

# Quantitative Photo-Acoustic Imaging of Small Absorbers\*

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## Abstract

In photo-acoustic imaging, energy absorption causes thermo-elastic expansion of optical absorbers, which in turn leads to propagation of a pressure wave. Recently, we have developed an efficient method for locating small absorbing regions inside a bounded domain from boundary measurements of the induced pressure wave and reconstructing the absorbed density. However, it is the absorption coefficient, not the absorbed energy, that is a fundamental physiological parameter. In this paper, we propose two methods for reconstructing the normalized optical absorption coefficient of a small absorber from the absorbed density.

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## 1 Introduction

Photo-acoustic imaging is an emerging imaging technique that combines high optical contrast and high ultrasound resolution in a single modality. It is based on the photo-acoustic effect which refers to the generation of acoustic waves by the absorption of the optical energy. See, for instance, [17, 8, 12, 13].

Let  $D$  be an absorbing domain inside the non-absorbing background bounded medium  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ . In an acoustically homogeneous medium, the photo-acoustic effect is described by the following equation:

$$\frac{\partial^2 p}{\partial t^2}(x, t) - c_P^2 \Delta p(x, t) = \gamma \frac{\partial H}{\partial t}(x, t), \quad x \in \Omega, \quad t \in ]-\infty, +\infty[, \quad (1.1)$$

where  $c_P$  is the acoustic speed in  $\Omega$ ,  $\gamma$  the dimensionless Grüneisen coefficient in  $\Omega$ , and  $H(x, t)$  a heat source function (absorbed energy per unit time per unit volume).

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Assuming the stress-confinement condition, the source term can be modelled as  $\gamma H(x, t) = \delta(t)\chi_D(x)A(x)$ . Here  $\chi_D$  is the indicator function of  $D$  and  $\delta(t)$  is the Dirac mass. Under this assumption, the pressure in an acoustically homogeneous medium obeys the following wave equation:

$$\frac{\partial^2 p}{\partial t^2}(x, t) - c_P^2 \Delta p(x, t) = 0, \quad x \in \Omega, \quad t \in ]0, T[, \quad (1.2)$$

for some final observation time  $T > \text{diam}(\Omega)/c_P$ . This says that the observation time is long enough for the wave initiated inside  $\Omega$  to reach the boundary  $\partial\Omega$ .

The pressure satisfies either the Dirichlet or the Neumann boundary condition

$$p = 0 \quad \text{or} \quad \frac{\partial p}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times ]0, T[ \quad (1.3)$$

and the initial conditions

$$p|_{t=0} = \chi_D(x)A(x) \quad \text{and} \quad \frac{\partial p}{\partial t} \Big|_{t=0} = 0 \quad \text{in } \Omega. \quad (1.4)$$

Here and throughout this paper,  $\nu$  denotes the outward normal to  $\partial\Omega$ .

The density  $A$  is related to the optical absorption coefficient,  $\mu_a$ , by the equation  $A = \gamma\mu_a\Phi$ , where  $\Phi$  is the light fluence. The function  $\Phi$  depends on the distribution of scattering and absorption within  $\Omega$ , as well as the light sources.

Quantitative photo-acoustic imaging is to reconstruct rather  $\mu_a$  than  $A$  from boundary measurements of the induced pressure  $p$ . In general, it is not possible to infer physiological parameters from  $A$ . It is the optical absorption coefficient  $\mu_a$  that directly correlates with tissue structural and functional information such as blood oxygenation. See [20, 10, 16].

Since  $A$  is a nonlinear function of  $\mu_a$ , the reconstruction of  $\mu_a$  from  $A$  is therefore a non-trivial task and of considerable practical interest.

In [13], a formula to reconstruct small changes in the absorption coefficient is given. A fixed-point algorithm has been designed in [6, 7] in a more general case. The algorithm starts with an initial guess for the absorption coefficient. Then, when the (reduced) scattering coefficient distribution is known *a priori*, the light fluence is calculated using the diffusion approximation to the light transport model. As long as the calculated and reconstructed absorption densities differ, these steps are repeated with an assumed absorption coefficient calculated from the quotient of the reconstructed absorption density and the computed fluence distribution. When the reduced scattering coefficient distribution is unknown, an optimal control approach for estimating the absorption and reduced scattering coefficient distributions from absorbed energies obtained at multiple wavelengths has been developed in [5]. See, for instance, [10] for the validity of a multi-wavelength approach.

However, such iterative approaches are not appropriate to reconstruct the absorption coefficient of a small absorber. Recently, we have addressed in [1] the inverse problem of determining the support  $D$  of nonzero optical absorption in  $\Omega$  and the absorbed optical energy density  $A(x)$  from boundary measurements of  $\frac{\partial p}{\partial \nu}$  on  $\partial\Omega \times ]0, T[$  if  $p$  satisfies the Dirichlet boundary condition and  $p$  on  $\partial\Omega \times ]0, T[$  if  $p$  satisfies the Neumann boundary condition. It turned out that, in the case of small spherical absorbers, only the normalized energy density of the absorber,  $\epsilon^2 A$  with  $\epsilon$  being the radius of  $D$ , can be reconstructed from pressure measurements. This has been already pointed out in [14].

The main purpose of this paper is to develop, in the context of small-volume absorbers, new efficient methods to recover  $\mu_a$  of the absorber  $D$  from the normalized energy density.

We distinguish two cases. The first case is the one where the reduced scattering coefficient inside the background medium is known *a priori*. In this case we develop an asymptotic approach to recover the normalized absorption coefficient,  $\epsilon^2\mu_a$ , from the normalized energy density using multiple measurements. We make use of inner expansions of the fluence distribution  $\Phi$  in terms of the size of the absorber. We also provide an approximate formula to separately recover  $\epsilon$  from  $\mu_a$ . However, this requires boundary measurements of  $\Phi$ .

The second case is when the reduced scattering coefficient is unknown. In such a case, we propose a formula to extract the absorption coefficient from multi-wavelength data. In this formula, we combine the multi-wavelength acoustic measurements with diffusing light measurements to separate the product of absorption coefficient and optical fluence. The feasibility of combining photo-acoustic and diffusing light measurements has been demonstrated in [18, 19]. Note that the approximate model we use in this case for the light transport allows us to estimate  $|D|$  independently from  $A$  and therefore, the multi-wavelength approach yields the absolute absorption coefficient.

For the sake of simplicity, we only consider the three-dimensional problem but stress that the techniques developed here apply directly to the two-dimensional case.

## 2 Mathematical Formulation

Suppose that the absorbing object  $D$  is small. We write

$$D = z + \epsilon B,$$

where  $z$  is the ‘‘center’’ of  $D$ ,  $B$  is a reference domain which contains the origin, and  $\epsilon$  is a small parameter.

The density  $A(x)$  is related to the optical absorption coefficient distribution  $\mu_a(x) = \mu_a\chi_D(x)$ , where  $\mu_a$  is a constant, by the equation  $A(x) = \mu_a(x)\Phi(x)$ , where  $\Phi$  is the fluence. The function  $\Phi$  depends on the distribution of scattering and absorption within  $\Omega$ , as well as the light sources. Suppose that  $\mu_s$ , the reduced scattering coefficient in  $\Omega$ , is constant. Based on the diffusion approximation to the transport equation,  $\Phi$  satisfies

$$\left( \mu_a(x) - \frac{1}{3} \nabla \cdot \frac{1}{\mu_a(x) + \mu_s} \nabla \right) \Phi(x) = 0 \quad \text{in } \Omega, \quad (2.1)$$

with the boundary condition

$$\frac{\partial \Phi}{\partial \nu} + l\Phi = g \quad \text{on } \partial\Omega. \quad (2.2)$$

Here,  $g$  denotes the light source and  $l$  a positive constant;  $1/l$  being an extrapolation length.

