Asymptotic Expansions for Eigenvalues of the Lamé System in the Presence of Small Inclusions

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Abstract

We derive a complete asymptotic expansion for eigenvalues of the Lamé system of the linear elasticity in domains with small inclusions in three dimensions. By an integral equation formulation of the solutions to the harmonic oscillatory linear elastic equation, we reduce this problem to the study of the characteristic values of integral operators in the complex planes. Generalized Rouché's theorem and other techniques from the theory of meromorphic operator-valued functions are combined with asymptotic analysis of integral kernels to obtain full asymptotic expansions for eigenvalues.

1 Introduction

Let Ω be an elastic media in $\mathbb{R}^3$ with a connected Lipschitz boundary whose Lamé constants are λ, μ. We consider the eigenvalue problem for the Lamé system of the linear elasticity:

$$\mu \triangle \mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \kappa \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1}$$

with the Neumann boundary condition $\partial \mathbf{u}/\partial \nu = 0$ on $\partial\Omega$. Here the conormal derivative $\partial \mathbf{u}/\partial \nu$ is defined to be

$$\frac{\partial \mathbf{u}}{\partial \nu} = \lambda(\nabla \cdot \mathbf{u})N + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^t)N, \tag{1.2}$$

where the superscript t denotes the transpose and N is the unit normal to the boundary $\partial\Omega$.

Suppose that Ω contains a small inclusion D of the form $D = z + \epsilon B$, where B is a bounded Lipschitz domain containing the origin, ϵ is a small parameter, and z indicates the location of the inclusion. Due to the presence of the inclusion D, the eigenvalues of the domain Ω are perturbed. Our goal is to find an asymptotic expansion for the perturbation of eigenvalues due to the presence of the inclusion. Let $\kappa_1 \leq \kappa_2 \leq \dots$ be the eigenvalues of

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(1.1) and $\kappa_1^\epsilon \leq \kappa_2^\epsilon \leq \ldots$ be the eigenvalues in the presence of the inclusion. The main result of this paper is a complete asymptotic expansion of $\kappa_j^\epsilon - \kappa_j$ as $\epsilon \to 0$.

The main ingredients in deriving the results of this paper are the integral equations and the theory of meromorphic operator-valued functions. Using integral representations of solutions to the harmonic oscillatory linear elastic equation, we reduce this problem to the study of characteristic values of integral operators in the complex planes. Generalized Rouché's theorem and other techniques from the theory of meromorphic operator-valued functions are combined with asymptotic analysis of integral kernels to obtain full asymptotic expansions for eigenvalues. This method was first used in [4] to obtain an asymptotic formula for the eigenvalues of Laplacian under the shape deformation. Another related work is [3] where complete asymptotic formulae for eigenvalues of the conductivity equation in the presence of small inclusions are derived. See also Rauch-Taylor [14], Ozawa [11, 12, 13], and Besson [6] where leading-order terms in the asymptotic expansion of the eigenvalues for the Laplacian in domains with small holes have been obtained.

The inclusions we deal with in this paper are of two kinds: hard and soft inclusions. A hard inclusion D is characterized by the boundary condition $u =$ constant on its boundary ∂D (we set the constant $= 0$), while a soft one is characterized by the transmission conditions on its boundary. The complete asymptotic expansions in both cases are given by infinite number of operators. We will calculate the leading order terms in both cases. It turns out that the leading order term for the hard inclusion is of order ϵ and expressed in terms of the eigenfunctions and some quantity related to the capacity of the inclusion. On the other hand, that for the soft inclusion is of order ϵ^3, the volume of the inclusion, and expressed in terms of the eigenfunctions and the elastic moment tensor, for which we refer to [2]. Even if it requires some endeavor (but straightforward), it is of some interest to compute the higher order terms. The second order terms of the perturbation of the eigenvalues for the Laplacian has been dereived by Ozawa [12] when the hole is of spherical shape.

The paper is organized as follows. We recall relevant definitions and state the generalized Rouché's theorem in the next section. In Section 3, we derive full asymptotic expansions for eigenvalues and calculate the leading order term explicitly when the inclusion is a hard one. In Section 4, we deal with the case of inclusions with finite Lamé constants.

Here we confine our attention to the eigenvalues of the Neumann boundary value problem. The Dirichlet boundary case can be treated in a similar way with only minor modifications of the techniques presented here. We intend to use this expansion in identifying the small inclusions by taking eigenvalue measurements.

2 Preliminaries

2.1 The generalized Rouché's theorem

In this section, we review the main results of Gohberg and Sigal in [9].

Let $\mathcal{G}$ and $\mathcal{H}$ be two Banach spaces and let $\mathcal{L}(\mathcal{G}, \mathcal{H})$ be the set of all bounded operators from $\mathcal{G}$ to $\mathcal{H}$. Let U be an open set in $\mathbb{C}$. Suppose that $A(\lambda)$ is an operator-valued function from U to $\mathcal{L}(\mathcal{G}, \mathcal{H})$. λ_0 is a *characteristic value* of $A(\lambda)$ if

- $A(\lambda)$ is holomorphic in some neighborhood of λ_0, except possibly for λ_0;
- There exists a function $\phi(\lambda)$ from a neighborhood of λ_0 to $\mathcal{G}$, holomorphic and nonzero at λ_0 such that $A(\lambda)\phi(\lambda)$ is holomorphic at λ_0 and $A(\lambda_0)\phi(\lambda_0) = 0$.

The function $\phi(\lambda)$ in the above definition is called a *root function* of $A(\lambda)$ associated to λ_0 and $\phi(\lambda_0)$ is called an *eigenvector.* The closure of the space of eigenvectors corresponding to λ_0 is denoted by $\operatorname{Ker} A(\lambda_0)$.

Let ϕ_0 be an eigenvector corresponding to λ_0. Let $V(\lambda_0)$ be a complex neighborhhod of λ_0. The *rank* of ϕ_0 is the largest integer m such that there exist $\phi : V(\lambda_0) \to \mathcal{G}$ and $\psi : V(\lambda_0) \to \mathcal{H}$ holomorphic satisfying

$$\begin{aligned} &A(\lambda)\phi(\lambda) = (\lambda - \lambda_0)^m \psi(\lambda), \\ &\phi(\lambda_0) = \phi_0, \quad \psi(\lambda_0) \neq 0. \end{aligned}$$

Suppose that $n = \dim \operatorname{Ker} A(\lambda_0) < +\infty$ and the ranks of all vectors in $\operatorname{Ker} A(\lambda_0)$ are finite. A system of eigenvectors ϕ_0^j, $j = 1, \cdots, n$, is called a *canonical system of eigenvectors* of $A(\lambda)$ associated to λ_0 if the rank of ϕ_0^j is the maximum of the ranks of all eigenvectors in some direct complement in $\operatorname{Ker} A(\lambda_0)$ of the linear span of the vectors $\phi_0^1, \cdots, \phi_0^{j-1}$. Then we define the *null multiplicity* of the characteristic value λ_0 of $A(\lambda)$ to be the sum of ranks of ϕ_0^j, $j = 1, \cdots, n$, which is denoted by $N(A(\lambda_0))$.

Suppose that $A^{-1}(\lambda)$ exists and is holomorphic in some neighborhood of λ_0, except possibly at this point itself. Then the number

$$M(A(\lambda_0)) = N(A(\lambda_0)) - N(A^{-1}(\lambda_0))$$

is called the *multiplicity* of the characteristic value λ_0 of $A(\lambda)$.

Suppose that the Laurent expansion of $A(\lambda)$ in λ_0 is given by

$$A(\lambda) = \sum_{j \geq -s} (\lambda - \lambda_0)^j A_j.$$

If the operators A_j, $j = -s, \cdots, -1$, are finite dimensional, then $A(\lambda)$ is called *finitely meromorphic* at λ_0. If the operator A_0 is a Fredholm operator, $A(\lambda)$ is said to be of *Fredholm type* at λ_0.

If $A(\lambda)$ is holomorphic at λ_0 and $A(\lambda_0)$ is invertible, the λ_0 is called a *regular point* of $A(\lambda)$. A point λ_0 is called a *normal point* of $A(\lambda)$ if $A(\lambda)$ is finitely meromorphic and of Fredholm type at λ_0 and there exists some neighborhood $V(\lambda_0)$ of λ_0 in which all the points except λ_0 are regular points of $A(\lambda)$.

Lemma 2.1 *Every normal point λ_0 of $A(\lambda)$ is a normal point of $A^{-1}(\lambda)$.*

An operator-valued function $A(\lambda)$ which is finitely meromorphic and of Fredholm type in $V(\lambda_0)$ and continuous at $\partial V(\lambda_0)$ is called *normal* with respect to $\partial V(\lambda_0)$ if $A(\lambda)$ is invertible in $\overline{V(\lambda_0)}$, except for a finite number of points of $V(\lambda_0)$ which are normal points of $A(\lambda)$.

