (\mathcal{D}_\Omega w)|_{\partial \Omega} + \frac{\partial}{\partial \nu} (\mathcal{D}_D (h \frac{\partial u}{\partial \nu} |_-)) |_{\partial \Omega} \right) \right].
\]

Thus, for \(x \in \partial \Omega\),
\[
w(x) + N_D (\lambda I - \mathcal{K}_D^*)^{-1} \left( \frac{\partial}{\partial \nu} \mathcal{D}_\Omega w |_{\partial \Omega} \right) (x) = - \int_{\partial D} \frac{\partial N}{\partial \nu} (x, y) h(y) \frac{\partial u}{\partial \nu} |_- (y) d\sigma(y)
\]
\[
+ \frac{1}{k-1} \int_{\partial D} N(x, y) (\lambda I - \mathcal{K}_D^*)^{-1} \left( \frac{\partial}{\partial T} h \frac{\partial u}{\partial T} \right) (y) d\sigma(y)
\]
\[
- \int_{\partial D} N(x, y) (\lambda I - \mathcal{K}_D^*)^{-1} \left( \frac{\partial}{\partial \nu} \mathcal{D}_D (h \frac{\partial u}{\partial \nu} |_-) \right) |_{\partial \Omega} d\sigma(y).
\]

Since
\[
\frac{\partial u}{\partial \nu} |_- = \frac{1}{k-1} \phi,
\]
and
\[
\frac{\partial}{\partial T} h \frac{\partial u}{\partial T} = h \left( - \langle (D^2 H) \nu, \nu \rangle - \tau \frac{\partial H}{\partial \nu} \right) + h' \frac{\partial H}{\partial T} + \frac{\partial}{\partial T} h \frac{\partial}{\partial T} S_D \phi,
\]
where \(\phi\) is defined by (3.4), then by using the expression of \(\mathcal{K}_D^{(1)}\) it is not difficult to see that (3.16) holds.

**Proof of Lemma 3.2.** Since \(\mathcal{E}\) is a compact operator, we can apply the Fredholm alternative. Suppose that \((I + \mathcal{E}) v = 0\). Then, first of all, \(v\) is smooth on \(\partial \Omega\). Since \((-\frac{1}{2} I + \mathcal{K}_\Omega) N_D = S_D\) on \(L^2(\partial D)\) as was proved in [1, 2, 3], we get
\[
\left( - \frac{1}{2} I + \mathcal{K}_\Omega \right) v + S_D (\lambda I - \mathcal{K}_D^*)^{-1} \left( \frac{\partial}{\partial \nu} \mathcal{D}_\Omega v |_{\partial \Omega} \right) = 0 \quad \text{on } \partial \Omega.
\]

Since
\[
(\mathcal{D}_\Omega f)|_- (x) = \left( \frac{1}{2} I + \mathcal{K}_\Omega \right) f(x), \quad x \in \partial \Omega,
\]
where the subscript \(-\) denotes the limit from the inside of \(\Omega\) [18], we get
\[
v(x) = (\mathcal{D}_\Omega v)|_- (x) + S_D (\lambda I - \mathcal{K}_D^*)^{-1} \left( \frac{\partial}{\partial \nu} \mathcal{D}_\Omega v |_{\partial \Omega} \right) (x), \quad x \in \partial \Omega.
\]
Thus \( v \) can be extended to whole \( \Omega \) to satisfy

\[
(3.17) \quad v(x) = (D_\Omega v)(x) + S_D(\lambda - K^*_D)^{-1}\left(\frac{\partial}{\partial v}(D_\Omega v)|_{\partial D}\right)(x), \quad x \in \Omega.
\]