In [1], we have provided an efficient method to reconstruct  $\epsilon^2 A(z) = \epsilon^2 \mu_a(z)\Phi(z)$  from boundary measurements of the induced pressure. In this paper, we develop two methods for reconstructing  $\epsilon^2 \mu_a(z)$  from  $\epsilon^2 A(z)$ . Then we show how to separate  $\epsilon$  from  $\mu_a(z)$ .

If  $\mu_s$  is known *a priori*, then we follow in the next section an asymptotic approach to recover the normalized absorption coefficient from the normalized absorbed energy.

In the case where  $\mu_s$  is unknown (and possibly varies in  $\Omega$ ), we provide in section 4 an algorithm to extract the absorption coefficient  $\mu_a$  from absorbed energies obtained at multiple wavelengths. We assume that the wavelength dependence of the scattering and absorption coefficients are known. In tissues, the wavelength-dependence of the scattering often approximates to a power law. Numerical results are presented in section 5 to show the validity of our inversion algorithms. The paper ends with a short discussion.

### 3 Asymptotic Approach

In this section, we consider a slightly more general equation than (2.1) and provide an asymptotic expansion of its solution as the size of the absorbing object  $D$  goes to zero.

Recall that the fluence  $\Phi$  is the integral over time of the fluence rate  $\Psi$  which satisfies

$$\left(\frac{1}{c}\partial_t + \mu_a(x) - \frac{1}{3}\nabla \cdot \frac{1}{\mu_a(x) + \mu_s}\nabla\right)\Psi(x) = 0 \quad \text{in } \Omega \times \mathbb{R}, \quad (3.1)$$

where  $c$  is the speed of light. Taking the Fourier transform of (3.1) yields that  $\Phi = \Phi_{\omega=0}$ , where, for a given frequency  $\omega$ ,  $\Phi_\omega$  is the solution to

$$\left(\frac{i\omega}{c} + \mu_a(x) - \frac{1}{3}\nabla \cdot \frac{1}{\mu_a(x) + \mu_s}\nabla\right)\Phi_\omega(x) = 0 \quad \text{in } \Omega, \quad (3.2)$$

with the boundary condition

$$\frac{\partial\Phi_\omega}{\partial\nu} + l\Phi_\omega = g \quad \text{on } \partial\Omega. \quad (3.3)$$

In the following, for any fixed  $\omega \geq 0$  we rigorously derive an asymptotic expansion of  $\Phi_\omega(z)$  as  $\epsilon$  goes to zero for  $\Phi_\omega$ , where  $z$  is the location of the absorbing object  $D$ . The results for nonzero  $\omega$  have their own mathematical and physical interests [8].

For simplicity, we assume that  $l \leq C\sqrt{\mu_s}$  for some constant  $C$  and drop in the notation the dependence with respect to  $\omega$ .

#### 3.1 Asymptotic Formula

In this section we assume that  $\mu_s$  is a constant and known *a priori*. As said before, we suppose that the space dimension is 3. Define  $\Phi^{(0)}$  by

$$\left(\frac{i\omega}{c} - \frac{1}{3\mu_s}\Delta\right)\Phi^{(0)}(x) = 0 \quad \text{in } \Omega,$$

subject to the boundary condition

$$\frac{\partial\Phi^{(0)}}{\partial\nu} + l\Phi^{(0)} = g \quad \text{on } \partial\Omega,$$

where  $g$  is a bounded function on  $\partial\Omega$ .

Throughout this paper we assume that the location  $z$  of the anomaly is away from the boundary  $\partial\Omega$ , namely

$$\text{dist}(z, \partial\Omega) \geq C_0 \quad (3.4)$$

for some constant  $C_0$ .

Let  $N$  be the Neumann function, that is, the solution to

$$\begin{cases} \left(\frac{i\omega}{c} - \frac{1}{3\mu_s}\Delta_x\right)N(x, y) = -\delta_y & \text{in } \Omega, \\ \frac{\partial N}{\partial\nu} + lN = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

Note that

$$\Phi^{(0)}(x) = -\frac{1}{3\mu_s} \int_{\partial\Omega} g(y)N(x, y) d\sigma(y), \quad x \in \Omega. \quad (3.6)$$

Thus, multiplying (3.2) by  $N$  and integrating by parts, we readily get the following lemma.

**Lemma 3.1** *For any  $x \in \Omega$ , the following representation formula of  $\Phi(x)$  holds:*

$$\begin{aligned} (\Phi - \Phi^{(0)})(x) &= \mu_a \int_D \Phi(y)N(x, y) dy \\ &+ \frac{1}{3} \left( \frac{1}{\mu_a + \mu_s} - \frac{1}{\mu_s} \right) \int_D \nabla\Phi(y) \cdot \nabla_y N(x, y) dy. \end{aligned} \quad (3.7)$$

We now derive an asymptotic expansion of  $(\Phi - \Phi^{(0)})(z)$ , where  $z$  is the location of  $D$ , as the size  $\epsilon$  of  $D$  goes to zero. The asymptotic expansion also takes the smallness of  $\mu_a/\mu_s$  into account.

Let us first recall that the (outgoing) fundamental solution to the operator  $\frac{i\omega}{c} - \frac{1}{3\mu_s}\Delta$  is given by

$$G(x, y) := \frac{3\mu_s}{4\pi} \frac{e^{-k|x-y|}}{|x-y|} \quad (3.8)$$

where

$$k = \exp\left(\frac{\pi}{4}i\right) \sqrt{\frac{3\mu_s\omega}{c}}. \quad (3.9)$$

In particular, we have

$$\left( \frac{i\omega}{c} - \frac{1}{3\mu_s}\Delta_x \right) G(x, y) = -\delta_y(x), \quad x \in \mathbb{R}^3.$$

Thus, the function  $R_1(x, y) := N(x, y) - G(x, y)$  is the solution to

$$\begin{cases} \left( \frac{i\omega}{c} - \frac{1}{3\mu_s}\Delta_x \right) R_1(x, y) = 0, & x \in \Omega, \\ \frac{\partial R_1}{\partial\nu} + lR_1 = -\frac{\partial G}{\partial\nu} - lG & \text{on } \partial\Omega. \end{cases}$$

Observe that if  $y \in D$ , then

$$l \|G(\cdot, y)\|_{L^\infty(\partial\Omega)} + \left\| \frac{\partial G}{\partial\nu_x}(\cdot, y) \right\|_{L^\infty(\partial\Omega)} \leq C\mu_s^{3/2},$$

since, by assumption,  $l \leq C'\sqrt{\mu_s}$  for some constant  $C'$ .