Suppose that $A(\lambda)$ is normal with respect to $\partial V(\lambda_0)$ and λ_i, $i = 1, \cdots, \sigma$, are all its characteristic values and poles lying in $V(\lambda_0)$, we put

$$\mathcal{M}(A(\lambda); \partial V(\lambda_0)) = \sum_{i=1}^{\sigma} M(A(\lambda_i)).$$

The generalization of Rouché's theorem is stated below.

Theorem 2.2 *Let* $A(\lambda)$ *be an operator-valued function which is normal with respect to* $\partial V(\lambda_0)$*. If an operator-valued function* $S(\lambda)$ *which is finitely meromorphic in* $V(\lambda_0)$ *and continuous at* $\partial V(\lambda_0)$ *satisfies the condition*

$$\|A^{-1}(\lambda)S(\lambda)\|_{\mathcal{L}(\mathcal{G},\mathcal{H})} < 1, \quad \lambda \in \partial V(\lambda_0),$$

then $A(\lambda) + S(\lambda)$ *is also normal with respect to* $\partial V(\lambda_0)$ *and*

$$\mathcal{M}(A(\lambda); \partial V(\lambda_0)) = \mathcal{M}(A(\lambda) + S(\lambda); \partial V(\lambda_0)).$$

The generalization of Steinberg theorem is given by

Theorem 2.3 *Suppose that* $A(\lambda)$ *is an operator-valued function which is finitely meromorphic and of Fredholm type in* $V(\lambda_0)$*. If* $A(\lambda)$ *is invertible at one point of* $V(\lambda_0)$*, then* $A(\lambda)$ *has a bounded inverse for all* $\lambda \in \partial V(\lambda_0)$*, except possibly for certain isolated points.*

Theorem 2.4 *Suppose that* $A(\lambda)$ *is an operator-valued function which is normal with respect to* $\partial V(\lambda_0)$*. Let* $f(\lambda)$ *be a scalar function which is holomorphic in* $V(\lambda_0)$ *and continuous in* $\overline{V(\lambda_0)}$*. Then we have*

$$\frac{1}{2\pi i} \operatorname{tr} \int_{\partial V(\lambda_0)} f(\lambda) A^{-1}(\lambda) \frac{d}{d\lambda} A(\lambda) d\lambda = \sum_{j=1}^{\sigma} M(A(\lambda_j)) f(\lambda_j),$$

where λ_j*,* $j = 1, \cdots, \sigma$*, are all the points in* $V(\lambda_0)$ *which are either poles or characteristic values of* $A(\lambda)$*.*

Here tr denotes the trace of operator which is the sum of all its nonzero eigenvalues. We mention the following property of the trace:

$$\operatorname{tr} \int_{\partial V(\lambda_0)} A(\lambda) B(\lambda) d\lambda = \operatorname{tr} \int_{\partial V(\lambda_0)} B(\lambda) A(\lambda) d\lambda, \tag{2.1}$$

where $A(\lambda)$ and $B(\lambda)$ are operator-valued functions which are finitely meromorphic in $V(\lambda_0)$, which contains no poles of $A(\lambda)$ and $B(\lambda)$ other than λ_0.

2.2 Integral representation of solutions to the Lamé system

Let Ω be a bounded domain in $\mathbb{R}^3$ with a connected Lipschitz boundary. Let λ, μ be the Lamé constants for Ω satisfying

$$\mu > 0 \ \text{ and } \ 3\lambda + 2\mu > 0.$$

The corresponding Lamé system is given by

$$\mathcal{L}^{\lambda,\mu}\mathbf{u} = \mu \triangle \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u},$$

and the conormal derivative is defined to be

$$\frac{\partial \mathbf{u}}{\partial \nu} = \lambda (\nabla \cdot \mathbf{u}) N + \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^t) N,$$

where N is the unit normal to the boundary.

The Kupradze matrix $\mathbf{\Gamma}^\omega = (\Gamma^\omega_{ij})_{i,j=1}^3$ of the fundamental solution to the operator $\mathcal{L}^{\lambda,\mu} + \omega^2$ is given by

$$\Gamma^\omega_{ij}(x) = -\frac{\delta_{ij}}{4\pi\mu|x|} e^{\frac{i\omega|x|}{c_T}} + \frac{1}{4\pi\omega^2}\partial_i\partial_j \frac{e^{\frac{i\omega|x|}{c_L}} - e^{\frac{i\omega|x|}{c_T}}}{|x|},$$

where δ_{ij} is the Kronecker's delta, ∂_j denotes $\partial/\partial x_j$, and

$$c_T = \sqrt{\mu}, \quad c_L = \sqrt{\lambda + 2\mu}.$$

See [10, Chap. 2]. One can easily see that Γ^ω_{ij} allows the following expansion:

$$\begin{aligned} \Gamma^\omega_{ij}(x) = &-\frac{1}{4\pi}\sum_{n=0}^{+\infty} \frac{i^n}{(n+2)n!}\Big(\frac{n+1}{c_T^{n+2}} + \frac{1}{c_L^{n+2}}\Big)\omega^n \delta_{ij}|x|^{n-1} \\ &+ \frac{1}{4\pi}\sum_{n=0}^{+\infty} \frac{i^n(n-1)}{(n+2)n!}\Big(\frac{1}{c_T^{n+2}} - \frac{1}{c_L^{n+2}}\Big)\omega^n |x|^{n-3} x_i x_j. \end{aligned} \tag{2.2}$$

If $\omega = 0$, then $\mathbf{\Gamma}^0$ is the Kelvin matrix of the fundamental solution to the Lamé system, *i.e.*,

$$\Gamma^0_{ij}(x) = -\frac{\gamma_1}{4\pi}\frac{\delta_{ij}}{|x|} - \frac{\gamma_2}{4\pi}\frac{x_i x_j}{|x|^3}, \tag{2.3}$$

where

$$\gamma_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad \gamma_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right). \tag{2.4}$$

The single and double layer potentials are defined by

$$\mathcal{S}^\omega_D\varphi(x) = \int_{\partial D} \mathbf{\Gamma}^\omega(x-y)\varphi(y)d\sigma(y), \tag{2.5}$$

$$\mathcal{D}^\omega_D\varphi(x) = \int_{\partial D} \frac{\partial}{\partial \nu_y}\mathbf{\Gamma}^\omega(x-y)\varphi(y)d\sigma(y), \tag{2.6}$$

for any $\varphi \in L^2(\partial D)^3$. The following formulae give the jump relations obeyed by the double layer potential and by the conormal derivative of the single layer potential:

$$\frac{\partial(\mathcal{S}^\omega_D\varphi)}{\partial\nu}\Big|_\pm(x) = \Big(\pm\frac{1}{2}I + (\mathcal{K}^\omega_\Omega)^*\Big)\varphi(x), \quad \text{a.e. } x \in \partial D, \tag{2.7}$$

$$(\mathcal{D}^\omega_D\varphi)\Big|_\pm(x) = \Big(\mp\frac{1}{2}I + \mathcal{K}^\omega_\Omega\Big)\varphi(x), \quad \text{a.e. } x \in \partial D, \tag{2.8}$$

where $\mathcal{K}^\omega_\Omega$ is the operator defined by

$$\mathcal{K}^\omega_\Omega(x) = \text{p.v.}\int_{\partial D} \frac{\partial\mathbf{\Gamma}^\omega(x-y)}{\partial\nu_y}\varphi(y)d\sigma(y), \tag{2.9}$$

and $(\mathcal{K}^\omega_\Omega)^*$ is the L^2-adjoint of $\mathcal{K}^\omega_\Omega$, that is,

$$(\mathcal{K}^\omega_\Omega)^*(x) = \text{p.v.}\int_{\partial D} \frac{\partial\mathbf{\Gamma}^\omega(x-y)}{\partial\nu_x}\varphi(y)d\sigma(y).$$

Here p.v. means the Cauchy principal value. See [10, 7].

Let Ψ be the vector space of all linear solutions to the equation $\mathcal{L}^{\lambda,\mu}\mathbf{u} = 0$ and $\dfrac{\partial \mathbf{u}}{\partial \nu} = 0$ on ∂D, or alternatively,

$$\Psi = \big\{\psi : \partial_i \psi_j + \partial_j \psi_i = 0,\ 1 \le i, j \le 3\big\}.$$

Define a subspace of $L^2(\partial D)^3$ by

$$L^2_\Psi(\partial D) = \Big\{\mathbf{f} \in L^2(\partial D)^3 : \int_{\partial D} \mathbf{f} \cdot \psi d\sigma = 0 \text{ for all } \psi \in \Psi\Big\}.$$

In particular, since Ψ contains constant functions, we get

$$\int_{\partial D} \mathbf{f} d\sigma = 0$$

for any $\mathbf{f} \in L^2_\Psi(\partial D)$. We also know that if $\mathbf{u}$ is smooth and satisfies $\mathcal{L}^{\lambda,\mu}\mathbf{u} = 0$ in D, then $\dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_{\partial D} \in L^2_\Psi(\partial D)$.