Let the space \( W^{1,2}(\Omega) \) be the set of functions \( f \in L^2(\Omega) \) such that \( \nabla f \in L^2(\Omega) \). We now recall the following facts from [14, 15]: The space \( W^{1,2}(\Omega) \) is given in (3.17) takes exactly the form in (3.18) with \( H = D_\Omega v \). By (3.3) and uniqueness of \( H \), we have \( S_D(\frac{\partial v}{\partial \nu}|_{\partial \Omega}) = 0 \) in \( \Omega \). It then follows that \( \frac{\partial v}{\partial \nu}|_{\partial \Omega} = 0 \) on \( \partial \Omega \), and hence \( v \) is constant in \( \Omega \). Since \( v \in L^2_0(\partial \Omega) \), we get \( v = 0 \). So, \( I + E \) is injective, and hence invertible. This completes the proof. \( \square \)

4. Reconstruction of the interface deformation

We finally give a reconstruction formula of the shape deformation \( h \) from measurements of boundary perturbations \( u_\epsilon - u \) on \( \partial \Omega \). Recall from the previous section that

\[
(4.1) \quad (u_\epsilon - u)(x) = \epsilon(k - 1)w(x) + O(\epsilon^2), \quad x \in \partial \Omega.
\]

For \( f \in L^2_0(\partial \Omega) \), let \( v \) be the solution to

\[
\begin{align*}
\nabla \cdot (1 + (k - 1)\chi_D) \nabla v = 0 & \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu}|_{\partial \Omega} = f & \quad \text{on } \partial \Omega, \quad v = 0.
\end{align*}
\]

Multiplying (4.1) by \( f \) and integrating over \( \partial \Omega \) yields

\[
(4.2) \quad \int_{\partial \Omega} f(u_\epsilon - u) \, d\sigma = \epsilon(1 - k) \int_{\partial D} h \left[ \frac{\partial v}{\partial T} \frac{\partial u}{\partial T} + k \frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} \right] \, d\sigma + O(\epsilon^2),
\]

where \( u \) is the solution to (1.1). Formula (4.2) can be used in order to reconstruct an approximation of \( h \) by appropriately choosing \( v \). Higher-order terms given in Theorem 3.3 can be used to get a better approximation of \( h \).

To illustrate this, we consider \( \Omega \) to be the unit disk centered at the origin, and \( D \) to be the disk centered at the origin with radius \( \alpha \). Set

\[
g(\theta) = e^{im\theta}, \quad f(\theta) = e^{in\theta}, \quad m, n \geq 1.
\]

It follows that

\[
u(r, \theta) = \frac{c_n}{\alpha^{|n|}} r^{|n|} e^{in\theta}, \quad v(r, \theta) = \frac{c_m}{\alpha^{|n|}} r^{|m|} e^{im\theta}, \quad \text{for } r < \alpha,
\]

where

\[
c_n = \frac{2}{|n| \alpha^{-|n|}(k + 1) + \alpha^{|n|}(k - 1)}, \quad n \geq 1.
\]

Therefore,

\[
\int_{\partial \Omega} f(u_\epsilon - u) \, d\sigma = \epsilon c_{n,m}(k) \int_{\partial D} h e^{i(n+m)\theta} \, d\theta + O(\epsilon^2),
\]

for some constant \( c_{n,m} \) depending only on \( n, m, \alpha \) and \( k \). This implies that the Fourier coefficient \( h_{np} \) can be determined from measurements of \( u_\epsilon - u \) on \( \partial \Omega \) provided that the order of magnitude of \( |h_{np}| \) is much larger than \( \epsilon \). To reconstruct higher Fourier coefficients of \( h \) or
more accurately the first ones, the high-order asymptotic expansions derived in this paper should be used. The idea is quite simple. We first use the leading-order term to determine the first Fourier coefficients of $h$, say $h_p$, for $|p| \leq L$. Then, we plug in these coefficients into the second-order term by approximating $h$ by $\sum_{|p| \leq L} h_p e^{ip\theta}$ and use the second-order asymptotic expansion in Theorem 3.3 to both determine further Fourier coefficients and get more accurate approximations of $(h_p)|_{p| \leq L}$. This procedure can be repeated recursively.

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