It then follows from Lemma A.1 in Appendix A that

$$\sup_{x, y \in D} \left( \mu_s^{-3/2} |R_1(x, y)| + C_1 \mu_s^{-2} |\nabla R_1(x, y)| + C_2 \mu_s^{-5/2} |\nabla\nabla R_1(x, y)| \right) \leq C_3 \quad (3.10)$$

for some constants (with different dimensions)  $C_i, i = 1, 2, 3$ , independent of  $\mu_s$ . Note that if  $\omega = 0$  then

$$\begin{aligned} \sup_{x, y \in D} (|R_1(x, y)| + C_1 |\nabla R_1(x, y)| + C_2 |\nabla\nabla R_1(x, y)|) &\leq C_3 \left( l \|G(\cdot, y)\|_{L^\infty(\partial\Omega)} + \left\| \frac{\partial G}{\partial\nu_x}(\cdot, y) \right\|_{L^\infty(\partial\Omega)} \right) \\ &\leq C_4 l, \end{aligned} \quad (3.11)$$

where  $C_i, i = 1, \dots, 4$ , are independent of  $\mu_s$ .

Let  $\Gamma(x) := -1/(4\pi|x|)$  be a fundamental solution of the Laplacian in three dimensions and let

$$R(x, y) = N(x, y) - 3\mu_s\Gamma(x - y). \quad (3.12)$$

Writing

$$R(x, y) = R_1(x, y) + (G(x, y) - 3\mu_s\Gamma(x - y)),$$

we obtain the following lemma as an immediate consequence of (3.10).

**Lemma 3.2** *Let  $R(x, y)$  be defined by (3.12). There are constants  $C_i, i = 1, \dots, 6$  (with different dimensions) depending on  $C_0$  given in (3.4) such that*

$$|R(x, y)| \leq C_1\mu_s^{3/2}, \quad (3.13)$$

$$|\nabla_x R(x, y)| \leq C_2\mu_s^2 + C_3\frac{\mu_s^{3/2}}{|x - y|}, \quad (3.14)$$

$$|\nabla_x \nabla_x R(x, y)| \leq C_4\mu_s^{5/2} + C_5\frac{\mu_s^2}{|x - y|} + C_6\frac{\mu_s^{3/2}}{|x - y|^2} \quad (3.15)$$

for all  $x, y \in D$  provided that  $\epsilon\sqrt{\mu_s}$  is sufficiently small.

Let us now introduce some notation. Let

$$n(x) := \int_D N(x, y) dy, \quad x \in D, \quad (3.16)$$

and define a multiplier  $\mathcal{M}$  by

$$\mathcal{M}[f](x) := \mu_a n(x) f(x). \quad (3.17)$$

We then define two operators  $\mathcal{N}$  and  $\mathcal{R}$  by

$$\begin{aligned} \mathcal{N}[f](x) &:= 3\mu_a\mu_s \int_D (f(y) - f(x))\Gamma(x - y) dy \\ &\quad + \mu_s \left( \frac{1}{\mu_a + \mu_s} - \frac{1}{\mu_s} \right) \int_D \nabla f(y) \cdot \nabla_y \Gamma(x - y) dy, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \mathcal{R}[f](x) &:= \mu_a \int_D (f(y) - f(x))R(x, y) dy \\ &\quad + \frac{1}{3} \left( \frac{1}{\mu_a + \mu_s} - \frac{1}{\mu_s} \right) \int_D \nabla f(y) \cdot \nabla_y R(x, y) dy. \end{aligned} \quad (3.19)$$

In view of Lemma 3.2, the equation (3.7) then can be rewritten as

$$(I - \mathcal{M})[\Phi] - (\mathcal{N} + \mathcal{R})[\Phi] = \Phi^{(0)} \quad \text{on } D, \quad (3.20)$$

where  $I$  is the identity operator.

The following lemma is proved in Appendix B.

**Lemma 3.3** *Let  $p > 3$  and  $q$  be its conjugate exponent, i.e.,  $1/p + 1/q = 1$ . Then there is a constant  $C$  depending on  $C_0$  given in (3.4) such that*

$$\|\mathcal{N}[f]\|_{L^p(D)} \leq C\epsilon \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right) \|\nabla f\|_{L^p(D)}, \quad (3.21)$$

$$\|\nabla \mathcal{N}[f]\|_{L^p(D)} \leq C \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right) \|\nabla f\|_{L^p(D)}, \quad (3.22)$$

$$\|\mathcal{R}[f]\|_{L^p(D)} \leq C\epsilon^2 \sqrt{\mu_s} \left( \mu_a \mu_s \epsilon^2 + \frac{\mu_a}{\mu_s} \right) \|\nabla f\|_{L^p(D)}, \quad (3.23)$$

$$\|\nabla \mathcal{R}[f]\|_{L^p(D)} \leq C\epsilon \sqrt{\mu_s} \left( \mu_a \mu_s \epsilon^2 + \frac{\mu_a}{\mu_s} \right) \|\nabla f\|_{L^p(D)}. \quad (3.24)$$

Note that

$$\|n\|_{L^\infty(D)} = O(\epsilon^2 \mu_s) \quad \text{and} \quad \|\nabla n\|_{L^\infty(D)} = O(\epsilon \mu_s). \quad (3.25)$$

We also note the simple fact that

$$(I - \mathcal{M})^{-1}[f](x) = \frac{f(x)}{1 - \mu_a n(x)}.$$

We may rewrite (3.20) as

$$\Phi - (I - \mathcal{M})^{-1}(\mathcal{N} + \mathcal{R})[\Phi] = (I - \mathcal{M})^{-1}[\Phi^{(0)}] \quad \text{on } D. \quad (3.26)$$

Moreover, one can see from Lemma 3.3 that

$$\|(I - \mathcal{M})^{-1}\mathcal{N}[f]\|_{W^{1,p}(D)} \leq C \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right) \|f\|_{W^{1,p}(D)}$$

and

$$\|(I - \mathcal{M})^{-1}\mathcal{R}[f]\|_{W^{1,p}(D)} \leq C\epsilon \sqrt{\mu_s} \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right) \|f\|_{W^{1,p}(D)}.$$

Here,  $W^{1,p}(D) := \{f \in L^p(D), \nabla f \in L^p(D)\}$ . So, if  $\epsilon^2 \mu_a \mu_s$  and  $\frac{\mu_a}{\mu_s}$  are sufficiently small, then the integral equation (3.26) can be solved by the Neumann series

$$\Phi = \sum_{j=0}^{\infty} ((I - \mathcal{M})^{-1}(\mathcal{N} + \mathcal{R}))^j (I - \mathcal{M})^{-1}[\Phi^{(0)}], \quad (3.27)$$

which converges in  $W^{1,p}(D)$ . It then follows from above two estimates that

$$\Phi = (I - \mathcal{M})^{-1}[\Phi^{(0)}] + (I - \mathcal{M})^{-1}(\mathcal{N} + \mathcal{R})(I - \mathcal{M})^{-1}[\Phi^{(0)}] + E_1, \quad (3.28)$$

where the error term  $E_1$  satisfies

$$\|E_1\|_{W^{1,p}(D)} \leq C \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right)^2 \|\Phi^{(0)}\|_{W^{1,p}(D)}.$$

We further have from (3.21), (3.22), and (3.25) that

$$\|\mathcal{M}(\mathcal{N} + \mathcal{R})[f]\|_{W^{1,p}(D)} \leq C\epsilon^2 \mu_s \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right) \|\Phi^{(0)}\|_{W^{1,p}(D)}.$$

Thus we get the following asymptotic expansion:

**Lemma 3.4** *The following estimate holds:*

$$\Phi(x) = (I - \mathcal{M})^{-1}[\Phi^{(0)}] + (\mathcal{N} + \mathcal{R})(I - \mathcal{M})^{-1}[\Phi^{(0)}](x) + E(x), \quad x \in D, \quad (3.29)$$

where the error term  $E$  satisfies

$$\|E\|_{W^{1,p}(D)} \leq C\epsilon^2\mu_s \left( \epsilon^2\mu_a\mu_s + \frac{\mu_a}{\mu_s} \right) \|\Phi^{(0)}\|_{W^{1,p}(D)}. \quad (3.30)$$

It is worth mentioning that since  $p > 3$ ,  $W^{1,p}(D)$  is continuously imbedded in  $L^\infty(D)$  by the Sobolev imbedding theorem, and hence the asymptotic formula (3.29) holds uniformly in  $D$ .