Let us formulate the *radiation conditions* for the elastic waves when $\operatorname{Im} \omega \ge 0$ and $\omega \ne 0$. Any solution $\mathbf{u}$ to $(\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{u} = 0$ admits the decomposition, see [10, Theorem 2.5],

$$\mathbf{u} = \mathbf{u}^{(p)} + \mathbf{u}^{(s)}, \tag{2.10}$$

where $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ are given by

$$\begin{aligned}\mathbf{u}^{(p)} &= (k_T^2 - k_L^2)^{-1}(\triangle + k_T^2)\mathbf{u},\\ \mathbf{u}^{(s)} &= (k_L^2 - k_T^2)^{-1}(\triangle + k_L^2)\mathbf{u},\end{aligned}$$

with $k_L = \frac{\omega}{c_L}$ and $k_T = \frac{\omega}{c_T}$. Then $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ satisfy the equations

$$\begin{cases}(\triangle + k_T^2)\mathbf{u}^{(p)} = 0, & \nabla \times \mathbf{u}^{(p)} = 0,\\ (\triangle + k_L^2)\mathbf{u}^{(s)} = 0, & \nabla \cdot \mathbf{u}^{(s)} = 0.\end{cases} \tag{2.11}$$

We impose on $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ the Sommerfeld radiation conditions for solutions of the Helmholtz equations by requiring that

$$\begin{cases}\partial_r \mathbf{u}^{(p)}(x) - ik_T \mathbf{u}^{(p)}(x) = o(r^{-1}),\\ \partial_r \mathbf{u}^{(s)}(x) - ik_L \mathbf{u}^{(s)}(x) = o(r^{-1}),\end{cases} \quad \text{as } r = |x| \to +\infty. \tag{2.12}$$

We say that $\mathbf{u}$ satisfies the radiation condition if it allows the decomposition (2.10) with $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(s)}$ satisfying (2.11) and (2.12). By a straightforward calculation one can see that single and double layer potentials satisfy the radiation condition. We refer to [1, 10] for details.

Let $\tilde{\lambda}, \tilde{\mu}$ be another pair of Lamé parameters such that

$$(\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) \ge 0, \quad (\lambda - \tilde{\lambda})^2 + (\mu - \tilde{\mu})^2 \ne 0. \tag{2.13}$$

Let $\tilde{\mathcal{S}}^\omega_D$ denote the single layer potential defined by (2.5) with λ, μ replaced by $\tilde{\lambda}, \tilde{\mu}$. We also denote by $\dfrac{\partial \mathbf{u}}{\partial \tilde{\nu}}$ the conormal derivative with respect to $\tilde{\lambda}, \tilde{\mu}$. Let the space $H^1(\partial D)$ be the set of functions $f \in L^2(\partial D)$ such that $\partial f / \partial T \in L^2(\partial D)$, where $\partial/\partial T$ denotes the tangential derivative on ∂D.

We now state the following solvability theorem.

Theorem 2.5 *Suppose that* $(\lambda-\tilde{\lambda})(\mu-\tilde{\mu}) \geq 0$ *and* $0 < \tilde{\lambda}, \tilde{\mu} < +\infty$. *Suppose that* $\operatorname{Im} \omega \geq 0$ *and* ω^2 *is not a Dirichlet eigenvalue for* $-\mathcal{L}^{\lambda,\mu}$ *on* D. *For any given* $(\mathbf{F}, \mathbf{G}) \in H^1(\partial D)^3 \times L^2(\partial D)^3$, *there exists a unique pair* $(\mathbf{f}, \mathbf{g}) \in L^2(\partial D)^3 \times L^2(\partial D)^3$ *such that*

$$\begin{cases} \tilde{\mathcal{S}}_D^\omega \mathbf{f}|_- - \mathcal{S}_D^\omega \mathbf{g}|_+ = \mathbf{F}, \\ \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_D^\omega \mathbf{f}|_- - \dfrac{\partial}{\partial \nu} \mathcal{S}_D^\omega \mathbf{g}|_+ = \mathbf{G}. \end{cases}$$

If $\omega = 0$, *we have that* $\mathbf{G} \in L^2_\Psi(\partial D)$ *implies* $\mathbf{g} \in L^2_\Psi(\partial D)$ *and moreover,* $\mathbf{F} \in \Psi$ *and* $\mathbf{G} = 0$ *imply* $\mathbf{g} = 0$.

Proof. For $\omega = 0$, the theorem is proved in [8].

The case $\omega \neq 0$ can be treated as a compact perturbation of the case $\omega = 0$. In fact, let us define an operator $T : L^2(\partial D)^3 \times L^2(\partial D)^3 \to H^1(\partial D)^3 \times L^2(\partial D)^3$ by

$$T(\mathbf{f}, \mathbf{g}) := \left(\tilde{\mathcal{S}}_D^\omega \mathbf{f}|_- - \mathcal{S}_D^\omega \mathbf{g}|_+ ,\ \frac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_D^\omega \mathbf{f}|_- - \frac{\partial}{\partial \nu} \mathcal{S}_D^\omega \mathbf{g}|_+ \right).$$

We also define T_0 by

$$T_0(\mathbf{f}, \mathbf{g}) := \left(\tilde{\mathcal{S}}_D^0 \mathbf{f}|_- - \mathcal{S}_D^0 \mathbf{g}|_+ ,\ \frac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_D^0 \mathbf{f}|_- - \frac{\partial}{\partial \nu} \mathcal{S}_D^0 \mathbf{g}|_+ \right).$$

It is easily checked that $T - T_0$ is a compact operator. Since we know that T_0 is invertible, by the Fredholm alternative, it is enough to show that T is injective. Suppose that $T(\mathbf{f}, \mathbf{g}) = 0$. Then the function $\mathbf{u}$ given by

$$\mathbf{u} := \begin{cases} \mathcal{S}_D^\omega \mathbf{g}(x), & x \in \mathbb{R}^3 \setminus D, \\ \tilde{\mathcal{S}}_D^\omega \mathbf{f}(x), & x \in D, \end{cases}$$

is the solution to the transmission problem

$$\begin{cases} \mathcal{L}^{\lambda,\mu} \mathbf{u} + \omega^2 \mathbf{u} = 0, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{L}^{\tilde{\lambda},\tilde{\mu}} \mathbf{u} + \omega^2 \mathbf{u} = 0, & \text{in } D, \\ \mathbf{u}\big|_+ - \mathbf{u}\big|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_+ - \dfrac{\partial \mathbf{u}}{\partial \tilde{\nu}}\Big|_- = 0, & \text{on } \partial D, \end{cases}$$

satisfying the radiation condition. By the uniqueness of a solution to this transmission problem, see for instance [10, Chap.3], we have $\mathbf{u} = 0$. From the assumption on ω, we conclude that $\mathbf{f} = \mathbf{g} = 0$. This completes the proof. □

In Section 4, we are primarily concerned with the following transmission problem:

$$\begin{cases} \mathcal{L}^{\lambda,\mu} \mathbf{u} + \omega^2 \mathbf{u} = 0, & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}^{\tilde{\lambda},\tilde{\mu}} \mathbf{u} + \omega^2 \mathbf{u} = 0, & \text{in } D, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = \mathbf{g}, & \text{on } \partial \Omega, \\ \mathbf{u}\big|_+ - \mathbf{u}\big|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial \mathbf{u}}{\partial \nu}\Big|_+ - \dfrac{\partial \mathbf{u}}{\partial \tilde{\nu}}\Big|_- = 0, & \text{on } \partial D. \end{cases} \tag{2.14}$$

For this problem we have the following representation formula.