Note that

$$\begin{aligned} & (I - \mathcal{M})^{-1}[\Phi^{(0)}] + (\mathcal{N} + \mathcal{R})(I - \mathcal{M})^{-1}[\Phi^{(0)}](x) \\ &= \frac{\Phi^{(0)}(x)}{1 - \mu_a n(x)} + \mu_a \int_D \frac{\Phi^{(0)}(y)}{1 - \mu_a n(y)} N(x - y) dy - \frac{\mu_a n(x) \Phi^{(0)}(x)}{1 - \mu_a n(x)} \\ & \quad + \frac{1}{3} \left( \frac{1}{\mu_a + \mu_s} - \frac{1}{\mu_s} \right) \int_D \nabla \left( \frac{\Phi^{(0)}(y)}{1 - \mu_a n(y)} \right) \cdot \nabla_y N(x - y) dy \\ & \approx \Phi^{(0)}(x) + 3\mu_a\mu_s \int_D \Phi^{(0)}(y) \Gamma(x - y) dy \\ & \quad + \mu_s \left( \frac{1}{\mu_a + \mu_s} - \frac{1}{\mu_s} \right) \int_D \nabla \Phi^{(0)}(y) \cdot \nabla_y \Gamma(x - y) dy \end{aligned}$$

where the error of the approximation satisfies (3.30). We then get

$$\int_D \Phi^{(0)}(y) \Gamma(x - y) dy = \Phi^{(0)}(x) \int_D \Gamma(x - y) dy + O(\epsilon^3 \|\nabla \Phi^{(0)}\|_{L^\infty(D)})$$

and

$$\begin{aligned} & \mu_s \left( \frac{1}{\mu_a + \mu_s} - \frac{1}{\mu_s} \right) \int_D \nabla \Phi^{(0)}(y) \cdot \nabla_y \Gamma(x - y) dy \\ &= \frac{\mu_a}{\mu_s} \nabla \Phi^{(0)}(x) \cdot \int_D \nabla_y \Gamma(x - y) dy + O\left(\epsilon^2 \frac{\mu_a}{\mu_s} \|\nabla \nabla \Phi^{(0)}\|_{L^\infty(D)} + \epsilon \left(\frac{\mu_a}{\mu_s}\right)^2 \|\nabla \Phi^{(0)}\|_{L^\infty(D)}\right). \end{aligned}$$

Let  $\hat{N}_B$  to be the Newtonian potential of  $B$ , which is given by

$$\hat{N}_B(x) := \int_B \Gamma(x - y) dy, \quad x \in \mathbb{R}^3, \quad (3.31)$$

and let  $\mathcal{S}_B$  be the single layer potential associated to  $B$ , which are given for a density  $\psi \in L^2(\partial B)$  by

$$\mathcal{S}_B[\psi](x) := \int_{\partial B} \Gamma(x - y) \psi(y) d\sigma(y), \quad x \in \mathbb{R}^3.$$

Then one can see by scaling  $x = \epsilon x' + z$  that

$$\int_D \Gamma(x - y) dy = \epsilon^2 \hat{N}_B(x'), \quad x' \in B,$$

and

$$\begin{aligned} \int_D \nabla_y \Gamma(x-y) dy &= \epsilon \int_B \nabla_y \Gamma(x'-y') dy' \\ &= -\epsilon \int_{\partial B} \Gamma(x'-y') \nu(y') d\sigma(y') = -\epsilon \mathcal{S}_B[\nu](x') \end{aligned}$$

where  $\nu(y)$  the outward normal to  $\partial B$  at  $y$ . Therefore we have

$$\Phi(x) \approx \Phi^{(0)}(x) + 3\epsilon^2 \mu_a \mu_s \Phi^{(0)}(z) \hat{N}_B \left( \frac{x-z}{\epsilon} \right) - \epsilon \frac{\mu_a}{\mu_s} \mathcal{S}_B[\nu] \left( \frac{x-z}{\epsilon} \right) \cdot \nabla \Phi^{(0)}(z) \quad (3.32)$$

with the approximation error satisfying (3.30). Since this approximation holds in  $W^{1,p}(D)$ , we have

$$\nabla \Phi(x) \approx 3\epsilon \mu_a \mu_s \Phi^{(0)}(z) \nabla \hat{N}_B \left( \frac{x-z}{\epsilon} \right) + \left( I - \frac{\mu_a}{\mu_s} \nabla \mathcal{S}_B[\nu] \left( \frac{x-z}{\epsilon} \right) \right) \nabla \Phi^{(0)}(z). \quad (3.33)$$

Note that, again by Lemma A.1,

$$\|\Phi^{(0)}\|_{W^{1,p}(D)} \leq \epsilon^{1/p} \sup_{x \in D} (|\Phi^{(0)}(x)| + |\nabla \Phi^{(0)}(x)|) \leq C \epsilon^{1/p} \sqrt{\mu_s},$$

and

$$\|\nabla \nabla \Phi^{(0)}\|_{L^\infty(D)} \leq C \mu_s.$$

Thus we have the following asymptotic formula, which is the main result of this section.

**Proposition 3.5** *We have*

$$(\Phi - \Phi^{(0)})(z) \approx 3\epsilon^2 \mu_a \mu_s \Phi^{(0)}(z) \hat{N}_B(0) - \epsilon \frac{\mu_a}{\mu_s} \mathcal{S}_B[\nu](0) \cdot \nabla \Phi^{(0)}(z), \quad (3.34)$$

where the error of the approximation is less than

$$C_1 \epsilon^{2+1/p} \mu_s^{3/2} \left( \epsilon^2 \mu_a \mu_s + \frac{\mu_a}{\mu_s} \right) + C_2 \mu_s^{1/2} \left( \epsilon^3 \mu_a \mu_s + \epsilon \left( \frac{\mu_a}{\mu_s} \right)^2 \right) + C_3 \epsilon^2 \mu_a$$

for  $p > 3$  and some constants  $C_1, C_2$ , and  $C_3$  (with different dimensions) depending on  $C_0$  given in (3.4) and on  $g$ .

Formula (3.34) shows that if  $\epsilon \Phi^{(0)}(z)$  is of the same order as  $(1/\mu_s^2(z)) \nabla \Phi^{(0)}(z)$  then we have two contributions in the leading-order term of the perturbations in  $\Phi$  that are due to  $D$ . The first contribution is coming from the source term  $\mu_a(x, \omega)$  and the second one from the jump conditions. If  $\epsilon \Phi^{(0)}(z)$  is much larger than  $(1/\mu_s^2(z)) \nabla \Phi^{(0)}(z)$  then we can neglect the second contribution. It worth emphasizing that formula (3.34) holds for any fixed  $\omega \geq 0$  as  $\epsilon$  goes to zero.