Theorem 2.6 *Let* $\operatorname{Im}\omega \geq 0$. *Suppose that* ω^2 *is not a Dirichlet eigenvalue for* $-\mathcal{L}^{\lambda,\mu}$ *on* D. *Let* $\mathbf{u}$ *be a solution of (2.14) and* $\mathbf{f} := \mathbf{u}|_{\partial\Omega}$. *Define*

$$\mathbf{H}(x) := \mathcal{D}_D^\omega(\mathbf{f})(x) - \mathcal{S}_D^\omega(\mathbf{g})(x), \quad x \in \mathbb{R}^3 \setminus \partial\Omega. \tag{2.15}$$

Then $\mathbf{u}$ *can be represented as*

$$\mathbf{u}(x) = \begin{cases} \mathbf{H}(x) + \mathcal{S}_D^\omega \psi(x), & x \in \Omega \setminus \overline{D}, \\ \tilde{\mathcal{S}}_D^\omega \phi(x), & x \in D, \end{cases} \tag{2.16}$$

where the pair $(\phi, \psi) \in L^2(\partial D)^3 \times L^2(\partial D)^3$ *is the unique solution of*

$$\begin{cases} \tilde{\mathcal{S}}_D^\omega \phi - \mathcal{S}_D^\omega \psi = \mathbf{H}|_{\partial D}, \\ \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_D^\omega \phi - \dfrac{\partial}{\partial \nu} \mathcal{S}_D^\omega \psi = \dfrac{\partial \mathbf{H}}{\partial \nu}\Big|_{\partial D}. \end{cases} \tag{2.17}$$

Moreover, we have

$$\mathbf{H}(x) + \mathcal{S}_D^\omega \psi(x) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}. \tag{2.18}$$

Proof. We consider the following two phases transmission problem:

$$\begin{cases} \mathcal{L}^{\lambda,\mu} \mathbf{v} + \omega^2 \mathbf{v} = 0, & \text{in } \ (\Omega \setminus \overline{D}) \bigcup (\mathbb{R}^3 \setminus \overline{\Omega}), \\ \mathcal{L}^{\tilde{\lambda},\tilde{\mu}} \mathbf{v} + \omega^2 \mathbf{v} = 0, & \text{in } \ D, \\ \mathbf{v}\big|_- - \mathbf{v}\big|_+ = \mathbf{f}, \ \dfrac{\partial \mathbf{v}}{\partial \nu}\Big|_- - \dfrac{\partial \mathbf{v}}{\partial \nu}\Big|_+ = \mathbf{g}, & \text{on } \ \partial\Omega, \\ \mathbf{v}\big|_- - \mathbf{v}\big|_+ = 0, \ \dfrac{\partial \mathbf{v}}{\partial \tilde{\nu}}\Big|_- - \dfrac{\partial \mathbf{v}}{\partial \nu}\Big|_+ = 0, & \text{on } \ \partial D \end{cases} \tag{2.19}$$

with the radiation condition. This problem has a unique solution. See [10, Chap.3]. It is easily checked that both $\mathbf{v}$ and $\tilde{\mathbf{v}}$ defined by

$$\mathbf{v}(x) = \begin{cases} \mathbf{u}(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases} \quad \text{and} \quad \tilde{\mathbf{v}}(x) = \begin{cases} \mathbf{H}(x) + \mathcal{S}_D^\omega \psi(x), & x \in \Omega \setminus \overline{D}, \\ \tilde{\mathcal{S}}_D^\omega \phi(x), & x \in D, \end{cases}$$

are solutions to (2.19). Hence $\mathbf{v} = \tilde{\mathbf{v}}$. This completes the proof. □

3 Hard inclusion case

Let κ be the eigenvalue of $-\mathcal{L}^{\lambda,\mu}$ in Ω with the Neumann condition on $\partial\Omega$ and let $\mathbf{u}$ denote the eigenfunction associated with κ, *i.e.*,

$$\begin{cases} \mathcal{L}^{\lambda,\mu} \mathbf{u} + \kappa \mathbf{u} = 0 & \text{in } \ \Omega \setminus \overline{D}, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{3.1}$$

We note that since $-\mathcal{L}^{\lambda,\mu}$ is elliptic, it has discrete eigenvalues of finite multiplicities. The following lemma from [10, Chapter 7] is of importance to us.

Lemma 3.1 *The necessary and sufficient condition for (3.1) to have a nontrivial solution is that κ is nonnegative and $\sqrt{\kappa}$ coincides with one of the characteristic values of $\frac{1}{2}I - \mathcal{K}_\Omega^\omega$. If $\kappa = \omega_0^2$ is a m-fold eigenvalue of (3.1), then $\big(\frac{1}{2}I - \mathcal{K}_\Omega^{\omega_0}\big)\mathbf{u} = 0$ has m linearly independent solutions. Moreover, for every eigenvalue $\kappa > 0$, $\sqrt{\kappa}$ is a simple pole of the operator-valued function $\omega \mapsto \big(\frac{1}{2}I - \mathcal{K}_\Omega^\omega\big)^{-1}$.*

Let $0 \leq \kappa_1 \leq \kappa_2 \leq \cdots$ be the eigenvalues of $-\mathcal{L}^{\lambda,\mu}$ in Ω with the Neumann condition on $\partial\Omega$. For $\omega \notin \{\sqrt{\kappa_j}\}_{j\geq 1}$, let $\mathbf{N}_\Omega^\omega(x,z)$ be the Neumann function for $\mathcal{L}^{\lambda,\mu} + \omega^2$ in Ω corresponding to a Dirac mass at z. That is, $\mathbf{N}_\Omega^\omega$ is the solution to

$$\begin{cases} (\mathcal{L}^{\lambda,\mu} + \omega^2)\mathbf{N}_\Omega^\omega(x,z) = -\delta_z(x)\mathbf{I}, & x \in \Omega, \\ \dfrac{\partial \mathbf{N}_\Omega^\omega}{\partial \nu}(x,z) = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Then the following relation holds (see [2]):

$$\Big(-\frac{1}{2}\mathbf{I} + \mathcal{K}_\Omega^\omega\Big)(\mathbf{N}_\Omega^\omega(\cdot,z)(x) = \mathbf{\Gamma}^\omega(x,z), \quad x \in \partial\Omega, \quad z \in \Omega. \tag{3.3}$$

Let $\mathbf{u}_j$ denotes the orthogonal eigenfunction associated with κ_j, with $\|\mathbf{u}_j\|_{L^2(\Omega)} = 1$. Then we have the following spectral decomposition:

$$\mathbf{N}_\Omega^\omega(x,z) = \sum_{j=1}^{+\infty} \frac{\mathbf{u}_j(x)\mathbf{u}_j(z)^t}{\kappa_j - \omega^2}. \tag{3.4}$$

Here we regard $\mathbf{u}_j$ as a column vector, and hence $\mathbf{u}_j(x)\mathbf{u}_j(z)^t$ is a 3×3 matrix-valued function. We refer the reader to [15] for its proof.

Suppose that Ω contains a small hard inclusion D of the form $D = z + \epsilon B$, where B is a bounded Lipschitz domain in $\mathbb{R}^3$ containing the origin. Let κ^ϵ be an Neumann eigenvalue of $-\mathcal{L}^{\lambda,\mu}$ in the presence of the inclusion, and let $\mathbf{u}^\epsilon$ be a corresponding eigenfunction, *i.e.*,

$$\begin{cases} \mathcal{L}^{\lambda,\mu}\mathbf{u}^\epsilon + \kappa^\epsilon \mathbf{u}^\epsilon = 0 & \text{in } \Omega \setminus \overline{D}, \\ \dfrac{\partial \mathbf{u}^\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \mathbf{u}^\epsilon = 0 & \text{on } \partial D. \end{cases} \tag{3.5}$$

Let $\psi := \mathbf{u}^\epsilon|_{\partial\Omega}$, $\phi := \frac{\partial \mathbf{u}^\epsilon}{\partial \nu}|_{\partial D}$, and $\omega = \sqrt{\kappa^\epsilon}$. By the Green's formula, one can see that the solution $\mathbf{u}^\epsilon$ of (3.5) can be represented as

$$\mathbf{u}^\epsilon = \mathcal{D}_\Omega^\omega \psi + \mathcal{S}_D^\omega \phi, \tag{3.6}$$

where ψ and ϕ satisfy

$$\begin{cases} (\dfrac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega)\psi - \mathcal{S}_D^\omega \phi = 0 & \text{on } \partial\Omega, \\ \mathcal{D}_\Omega^\omega \psi + \mathcal{S}_D^\omega \phi = 0 & \text{on } \partial D. \end{cases} \tag{3.7}$$

Conversely, if a nonzero pair $(\psi, \phi) \in L^2(\partial\Omega)^3 \times L^2(\partial\Omega)^3$ satisfies (3.7), then $\mathbf{u}$ defined by (3.6) is the solution to (3.5). We can assume that ω^2 is not a Dirichlet eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ on D since the Dirichlet-eigenvalues for D tend to $+\infty$ as ϵ tends to 0. In this case, the resulting $\mathbf{u}$ is also nonzero. Therefore the square roots of the eigenvalues correspond exactly to the characteristic values of the following operator-valued function:

$$\omega \mapsto \begin{pmatrix} (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega) & -\mathcal{S}_D^\omega \\ \mathcal{D}_\Omega^\omega & \mathcal{S}_D^\omega \end{pmatrix}.$$

Lemma 3.2 *For $\psi \in L^2(\partial\Omega)^3$ and $\phi \in L^2(\partial D)^3$ let $\tilde{\phi}(x) = \epsilon\phi(\epsilon x + z)$ for $x \in \partial B$. Then we have*

$$\mathcal{S}_D^\omega \phi(x) = \sum_{n=0}^{+\infty} (-1)^n \epsilon^{n+1} \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \mathbf{\Gamma}^\omega (x-z) \int_{\partial B} y^\alpha \tilde{\phi}(y) d\sigma(y), \quad x \in \partial\Omega, \tag{3.8}$$

$$\mathcal{D}_\Omega^\omega \psi(\epsilon x + z) = \sum_{n=0}^{+\infty} \epsilon^n \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha (\mathcal{D}_\Omega^\omega \psi)(z) x^\alpha, \quad x \in \partial B, \tag{3.9}$$

and for $x \in \partial B$ and $i = 1,2,3$,

$$\begin{aligned} \mathcal{S}_D^\omega(\phi)_i(\epsilon x + z) = &-\frac{1}{4\pi} \sum_{n=0}^{+\infty} \frac{i^n}{(n+2)n!} \Big(\frac{n+1}{c_T^{n+2}} + \frac{1}{c_L^{n+2}} \Big) (\epsilon\omega)^n \delta_{ij} \int_{\partial B} |x-y|^{n-1} \tilde{\phi}_j(y) d\sigma(y) \\ &+ \frac{1}{4\pi} \sum_{n=0}^{+\infty} \frac{i^n (n-1)}{(n+2)n!} \Big(\frac{1}{c_T^{n+2}} - \frac{1}{c_L^{n+2}} \Big) (\epsilon\omega)^n \int_{\partial B} |x-y|^{n-3} (x_i - y_i)(x_j - y_j) \tilde{\phi}_j(y) d\sigma(y), \end{aligned} \tag{3.10}$$

where $\mathcal{S}_D^\omega(\phi)_i$ denotes the i-th component of $\mathcal{S}_D^\omega(\phi)$.