**Remark.** If the reduced scattering coefficient  $\mu_s$  is not constant, then the expected asymptotic formula would be

$$(\Phi - \Phi^{(0)})(z) \approx 3\epsilon^2 \mu_a \mu_s(z) \Phi^{(0)}(z) \hat{N}_B(0) - \epsilon \frac{\mu_a}{\mu_s(z)} \mathcal{S}_B[\nu](0) \cdot \nabla \Phi^{(0)}(z). \quad (3.35)$$

To prove it, one needs to prove (3.13) - (3.15) with variable  $\mu_s$ . Even though these estimates are most likely true, we do not attempt to prove them since this is out of scope of the paper.

### 3.2 Reconstruction of the Absorption Coefficient

We now turn to the reconstruction of the absorption coefficient. Given the light source  $g$ , it has been shown in [1] that the location  $z$  and  $\alpha := \epsilon^2 \mu_a \Phi(z)$  can be reconstructed from photo-acoustic measurements. Here  $\Phi = \Phi_{\omega=0}$ .

Suppose that  $B$  is the unit sphere. Since  $\mathcal{S}_B[\nu](0) = 0$ , formula (3.34) reads

$$(\Phi - \Phi^{(0)})(z) \approx 3\epsilon^2 \mu_a \mu_s \Phi^{(0)}(z) \hat{N}_B(0) \approx 3\alpha \mu_s \hat{N}_B(0). \quad (3.36)$$

Thus one can easily see that

$$\epsilon^2 \mu_a \approx \frac{\alpha}{3\alpha \mu_s \hat{N}_B(0) + \Phi^{(0)}(z)}. \quad (3.37)$$

Let us see how one may separate  $\epsilon$  from  $\mu_a$ . Because of (3.6), it follows from (3.7) that

$$\frac{1}{3} \int_{\partial\Omega} g(\Phi - \Phi^{(0)}) d\sigma \approx -\mu_a \Phi(z) \Phi^{(0)}(z) |D| + \frac{1}{3} \left( \frac{1}{\mu_s + \mu_a} - \frac{1}{\mu_s} \right) \int_D \nabla \Phi(y) \cdot \nabla \Phi^{(0)}(y) dy.$$

Thus we get from (3.33) that

$$\begin{aligned} & \frac{1}{3} \int_{\partial\Omega} g(\Phi - \Phi^{(0)}) d\sigma \\ & \approx -\mu_a \Phi(z) \Phi^{(0)}(z) |D| - \frac{\mu_a}{3\mu_s^2} \nabla \Phi^{(0)}(z) \cdot \left[ 3\epsilon^4 \mu_a \mu_s \Phi^{(0)}(z) \int_B \nabla \hat{N}_B(y) dy + \right. \\ & \quad \left. + \epsilon^3 \int_B \left( I - \frac{\mu_a}{\mu_s} \nabla \mathcal{S}_B[\nu] \right)(y) dy \nabla \Phi^{(0)}(z) \right] \\ & \approx -\epsilon \alpha |B| \Phi^{(0)}(z) - \frac{\mu_a \epsilon^3}{3\mu_s^2} \nabla \Phi^{(0)}(z) \cdot \left[ 3\epsilon \mu_a \mu_s \Phi^{(0)}(z) \int_B \nabla \hat{N}_B(y) dy + \right. \\ & \quad \left. + \int_B \left( I - \frac{\mu_a}{\mu_s} \nabla \mathcal{S}_B[\nu] \right)(y) dy \nabla \Phi^{(0)}(z) \right]. \end{aligned} \quad (3.38)$$

One may use this approximation to separately recover  $\epsilon$  from  $\mu_a$  even in the general case, where  $B$  not necessary a unit sphere by combining (3.38) together with (3.34). However, this approach requires boundary measurements of  $\Phi$  on  $\partial\Omega$ .

## 4 Multi-Wavelength Approach

We now deal with the problem of estimating both the absorption coefficient  $\mu_a$  and the reduced scattering coefficient  $\mu_s$  from  $A = \mu_a \Phi$  where  $\Phi$  satisfies (3.2) and the boundary condition (3.3). It is known that this problem at fixed wavelength  $\lambda$  is a severe ill-posed problem. However, if the wavelength dependence of both the scattering and the absorption are known, then the ill-posedness of the inversion can be dramatically reduced.

Let  $\mu_s(x, \lambda_j)$  and  $\mu_a(x, \lambda_j)$  be the reduced scattering and absorption coefficients at the wavelength  $\lambda_j$  for  $j = 1, 2$ , respectively. Note that  $\mu_a(\cdot, \lambda_j)$  is supported in the absorbing region  $D$  which is of the form  $D = z + \epsilon B$  for  $\epsilon$  of small magnitude. We assume that  $\mu_s(x, \lambda)$  and  $\mu_a(x, \lambda)$  depend on the wavelength in the following way:

$$\mu_s(x, \lambda) = f_s(x) g_s(\lambda), \quad (4.1)$$

and

$$\mu_a(x, \lambda) = f_a(x)g_a(\lambda), \quad (4.2)$$

for some functions  $f_a, f_s, g_a, g_s$ . Denote

$$C_s := \frac{\mu_s(x, \lambda_1)}{\mu_s(x, \lambda_2)} = \text{constant in the } x \text{ variable in } \Omega,$$

and

$$C_a := \frac{\mu_a(x, \lambda_1)}{\mu_a(x, \lambda_2)} = \text{constant in the } x \text{ variable in } D.$$

Assumptions (4.1) and (4.2) are physically acceptable. See, for instance, [5].

Let  $A_j$  be the optical absorption density at  $\lambda_j, j = 1, 2$ . Let  $l'_1, l'_2$  be two positive constants. Let  $\Phi_j$  be the solution of

$$\left( \mu_a(x, \lambda_j) - \frac{1}{3} \nabla \cdot \frac{1}{\mu_s(x, \lambda_j)} \nabla \right) \Phi_j(x) = 0, \quad (4.3)$$

with the boundary condition

$$\frac{1}{\mu_s} \frac{\partial \Phi_j}{\partial \nu} + l'_j \Phi_j = g'_j \quad \text{on } \partial \Omega. \quad (4.4)$$

Note that the boundary condition (4.4) is slightly different from (2.2) because  $\mu_s$  is assumed variable possibly up to the boundary.

Multiplying (4.3) for  $j = 1$  by  $\Phi_2$  and integrating by parts over  $\Omega$ , we obtain that

$$\begin{aligned} 0 &= \int_{\Omega} \left( \mu_a(x, \lambda_1) - \frac{1}{3} \nabla \cdot \frac{1}{\mu_s(x, \lambda_1)} \nabla \right) \Phi_1(x) \Phi_2(x) dx \\ &= \int_{\Omega} \mu_a(x, \lambda_1) \Phi_1 \Phi_2 dx - \frac{1}{3} \int_{\partial \Omega} (g'_1 \Phi_2 - l'_1 \Phi_1 \Phi_2) d\sigma \\ &\quad + \frac{1}{3} \int_{\Omega} \frac{1}{\mu_s(x, \lambda_1)} \nabla \Phi_1(x) \cdot \nabla \Phi_2(x) dx. \end{aligned}$$

We then replace  $\mu_s(x, \lambda_1)$  by  $C_s \mu_s(x, \lambda_2)$  and integrate by parts further to obtain

$$\begin{aligned} &\frac{1}{3} \int_{\partial \Omega} \left( g'_1 \Phi_2 - \frac{1}{C_s} g'_2 \Phi_1 \right) (x) d\sigma(x) + \frac{1}{3} \int_{\partial \Omega} \left( \frac{l'_2}{C_s} - l'_1 \right) \Phi_1(x) \Phi_2(x) d\sigma(x) \\ &= \int_D \left( -\frac{\mu_a(x, \lambda_2)}{C_s} + \mu_a(x, \lambda_1) \right) \Phi_1(x) \Phi_2(x) dx. \end{aligned}$$

Since  $D = z + \epsilon B$ , we have the following proposition.