Proof. The series (3.9) is exactly a Taylor expansion of $\mathcal{D}_\Omega^\omega(\psi)(\epsilon x + z)$ at z. By a change of variables, we have that for any $x \in \partial\Omega$,

$$\mathcal{S}_D^\omega \phi(x) = \int_{\partial D} \mathbf{\Gamma}^\omega (x - \tilde{y}) \phi(\tilde{y}) d\sigma(\tilde{y}) = \epsilon \int_{\partial B} \mathbf{\Gamma}^\omega (x - z - \epsilon y) \tilde{\phi}(y) d\sigma(y).$$

Using the Taylor expansion of $\mathbf{\Gamma}^\omega(x - z - \epsilon y)$ at $x - z$, we obtain (3.8). Similarly, (3.10) immediately follows from a change of variables and (2.2). This completes the proof. □

By Lemma 3.2, (3.7) can be written as

$$\mathcal{A}_\epsilon^\omega \begin{pmatrix} \psi \\ \tilde{\phi} \end{pmatrix} = 0, \qquad \mathcal{A}_\epsilon^\omega = \sum_{n=0}^{+\infty} (\omega\epsilon)^n \mathcal{A}_n^\omega,$$

where

$$\mathcal{A}_0^\omega = \begin{pmatrix} (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega) & 0 \\ \mathcal{D}_\Omega^\omega(\cdot)(z) & \mathcal{S}_B^0 \end{pmatrix},$$

and for $n = 1, 2, \ldots,$

$$\mathcal{A}_n^\omega = \begin{pmatrix} 0 & \dfrac{(-1)^n}{\omega^n} \displaystyle\sum_{|\alpha|=n-1} \frac{1}{\alpha!} \partial^\alpha \mathbf{\Gamma}^\omega (x-z) \int_{\partial B} y^\alpha \cdot d\sigma(y) \\ \dfrac{1}{\omega^n} \displaystyle\sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha \mathcal{D}_\Omega^\omega(\cdot)(z) x^\alpha & \mathbf{S}_n \end{pmatrix}.$$

Here the operator $\mathbf{S}_n$ is given by

$$\mathbf{S}_n(\tilde{\phi})_i = \sum_{j=1}^{3} \big(\mathbf{S}_n\big)_{ij}(\tilde{\phi}_j)$$

for $\tilde{\phi} \in L^2(\partial B)^3$ and $i = 1, 2, 3$ where

$$\begin{aligned}\big(\mathbf{S}_n\big)_{ij} = &-\frac{1}{4\pi}\frac{i^n}{(n+2)n!}\Big(\frac{n+1}{c_T^{n+2}} + \frac{1}{c_L^{n+2}}\Big)\delta_{ij}\int_{\partial B}|x-y|^{n-1}\cdot d\sigma(y)\\ &+\frac{1}{4\pi}\frac{i^n(n-1)}{(n+2)n!}\Big(\frac{1}{c_T^{n+2}} - \frac{1}{c_L^{n+2}}\Big)\int_{\partial B}|x-y|^{n-3}(x_i-y_i)(x_j-y_j)\cdot d\sigma(y).\end{aligned}$$

Then the following assertion holds.

Lemma 3.3 *For each eigenvalue κ_j of $-\mathcal{L}^{\lambda,\mu}$ and sufficiently small ϵ, there exists a small neighborhood V_j of $\sqrt{\kappa_j}$ such that $\mathcal{A}_\epsilon^\omega$ is normal with respect to ∂V_j and $\mathcal{M}(\mathcal{A}_\epsilon^\omega, \partial V_j) =$* $\dim \operatorname{Ker}\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^{\sqrt{\kappa_j}}\big)$.

Proof. We note that each operator $\mathcal{A}_n^\omega$ maps $L^2(\partial\Omega)^3 \times L^2(\partial B)^3$ into $L^2(\partial\Omega)^3 \times H^1(\partial B)^3$. We also know that $\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^0$ is a Fredholm operator of index 0 and $\mathcal{S}_B^0 : L^2(\partial B)^3 \rightarrow H^1(\partial B)^3$ is invertible (See [1]). Combining these facts with Lemma 3.1 we see that $\mathcal{A}_0^\omega$ is normal in a small neighborhood V_j of $\sqrt{\kappa_j}$. Moreover, the order of $\sqrt{\kappa_j}$ as a pole of $\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega\big)^{-1}$ is precisely the maximum of the ranks of the eigenvectors in $\operatorname{Ker}\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^{\sqrt{\kappa_j}}\big)$. Hence $\mathcal{M}(\mathcal{A}_0^\omega, \partial V_j)$ is equal to $\dim\operatorname{Ker}\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^{\sqrt{\kappa_j}}\big)$. By Theorem 2.2, the proof is complete. □

Finally we obtain the following full asymptotic formula.

Theorem 3.4 *Let κ_j be a simple Neumann eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ in Ω without the inclusion and κ_j^ϵ be that with the inclusion. Let $\omega_0 := \sqrt{\kappa_j}$ and $\omega_\epsilon := \sqrt{\kappa_j^\epsilon}$. Then we have*

$$\omega_\epsilon - \omega_0 = \frac{1}{2\pi i}\sum_{p=1}^{+\infty}\frac{1}{p}\sum_{n=p}^{+\infty}\epsilon^n \operatorname{tr}\int_{\partial V_j}\mathcal{B}_{n,p}(\omega)d\omega \tag{3.11}$$

where

$$\mathcal{B}_{n,p}(\omega) = (-1)^p \sum_{\substack{n_1+\cdots+n_p=n\\ n_i\geq 1}} (\mathcal{A}_0^\omega)^{-1}\mathcal{A}_{n_1}^\omega \cdots (\mathcal{A}_0^\omega)^{-1}\mathcal{A}_{n_p}^\omega \omega^n. \tag{3.12}$$

Proof. By Lemma 3.3, ω_ϵ is a characteristic value of $\mathcal{A}_\epsilon^\omega$ with multiplicity 1. Applying Theorem 2.4, we get

$$\omega_\epsilon - \omega_0 = \frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega - \omega_0)(\mathcal{A}_\epsilon^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_\epsilon^\omega d\omega.$$

For ϵ, small enough, the following Neumann series converges uniformly in ω in ∂V_j:

$$(\mathcal{A}_\epsilon^\omega)^{-1} = \sum_{p=0}^{+\infty}\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega - \mathcal{A}_\epsilon^\omega)\Big]^p(\mathcal{A}_0^\omega)^{-1}.$$

By (2.1) and the relation

$$\frac{d}{d\omega}(\mathcal{A}_0^\omega)^{-1} = -(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_0^\omega(\mathcal{A}_0^\omega)^{-1},$$

we have

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\frac{1}{p}\frac{d}{d\omega}\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^p d\omega\\
&=\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^{p-1}(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)d\omega\\
&\quad-\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^{p}(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_0^\omega d\omega.
\end{aligned}$$

Summing over p, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\sum_{p=1}^{+\infty}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\frac{1}{p}\frac{d}{d\omega}\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^p d\omega\\
&=-\frac{1}{2\pi i}\sum_{p=0}^{+\infty}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^{p}(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_\epsilon^\omega d\omega\\
&\quad+\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_0^\omega d\omega.
\end{aligned}$$

Since

$$\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_0^\omega d\omega=0,$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\frac{d}{d\omega}\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^p d\omega\\
&=-\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^p d\omega,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2\pi i}\sum_{p=1}^{+\infty}\operatorname{tr}\int_{\partial V_j}\frac{1}{p}\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^p d\omega\\
&=\frac{1}{2\pi i}\sum_{p=0}^{+\infty}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)\Big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\Big]^{p}(\mathcal{A}_0^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_\epsilon^\omega d\omega.\\
&=\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_j}(\omega-\omega_0)(\mathcal{A}_\epsilon^\omega)^{-1}\frac{d}{d\omega}\mathcal{A}_\epsilon^\omega d\omega.
\end{aligned}$$

By expanding $\big[(\mathcal{A}_0^\omega)^{-1}(\mathcal{A}_0^\omega-\mathcal{A}_\epsilon^\omega)\big]^p$, we obtain the desired result. □

We emphasize that the expansion (3.11) is complete, and all the terms can be computed even if the computation requires some endeavor. Here let us compute the leading order term.