**Proposition 4.1** *The following approximation holds:*

$$\begin{aligned} |D|(1 + o(1)) \left( -\frac{1}{C_s C_a} + 1 \right) \frac{A_1(z) A_2(z)}{\mu_a(x, \lambda_2)} &= \frac{1}{3} \int_{\partial \Omega} \left( g'_1 \Phi_2 - \frac{1}{C_s} g'_2 \Phi_1 \right) d\sigma \\ &\quad + \frac{1}{3} \int_{\partial \Omega} \left( \frac{l'_2}{C_s} - l'_1 \right) \Phi_1 \Phi_2 d\sigma. \end{aligned} \quad (4.5)$$

Proposition 4.1 yields approximations of  $\mu_a(z, \lambda_2)$  and  $\mu_a(z, \lambda_1) = C_a \mu_a(z, \lambda_2)$  from  $A_1$  and  $A_2$  provided that  $|D|$  is known. To estimate  $|D|$  one can use the following identity

$$\int_D \mu_a(x, \lambda_j) \Phi_j(x) dx = \frac{1}{3} \int_{\partial \Omega} \left( g'_j - l'_j \Phi_j \right) d\sigma. \quad (4.6)$$

## 5 Numerical Examples

To explore the performances of approaches in the previous sections, we run numerical simulations in a simple 3D setting. The medium we considered is a cube in  $\mathbb{R}^3$  of edge 4cm ( $[0\text{cm}, 4\text{cm}]^3$ ), containing a small spherical inclusion  $D$  located at  $(17, 25, 8)$ . The background medium was assumed to be non-absorbing and to have a constant realistic reduced scattering coefficient  $\mu_s = 10\text{cm}^{-1}$ . The absorption coefficient  $\mu_a|_D$  inside the inclusion was set to different (positive) values, from asymptotically small to realistic ones ( $\mu_a = 0.01, 0.05$  and  $0.1\text{cm}^{-1}$ ). The inclusion radius  $\epsilon$  ranges from 0.03cm to 0.3cm (to 0.5cm in the multi-wavelength setting).

Simulations were conducted using FreeFEM++ (<http://www.freefem.org>). We solved the direct problem (3.2)-(3.3) using P1-elements on an adapted mesh (characteristic size of the mesh  $h \approx 0.1\text{cm}$  on the boundary of the medium;  $h \approx \frac{\epsilon}{100}$  in the neighborhood of the inclusion).

### 5.1 Asymptotic Approach

Following our work in [1], we assumed that we can accurately estimate the position  $z$  of the inclusion and the quantity  $\alpha = \mu_a \epsilon^2 \Phi(z)$  from photoacoustic inversion. Both quantities, along with the reduced scattering coefficient  $\mu_s$  were assumed to be known with no error.

We computed  $\Phi^{(0)}$  solving (3.6) on the same mesh. Applying formula (3.37), we obtained an estimate on  $\mu_a \epsilon^2$ . The error on this estimate for the values of the parameters can be seen in Fig.1.

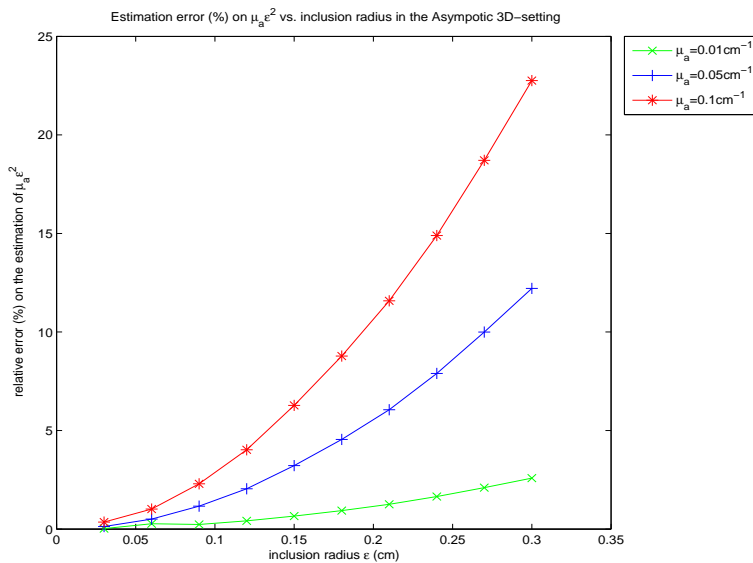


Figure 1: Error on the first estimate  $\mu_a \epsilon^2$  using the asymptotic approach.

To separate size from attenuation, we furthermore assumed that we could access total

boundary measurements  $\Phi|_{\partial\Omega}$ . Using the leading-order term in (3.38), we can write

$$\frac{1}{3} \int_{\partial\Omega} g(\Phi - \Phi^{(0)}) d\sigma \approx -\frac{4}{3} \pi \epsilon \alpha \Phi^{(0)}(z),$$

which yields a direct estimation for  $\epsilon$ . Using both estimates, we derived an estimation of  $\mu_a$ . Errors on these estimates are given in Figs. 2 and 3.

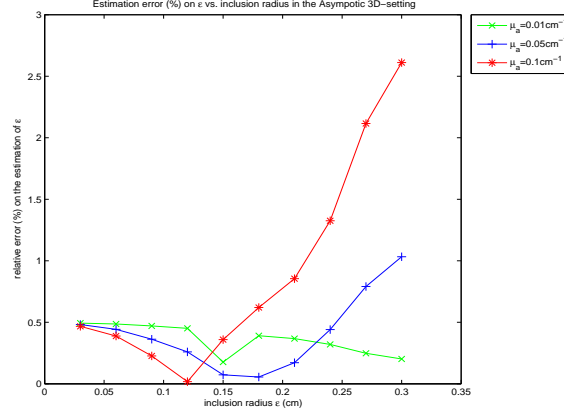


Figure 2: Error on the reconstruction of  $\epsilon$  using the asymptotic approach.

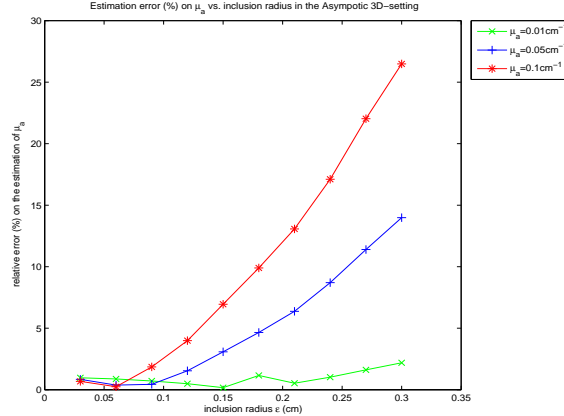


Figure 3: Error on the reconstruction of  $\mu_a$  using the asymptotic approach.