Theorem 3.5 *Let κ_j be a simple Neumann eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ in Ω without the inclusion and κ_j^ϵ be that with the inclusion and let $\mathbf{u}_j$ be the corresponding eigenfunction such that $\|\mathbf{u}_j\|_{L^2(\Omega)} = 1$. Then we have*

$$\kappa_j^\epsilon - \kappa_j = -\epsilon \mathbf{u}_j(z)^t \Big(\int_{\partial B} (\mathcal{S}_B^0)^{-1}(\mathbf{I}) d\sigma \Big) \mathbf{u}_j(z) + O(\epsilon^2). \tag{3.13}$$

Proof. By theorem 3.4, we have

$$\omega_\epsilon - \omega_0 = -\frac{\epsilon}{2\pi i} \operatorname{tr} \int_{\partial V_j} (\mathcal{A}_0^\omega)^{-1} \mathcal{A}_1^\omega \omega d\omega + O(\epsilon^2).$$

Since

$$\mathcal{A}_1(\omega) = \begin{pmatrix} 0 & -\frac{1}{\omega} \mathbf{\Gamma}^\omega(x - z) \int_{\partial B} \cdot \, d\sigma \\ \frac{1}{\omega} \nabla \mathcal{D}_\Omega^\omega(\cdot)(z) \cdot x & C \int_{\partial B} \cdot \, d\sigma \end{pmatrix}$$

and

$$(\mathcal{A}_0^\omega)^{-1} = \begin{pmatrix} (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega)^{-1} & 0 \\ -(\mathcal{S}_B^0)^{-1}(\mathbf{I}) \big[\mathcal{D}_\Omega^\omega (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega)^{-1}(\cdot) \big](z) & (\mathcal{S}_B^0)^{-1} \end{pmatrix},$$

we have

$$\begin{aligned} \omega_\epsilon - \omega_0 &= -\frac{\epsilon}{2\pi i} \operatorname{tr} \int_{\partial V_j} (\mathcal{S}_B^0)^{-1}(\mathbf{I}) \big[\mathcal{D}_\Omega^\omega (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega)^{-1} \mathbf{\Gamma}^\omega(\cdot - z) \big](z) \int_{\partial B} \cdot \, d\sigma d\omega + O(\epsilon^2) \\ &= \frac{\epsilon}{2\pi i} \operatorname{tr} \int_{\partial B} (\mathcal{S}_B^0)^{-1}(\mathbf{I}) d\sigma \int_{\partial V_j} \big(\mathcal{D}_\Omega^\omega \mathbf{N}_\Omega^\omega(\cdot, z) \big)(z) d\omega + O(\epsilon^2). \end{aligned}$$

By Green's formula, the following relation holds:

$$\mathcal{D}_\Omega^\omega(\mathbf{u}_j)(z) = \mathbf{u}_j(z) + (\kappa_j - \omega^2) \int_\Omega \mathbf{\Gamma}^\omega(z - y) \mathbf{u}_j(y) dy. \tag{3.14}$$

Combining (3.14) with (3.4), we obtain

$$\frac{1}{2\pi i} \int_{\partial V_j} \big(\mathcal{D}_\Omega^\omega \mathbf{N}_\Omega^\omega(x, z) \big)(z) d\omega = \frac{1}{2\pi i} \mathbf{u}_j(z) \mathbf{u}_j(z)^t \int_{\partial V_j} \frac{1}{\kappa_j - \omega^2} d\omega = \frac{-1}{2\sqrt{\kappa_j}} \mathbf{u}_j(z) \mathbf{u}_j(z)^t. \tag{3.15}$$

Therefore, we have

$$\begin{aligned} \kappa_j^\epsilon - \kappa_j &= -\epsilon \ \operatorname{tr} \left[\int_{\partial B} (\mathcal{S}_B^0)^{-1}(\mathbf{I}) d\sigma \Big(\mathbf{u}_j(z) \mathbf{u}_j(z)^t \Big) \right] + O(\epsilon^2) \\ &= -\epsilon \mathbf{u}_j(z)^t \Big(\int_{\partial B} (\mathcal{S}_B^0)^{-1}(\mathbf{I}) d\sigma \Big) \mathbf{u}_j(z) + O(\epsilon^2), \end{aligned}$$

which completes the proof of the corollary. □

If B is a ball, then (3.13) takes a particularly simple form. It is easy to see from the symmetry of the ball and (2.3) that for $i, j = 1, 2, 3$,

$$\begin{aligned}\int_{\partial B} \Gamma^0_{ij}(x-y)d\sigma(y) &= -\frac{\gamma_1 \delta_{ij}}{4\pi} \int_{\partial B} \frac{1}{|x-y|} d\sigma(y) - \frac{\gamma_2}{4\pi} \int_{\partial B} \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} d\sigma(y) \\ &= -\frac{\gamma_1 \delta_{ij}}{4\pi} \int_{\partial B} \frac{1}{|x-y|} d\sigma(y) - \frac{\gamma_2 \delta_{ij}}{4\pi} \int_{\partial B} \frac{(x_i - y_i)^2}{|x-y|^3} d\sigma(y).\end{aligned}$$

By symmetry again, we have

$$\int_{\partial B} \frac{(x_i - y_i)^2}{|x-y|^3} d\sigma(y) = \frac{1}{3} \int_{\partial B} \frac{1}{|x-y|} d\sigma(y),$$

and hence

$$\int_{\partial B} \Gamma^0_{ij}(x-y)d\sigma(y) = -\delta_{ij}\big(\gamma_1 + \frac{1}{3}\gamma_2\big)\frac{1}{4\pi} \int_{\partial B} \frac{1}{|x-y|} d\sigma(y).$$

Since $\frac{1}{4\pi}\int_{\partial B} \frac{1}{|x-y|} d\sigma(y) = r$, the radius of B, for all $x \in \overline{B}$, it follows from (2.4) that

$$\int_{\partial B} \Gamma^0_{ij}(x-y)d\sigma(y) = -r\delta_{ij} \frac{3\mu + 2\lambda}{3\mu(2\mu + \lambda)},$$

which in turn implies that

$$(\mathcal{S}^0_B)^{-1}(\mathbf{I}) = -\frac{3\mu(2\mu + \lambda)}{r(3\mu + 2\lambda)}\mathbf{I}.$$

It then follows from (3.13) that if B is the unit ball, then

$$\kappa^\epsilon_j - \kappa_j = \epsilon \frac{3\mu(2\mu + \lambda)}{(3\mu + 2\lambda)} 4\pi |\mathbf{u}_j(z)|^2 + O(\epsilon^2).$$

Note that 4π is the capacity of the unit ball B. See [3].

4 The transmission problem

We now investigate the perturbation of eigenvalues due to the presence of small elastic inclusions which are not hard. Suppose that the elastic medium Ω contains a small inclusion D of the form $D = z + \epsilon B$, whose Lamé constants are $\tilde{\lambda}, \tilde{\mu}$ satisfying $(\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) \geq 0$ and $0 < \tilde{\lambda}, \tilde{\mu} < +\infty$. Let κ_k be an eigenvalue of $-\mathcal{L}^{\lambda,\mu}$ and κ^ϵ_k be the perturbed eigenvalue in the presence of the inclusion. Then the eigenfunction $\mathbf{u}^\epsilon_k$ corresponding to the (simple) eigenvalue κ^ϵ_k is the solution to (2.14) with $\omega^2 = \kappa^\epsilon_k$. Recall that $\tilde{\mathcal{S}}^\omega_D$ and $\tilde{\mathcal{K}}^\omega_D$ denote the single layer potential and the boundary integral operator respectively defined by (2.5) and (2.9) with λ, μ replaced by $\tilde{\lambda}, \tilde{\mu}$.

Let $\mathbf{u}$ be the solution to (2.14). We may assume that ω^2 is not a Dirichlet eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ on D since the Dirichlet eigenvalues on D go to infinity as ϵ tends to 0. By Theorem 2.6, $\mathbf{u}$ can be represented as

$$\mathbf{u} = \begin{cases} \mathcal{D}^\omega_\Omega \psi + \mathcal{S}^\omega_D \phi & \text{in } \Omega \setminus \overline{D}, \\ \tilde{\mathcal{S}}^\omega_D \theta & \text{in } D, \end{cases} \tag{4.1}$$

where ψ, ϕ, and θ satisfy

$$\begin{cases} (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega)\psi - \mathcal{S}_D^\omega \phi = 0, & \text{on } \partial\Omega, \\ \mathcal{D}_\Omega^\omega \psi + \mathcal{S}_D^\omega \phi - \tilde{\mathcal{S}}_D^\omega \theta = 0, & \text{on } \partial D, \\ \dfrac{\partial(\mathcal{D}_\Omega^\omega \psi)}{\partial \nu} + \dfrac{\partial(\mathcal{S}_D^\omega \phi)}{\partial \nu}\Big|_+ - \dfrac{\partial(\tilde{\mathcal{S}}_D^\omega \theta)}{\partial \tilde{\nu}}\Big|_- = 0, & \text{on } \partial D. \end{cases} \tag{4.2}$$

Conversely, $(\phi, \psi, \theta) \in L^2(\partial\Omega)^3 \times L^2(\partial D)^3 \times L^2(\partial D)^3$ satisfying (4.2) yields the solution to (2.14) via the representation formula (4.1).

Let $\varphi(x) = \epsilon\phi(\epsilon x + z)$ and $\vartheta(x) = \epsilon\theta(\epsilon x + z)$. Then using Lemma 3.2, (4.2) can be written as

$$\mathcal{A}_\epsilon^\omega \begin{pmatrix} \psi \\ \varphi \\ \vartheta \end{pmatrix} = 0, \qquad \mathcal{A}_\epsilon^\omega = \sum_{n=0}^{+\infty} (\omega\epsilon)^n \mathcal{A}_n^\omega,$$

where

$$\mathcal{A}_0^\omega = \begin{pmatrix} (\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega) & 0 & 0 \\ \mathcal{D}_\Omega^\omega(\cdot)(z) & \mathcal{S}_B^0 & -\tilde{\mathcal{S}}_B^0 \\ 0 & \frac{1}{2}\mathbf{I} + (\mathcal{K}_B^0)^* & \frac{1}{2}\mathbf{I} - (\tilde{\mathcal{K}}_B^0)^* \end{pmatrix},$$

and for $n = 1, 2, \ldots$

$$\mathcal{A}_n^\omega = \begin{pmatrix} 0 & \dfrac{(-1)^n}{\omega^n} \displaystyle\sum_{|\alpha|=n-1} \frac{1}{\alpha!} \partial^\alpha \mathbf{\Gamma}^\omega(x-z) \int_{\partial B} y^\alpha \cdot d\sigma(y) & 0 \\ \dfrac{1}{\omega^n} \displaystyle\sum_{|\alpha|=n} \frac{1}{\alpha!} x^\alpha \partial^\alpha \mathcal{D}_\Omega^\omega(\cdot)(z) & \mathbf{S}_n & -\tilde{\mathbf{S}}_n \\ \dfrac{1}{\omega^n} \displaystyle\sum_{|\alpha|=n} \frac{1}{\alpha!} \frac{\partial(x^\alpha I)}{\partial \nu} \partial^\alpha \mathcal{D}_\Omega^\omega(\cdot)(z) & \mathbf{K}_n & -\tilde{\mathbf{K}}_n \end{pmatrix}.$$

Here $\mathbf{S}_n$ is the operator from $L^2(\partial B)^3$ into $H^1(\partial B)^3$ defined by

$$\mathbf{S}_n(\varphi)_i = \sum_{j=1}^{3} \left(\mathbf{S}_n\right)_{ij} \varphi_j$$

with

$$\begin{aligned} \left(\mathbf{S}_n\right)_{ij} = &- \frac{1}{4\pi} \frac{i^n}{(n+2)n!} \Big(\frac{n+1}{c_T^{n+2}} + \frac{1}{c_L^{n+2}} \Big) \delta_{ij} \int_{\partial B} |x-y|^{n-1} \cdot d\sigma(y) \\ &+ \frac{1}{4\pi} \frac{i^n(n-1)}{(n+2)n!} \Big(\frac{1}{c_T^{n+2}} - \frac{1}{c_L^{n+2}} \Big) \int_{\partial B} |x-y|^{n-3} (x_i - y_i)(x_j - y_j) \cdot d\sigma(y), \end{aligned}$$

and $\mathbf{K}_n$ given similarly by

$$\begin{aligned} \left(\mathbf{K}_n\right)_{ij} = &- \frac{1}{4\pi} \frac{i^n}{(n+2)n!} \Big(\frac{n+1}{c_T^{n+2}} + \frac{1}{c_L^{n+2}} \Big) \delta_{ij} \int_{\partial B} \frac{\partial}{\partial \nu_x} |x-y|^{n-1} \cdot d\sigma(y) \\ &+ \frac{1}{4\pi} \frac{i^n(n-1)}{(n+2)n!} \Big(\frac{1}{c_T^{n+2}} - \frac{1}{c_L^{n+2}} \Big) \int_{\partial B} \frac{\partial}{\partial \nu_x} |x-y|^{n-3} (x_i - y_i)(x_j - y_j) \cdot d\sigma(y). \end{aligned}$$

Operators $\tilde{\mathbf{S}}_n$ and $\tilde{\mathbf{K}}_n$ are defined in the exactly same way with c_T and c_L replaced by $\tilde{c}_T = \sqrt{\tilde{\mu}}$ and $\tilde{c}_L = \sqrt{\tilde{\lambda} + 2\tilde{\mu}}$.

With these operators in hand, Theorem 3.4 is valid in this case. We now derive from (3.11) the leading order term. Unlike the hard inclusion case, the leading order term in this case turns out to be of order ϵ^3, the order of the volume of the inclusion. Moreover it is determined by the elastic moment tensor associated with the inclusion. The elastic moment tensor $M = \left(m_{pq}^{ij}\right)_{i,j,p,q=1,2,3}$ is defined by

$$m_{pq}^{ij} = \int_{\partial B} x_p \mathbf{e}_q \cdot \mathbf{g}_i^j \, d\sigma,$$

where $(\mathbf{f}_i^j, \mathbf{g}_i^j) \in L^2(\partial B)^3 \times L^2(\partial B)^3$ is the solution to

$$\begin{cases} \tilde{\mathcal{S}}_B^0 \mathbf{f}_i^j|_- - \mathcal{S}_B^0 \mathbf{g}_i^j|_+ = x_i \mathbf{e}_j|_{\partial B}, \\ \dfrac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_B^0 \mathbf{f}_i^j \Big|_- - \dfrac{\partial}{\partial \nu} \mathcal{S}_B^0 \mathbf{g} \Big|_+ = \dfrac{\partial (x_i \mathbf{e}_j)}{\partial \nu} \Big|_{\partial B}. \end{cases}$$

Here $\mathbf{e}_j$ is the standard basis for $\mathbb{R}^3$. See [2]. We note that the eigenvalue perturbation for the Laplacian due to the presence of a soft inclusion is expressed in terms of the polarization tensors. This was first obtained in [5]. See also [3].

Theorem 4.1 *Let κ_k be a simple Neumann eigenvalue for $-\mathcal{L}^{\lambda,\mu}$ in Ω without the inclusion and κ_k^ϵ be that with the inclusion and let $\mathbf{u}_k$ be the corresponding eigenfunction such that $\|\mathbf{u}_k\|_{L^2(\Omega)} = 1$. Then we have*

$$\kappa_k^\epsilon - \kappa_k = \epsilon^3 \sum_{i,j,p,q=1}^{3} m_{pq}^{ij} \partial_i (\mathbf{u}_k)_j (z) \partial_p (\mathbf{u}_k)_q (z) + O(\epsilon^4), \tag{4.3}$$

where $(\mathbf{u}_k)_j$ denotes the j-th component of $\mathbf{u}_k$ and $M = (m_{pq}^{ij})$ is the elastic moment tensor associated with B.

We note that because of the symmetry of the elastic moment tensor $m_{pq}^{ij} = m_{pq}^{ji} = m_{qp}^{ij}$ (see [2]), (4.3) can be written in more dense form using the standard notations of the contraction and the strain tensor:

$$\kappa_k^\epsilon - \kappa_k = \epsilon^3 \mathcal{E}(\mathbf{u}_k)(z) : M \mathcal{E}(\mathbf{u}_k)(z) + O(\epsilon^4), \tag{4.4}$$

where $a : b = \sum_{ij} a_{ij} b_{ij}$ for two matrices a and b, and $\mathcal{E}(\mathbf{u}_k) = \frac{1}{2}(\nabla \mathbf{u}_k + \nabla \mathbf{u}_k^T)$. It is worth mentioning that if the inclusion is harder (softer, resp.) than the background, *i.e.*, $\tilde{\mu} > \mu$ and $\widetilde{\lambda} \geq \lambda$ ($\tilde{\mu} < \mu$ and $\widetilde{\lambda} \leq \lambda$, resp.), then M is positive (negative, resp.) definite, and hence $\kappa_k^\epsilon > \kappa_k$ ($\kappa_k^\epsilon > \kappa_k$, resp) provided that ϵ is small enough and $\mathcal{E}(\mathbf{u}_k)(z) \neq 0$.