As expected, reconstruction is very accurate in the asymptotic limit and degrades as  $\mu_a$  and/or  $\epsilon$  increase.

We mention that the asymptotic estimations of  $\epsilon$  and  $\mu_a \epsilon^2$  are independent. The latter is only based on the photoacoustic measurements. The former requires also diffusing light

measurements and seems to be more accurate (error < 3% in the range of our parameters vs. < 30% for the estimation of  $\mu_a \epsilon^2$ ).

The estimate on  $\mu_a$  is obtained using these two estimates, thus the quality of the attenuation reconstruction is limited by the least-quality estimate, *i.e.*,  $\mu_a \epsilon^2$ .

## 5.2 Multi-Wavelength Approach

In this setting, we assumed that we could access the position  $z$  of the inclusion, the quantities  $\alpha_j = \mu_a(\lambda_j)\epsilon^2\Phi_j(z)$  and the factors  $C_s$  and  $C_a$ , with no error. The factors  $C_s$  and  $C_a$  were arbitrarily fixed at values 1.1 and 1.25 and  $l'_j, j = 1, 2$ , set to 0.

We first used identity (4.6) to get estimations of  $\epsilon$ . Indeed, in an asymptotic setting, (4.6) can be written as follows:

$$\frac{4}{3}\pi\epsilon^3\mu_a(\lambda_j)\Phi_j(z) = \frac{4}{3}\pi\epsilon\alpha_j \approx \frac{1}{3}\int_{\partial\Omega} g'_j.$$

This way to estimate the inclusion size  $\epsilon$  is slightly different from the one used in the previous subsection.

Assuming we had total boundary measurements, we then applied formula (4.5) to extract an estimation on  $\mu_a$ . Errors on the estimations in this multi-wavelength setting are given in Figs. 4 and 5.

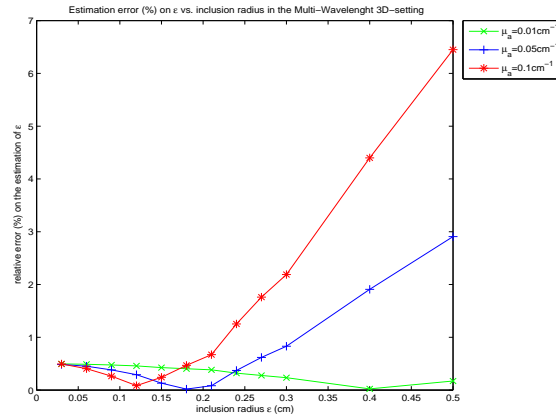


Figure 4: Error on the reconstruction of  $\epsilon$  in the multi-wavelength setting.

## 6 Concluding Remarks

Assuming that the reduced scattering coefficient is known, we have provided an asymptotic approach to estimate the normalized absorption coefficient of a small absorber from the normalized absorbed energy. We have also shown how to separate the size of the absorber from the absolute absorption coefficient by combining photo-acoustic and diffusing light measurements. In the case where the reduced scattering coefficient is unknown, we have developed a

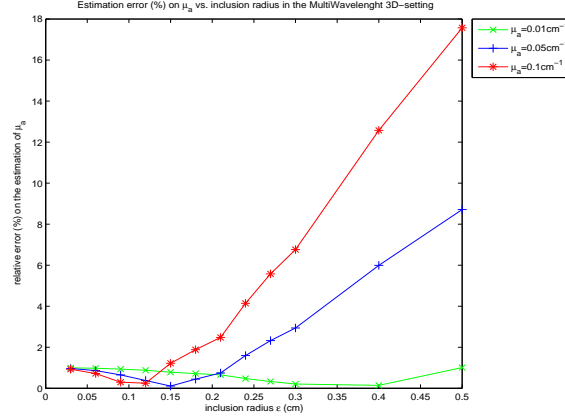


Figure 5: Error on the reconstruction of  $\mu_a$  in the multi-wavelength setting.

multi-wavelength approach to estimate the absolute absorption coefficient. Finally, it would be of interest to use the radiative transfer equation as a light transport model when the diffusion approximation breaks down. Another subject of future work is to develop a method, similar to the one derived in this paper, to estimate the normalized absorption coefficient from the normalized absorbed energy for the half-space problem. The Half-space model is of considerable practical interest in photo-acoustic imaging of the skin. See [4, 15].

## A Proof of Lemma 3.2

In this section we prove estimate (3.10). To this end, we consider

$$\begin{cases} \Delta u - i\alpha u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u = g & \text{on } \partial\Omega, \end{cases} \quad (\text{A.1})$$

where  $\alpha \geq 0$  and  $\beta > 0$ . The estimate (3.10) is an easy consequence of the following lemma.

**Lemma A.1** *Let  $y \in \Omega$  and  $r > 0$  be such that  $B_{3r} := B_{3r}(y)$  is a subset of  $\Omega$  with  $\text{dist}(B_{3r}, \partial\Omega) > c_0$  for some positive constant  $c_0$ . Here  $B_r(y)$  denotes the ball of radius  $r$  centered at  $y$ . The following estimates hold:*

- *If  $\alpha > 1$ , then there is a constant  $C$  independent of  $\alpha$  such that if  $u$  be the solution to (A.1), then*

$$\|u\|_{L^\infty(B_r)} + \alpha^{-1/2} \|\nabla u\|_{L^\infty(B_r)} + \alpha^{-1} \|\nabla \nabla u\|_{L^\infty(B_r)} \leq C \|g\|_{L^\infty(\partial\Omega)}. \quad (\text{A.2})$$

- *If  $0 \leq \alpha \leq 1$ , then is a constant  $C$  independent of  $\alpha$  such that if  $u$  be the solution to (A.1), then*

$$\|u\|_{L^\infty(B_r)} + \|\nabla u\|_{L^\infty(B_r)} + \|\nabla \nabla u\|_{L^\infty(B_r)} \leq C \|g\|_{L^\infty(\partial\Omega)}. \quad (\text{A.3})$$

*Proof.* Multiplying both sides of the equation in (A.1) by  $\bar{u}$  and integrating by parts yield

$$\beta \int_{\partial\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 dx + i\alpha \int_{\Omega} |u|^2 dx = \int_{\partial\Omega} g\bar{u}.$$

By taking the real and imaginary parts of the above identity and using the inequality  $ab < a^2/\epsilon + \epsilon b^2$ , we have

$$\beta \int_{\partial\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 dx + \alpha \int_{\Omega} |u|^2 dx \leq \frac{C}{\epsilon} \|g\|_{L^\infty(\partial\Omega)}^2 + C\epsilon \|u\|_{L^1(\partial\Omega)}^2 \quad (\text{A.4})$$

for some constant  $C$  where  $\epsilon$  is a small constant to be determined later. Since

$$\|u\|_{L^1(\partial\Omega)}^2 \leq |\partial\Omega| \|u\|_{L^2(\partial\Omega)}^2,$$

then by choosing  $\epsilon$  so small that  $C\epsilon|\partial\Omega| < \frac{\beta}{2}$ , we have

$$\frac{\beta}{2} \int_{\partial\Omega} |u|^2 d\sigma(x) + \int_{\Omega} |\nabla u|^2 dx + \alpha \int_{\Omega} |u|^2 dx \leq C \|g\|_{L^\infty(\partial\Omega)}^2. \quad (\text{A.5})$$

Let  $\varphi$  be a smooth function with a support in  $B_{3r}$  such that  $\varphi \equiv 1$  on  $B_{2r}$ , and let  $w := \varphi u$ . Then  $w$  is a smooth function (with a compact support) satisfying

$$(\Delta - i\alpha)w = 2\nabla\varphi \cdot \nabla u + \Delta\varphi u.$$

Recall that the fundamental solution to the operator  $\Delta - i\alpha$  is given by

$$\Gamma_\alpha(x) := \frac{\exp(-e^{\frac{\pi}{4}i}\sqrt{\alpha}|x|)}{4\pi|x|}, \quad x \neq 0.$$

Therefore, we have

$$w(x) = \int_{\Omega} \Gamma_\alpha(x-y)(2\nabla\varphi \cdot \nabla u + \Delta\varphi u)(y) dy, \quad x \in \Omega.$$

Note that  $2\nabla\varphi \cdot \nabla u + \Delta\varphi u$  is supported in  $B_{3r} \setminus B_{2r}$ . So (A.2) and (A.3) for  $\alpha \neq 0$  follow immediately. For  $\alpha = 0$ , we shall make use of the inequality

$$\left\| u - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u d\sigma \right\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

to obtain the desired estimate. This completes the proof.

## B Proof of Lemma 3.3

We first note that since  $p > 3$ ,  $q = \frac{p}{p-1} < \frac{3}{2}$ . For  $-1 \leq \alpha \leq 2$ , define

$$T_\alpha[f](x) = \int_D \frac{f(y)}{|x-y|^\alpha} dy, \quad x \in D.$$

By Hölder's inequality, we have

$$|T_\alpha[f](x)| \leq \|f\|_{L^p(D)} \left( \int_D \frac{1}{|x-y|^{\alpha q}} dy \right)^{1/q} \leq C \epsilon^{\frac{3}{q}-\alpha} \|f\|_{L^p(D)}.$$

It then follows that

$$\|T_\alpha[f]\|_{L^p(D)} \leq C \epsilon^{3-\alpha} \|f\|_{L^p(D)}. \quad (\text{B.1})$$

Let us put

$$\begin{aligned} \mathcal{N}_1[f](x) &:= \int_D (f(y) - f(x)) \Gamma(x-y) dy, \\ \mathcal{N}_2[f](x) &:= \int_D \nabla f(y) \cdot \nabla_y \Gamma(x-y) dy, \\ \mathcal{R}_1[f](x) &:= \int_D (f(y) - f(x)) R(x,y) dy, \\ \mathcal{R}_2[f](x) &:= \int_D \nabla f(y) \cdot \nabla_y R(x,y) dy, \end{aligned}$$

so that

$$\mathcal{N} = 3\mu_a\mu_s\mathcal{N}_1 - \frac{\mu_a}{\mu_a + \mu_s}\mathcal{N}_2 \quad \text{and} \quad \mathcal{R} = \mu_a\mathcal{R}_1 - \frac{\mu_a}{3(\mu_a + \mu_s)\mu_s}\mathcal{R}_2.$$

Note that

$$\mathcal{N}_1[f](x) = \int_D \frac{f(y) - f(x)}{|y-x|} |y-x| \Gamma(x-y) dy.$$

Since  $|y-x|\Gamma(x-y) \leq C$ , we have as in (B.1)

$$\|\mathcal{N}_1[f]\|_{L^p(D)} \leq C \epsilon^{\frac{3}{q}} \left( \int_D \int_D \frac{|f(y) - f(x)|^p}{|y-x|^p} dx dy \right)^{1/p} \leq C \epsilon^3 \|\nabla f\|_{L^p(D)}.$$

Similarly, we have

$$\|\nabla \mathcal{N}_1[f]\|_{L^p(D)} \leq C \epsilon^2 \|\nabla f\|_{L^p(D)}.$$

We also have

$$\|\mathcal{N}_2[f]\|_{L^p(D)} \leq C \epsilon \|\nabla f\|_{L^p(D)}.$$

Note that  $\nabla_x \nabla_y \Gamma(x-y)$  is a Calderón-Zygmund kernel and hence the operator  $\nabla \mathcal{N}_2$  is bounded on  $L^p$ ,  $1 < p < \infty$ . Therefore, we have

$$\|\nabla \mathcal{N}_2[f]\|_{L^p(D)} \leq C \|\nabla f\|_{L^p(D)}.$$

Therefore, we obtain (3.21) and (3.22).

Similarly, we have using (3.13)-(3.15) that

$$\begin{aligned} \|\mathcal{R}_1[f]\|_{L^p(D)} &\leq C \mu_s^{3/2} \epsilon^4 \|\nabla f\|_{L^p(D)}, \\ \|\nabla \mathcal{R}_1[f]\|_{L^p(D)} &\leq C (\mu_s^2 \epsilon^4 + \mu_s^{3/2} \epsilon^3) \|\nabla f\|_{L^p(D)}, \\ \|\mathcal{R}_2[f]\|_{L^p(D)} &\leq C (\mu_s^2 \epsilon^3 + \mu_s^{3/2} \epsilon^2) \|\nabla f\|_{L^p(D)}, \\ \|\nabla \mathcal{R}_2[f]\|_{L^p(D)} &\leq C (\mu_s^{5/2} \epsilon^3 + \mu_s^2 \epsilon^2 + \mu_s^{3/2} \epsilon) \|\nabla f\|_{L^p(D)}. \end{aligned}$$

Therefore we get

$$\begin{aligned}\|\mathcal{R}[f]\|_{L^p(D)} &\leq C \left( \mu_a \mu_s^{3/2} \epsilon^4 + \frac{\mu_a}{\mu_s^2} (\mu_s^2 \epsilon^3 + \mu_s^{3/2} \epsilon^2) \right) \|\nabla f\|_{L^p(D)}, \\ \|\nabla \mathcal{R}[f]\|_{L^p(D)} &\leq C \left( \mu_a (\mu_s^2 \epsilon^4 + \mu_s^{3/2} \epsilon^3) + \frac{\mu_a}{\mu_s^2} (\mu_s^{5/2} \epsilon^3 + \mu_s^2 \epsilon^2 + \mu_s^{3/2} \epsilon) \right) \|\nabla f\|_{L^p(D)}.\end{aligned}$$

Suppose that  $\epsilon\sqrt{\mu_s}$  is small, we obtain

$$\mu_a \mu_s^{3/2} \epsilon^4 + \frac{\mu_a}{\mu_s^2} (\mu_s^2 \epsilon^3 + \mu_s^{3/2} \epsilon^2) \leq \epsilon^2 \sqrt{\mu_s} \left( \mu_a \mu_s \epsilon^2 + \frac{\mu_a}{\mu_s} \right)$$

and

$$\mu_a (\mu_s^2 \epsilon^4 + \mu_s^{3/2} \epsilon^3) + \frac{\mu_a}{\mu_s^2} (\mu_s^{5/2} \epsilon^3 + \mu_s^2 \epsilon^2 + \mu_s^{3/2} \epsilon) \leq \epsilon \sqrt{\mu_s} \left( \mu_a \mu_s \epsilon^2 + \frac{\mu_a}{\mu_s} \right),$$

which yields (3.24) and (3.23) and thus completes the proof.  $\square$

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