Proof of Theorem 4.1. We first observe from (3.11) that the ϵ order term is given by

$$-\frac{1}{2\pi i} \operatorname{tr} \int_{\partial V_k} (\mathcal{A}_0^\omega)^{-1} \mathcal{A}_1^\omega \omega \, d\omega, \tag{4.5}$$

the ϵ^2 order term is given by

$$\frac{1}{2\pi i} \operatorname{tr} \int_{\partial V_k} \left[-(\mathcal{A}_0^\omega)^{-1} \mathcal{A}_2^\omega + \frac{1}{2} \left((\mathcal{A}_0^\omega)^{-1} \mathcal{A}_1^\omega \right)^2 \right] \omega^2 \, d\omega, \tag{4.6}$$

and the ϵ^3 order term is given by

$$\frac{1}{2\pi i}\,\mathrm{tr}\int_{\partial V_k}\big[-(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_3^\omega+(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_1^\omega(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_2^\omega-\frac{1}{3}\big((\mathcal{A}_0^\omega)^{-1}\mathcal{A}_1^\omega\big)^3\big]\omega^3 d\omega. \tag{4.7}$$

Let

$$\begin{pmatrix} \mathcal{S}_B^0 & -\tilde{\mathcal{S}}_B^0 \\ \frac{1}{2}\mathbf{I}+(\mathcal{K}_B^0)^* & \frac{1}{2}\mathbf{I}-(\tilde{\mathcal{K}}_B^0)^* \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix}, \tag{4.8}$$

where the invertibility is guaranteed by Theorem 2.5. As a direct consequence of Theorem 2.5, we have that

$$\mathbf{A}_1(\mathbf{f}),\ \mathbf{A}_2(\mathbf{g})\in L^2_\Psi(\partial B) \tag{4.9}$$

for any $\mathbf{f}\in H^1(\partial B)^3$ and $\mathbf{g}\in L^2_\Psi(\partial B)$, and

$$\mathbf{A}_1(\mathbf{f})=0\ \text{ for any }\ \mathbf{f}\in\Psi. \tag{4.10}$$

Explicit calculations show that $\big(\mathcal{A}_0^\omega\big)^{-1}$ takes the following form

$$\big(\mathcal{A}_0^\omega\big)^{-1}=\begin{pmatrix} (\frac{1}{2}\mathbf{I}-\mathcal{K}_\Omega^\omega)^{-1} & 0 & 0 \\ 0 & \mathbf{A}_1 & \mathbf{A}_2 \\ (\tilde{\mathcal{S}}_B^0)^{-1}(\mathbf{I})\big[\mathcal{D}_\Omega^\omega(\frac{1}{2}\mathbf{I}-\mathcal{K}_\Omega^\omega)^{-1}(\cdot)\big](z) & \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix}.$$

Since $\mathbf{A}_i$, $i=1,2,3,4$, are independent of ω, we have

$$\mathrm{tr}\int_{\partial V_k}(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_n^\omega\omega^n d\omega=0, \tag{4.11}$$

for every integer n.

By (4.9) and (4.10) we have

$$\big(\mathcal{A}_0^\omega\big)^{-1}\mathcal{A}_1^\omega\omega=\begin{pmatrix} 0 & \mathbf{T}_1 & 0 \\ \displaystyle\sum_{|\alpha|=1}\Big(\mathbf{A}_1(x^\alpha\mathbf{I})+\mathbf{A}_2\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\Big)\partial^\alpha\mathcal{D}_\Omega^\omega(\cdot)(z) & 0 & 0 \\ \displaystyle\sum_{|\alpha|=1}\Big(\mathbf{A}_3(x^\alpha\mathbf{I})+\mathbf{A}_4\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\Big)\partial^\alpha\mathcal{D}_\Omega^\omega(\cdot)(z) & \mathbf{T}_2 & -\omega\mathbf{A}_3\tilde{\mathbf{S}}_1 \end{pmatrix},$$

where

$$\mathbf{T}_1=-\big(\frac{1}{2}\mathbf{I}-\mathcal{K}_\Omega^\omega\big)^{-1}\mathbf{\Gamma}^\omega(x-z)\int_{\partial B}\cdot\,d\sigma(y),$$

$$\mathbf{T}_2=-(\tilde{\mathcal{S}}_B^0)^{-1}\mathcal{D}_\Omega^\omega\Big[\big(\frac{1}{2}\mathbf{I}-\mathcal{K}_\Omega^\omega\big)^{-1}\mathbf{\Gamma}^\omega(x-z)\Big](z)\int_{\partial B}\cdot\,d\sigma(y)+\mathbf{A}_3\mathbf{S}_1.$$

By (4.9) we have

$$\int_{\partial B}\mathbf{A}_1(x^\alpha\mathbf{I})+\mathbf{A}_2\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\,d\sigma(y)=0\quad\text{if }|\alpha|=1.$$

It is then easy to check that

$$\operatorname{tr}\int_{\partial V_k}\Big((\mathcal{A}_0^\omega)^{-1}\mathcal{A}_1^\omega\Big)^n\omega^n d\omega = 0, \tag{4.12}$$

for every integer n. It now follows from (4.5)-(4.7), (4.11), and (4.12) that

$$\omega_\epsilon - \omega_0 = \frac{\epsilon^3}{2\pi i}\operatorname{tr}\int_{\partial V_k}(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_1^\omega(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_2^\omega\omega^3 d\omega + O(\epsilon^4). \tag{4.13}$$

We know

$$\big(\mathcal{A}_0^\omega\big)^{-1}\mathcal{A}_2^\omega\omega^2 = \begin{pmatrix} 0 & \mathbf{T}_3 & 0 \\ \mathbf{T}_5 & \omega^2(\mathbf{A}_1\mathbf{S}_2 + \mathbf{A}_2\mathbf{K}_2) & -\omega^2(\mathbf{A}_1\tilde{\mathbf{S}}_2 + \mathbf{A}_2\tilde{\mathbf{K}}_2) \\ \mathbf{T}_6 & \mathbf{T}_4 & -\omega^2(\mathbf{A}_3\tilde{\mathbf{S}}_2 + \mathbf{A}_4\tilde{\mathbf{K}}_2) \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{T}_3 &= \sum_{|\alpha|=1}\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega\big)^{-1}\partial^\alpha\mathbf{\Gamma}^\omega(x-z)\int_{\partial B} y^\alpha\cdot d\sigma(y),\\
\mathbf{T}_4 &= \sum_{|\alpha|=1}(\tilde{\mathcal{S}}_B^0)^{-1}\mathcal{D}_\Omega^\omega(\partial^\alpha\mathbf{\Gamma}^\omega(x-z))(z)\int_{\partial B} y^\alpha\cdot d\sigma(y) + \omega^2(\mathbf{A}_3\mathbf{S}_2 + \mathbf{A}_4\mathbf{K}_2),\\
\mathbf{T}_5 &= \sum_{|\alpha|=2}\frac{1}{\alpha!}\Big(\mathbf{A}_1(x^\alpha\mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\Big)\partial^\alpha\mathcal{D}_\Omega^\omega(\cdot)(z),\\
\mathbf{T}_6 &= \sum_{|\alpha|=2}\frac{1}{\alpha!}\Big(\mathbf{A}_3(x^\alpha\mathbf{I}) + \mathbf{A}_4\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\Big)\partial^\alpha\mathcal{D}_\Omega^\omega(\cdot)(z).
\end{aligned}$$

Then we will prove the following identity at the end of this section:

$$\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_k}\Big(\mathbf{T}_1\mathbf{T}_5 - \omega^2\mathbf{T}_2(\mathbf{A}_1\tilde{\mathbf{S}}_2 + \mathbf{A}_2\tilde{\mathbf{K}}_2)\Big)d\omega = 0. \tag{4.14}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_k}(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_1^\omega(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_2^\omega\omega^3 d\omega\\
&= \frac{1}{2\pi i}\operatorname{tr}\sum_{|\alpha|=1}\int_{\partial V_k}\Big(\mathbf{A}_1(x^\alpha\mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\Big)\partial^\alpha\mathcal{D}_\Omega^\omega(\mathbf{T}_3(\cdot))(z)d\omega\\
&= \frac{1}{2\pi i}\operatorname{tr}\sum_{|\alpha|=1}\Big(\mathbf{A}_1(x^\alpha\mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha\mathbf{I})}{\partial\nu}\Big)\Big)\int_{\partial V_k}\partial^\alpha\mathcal{D}_\Omega^\omega(\mathbf{T}_3(\cdot))(z)d\omega. \qquad (4.15)
\end{aligned}$$

By (3.3) and (3.4) we have

$$\begin{aligned}
\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega\big)^{-1}\partial^\alpha\mathbf{\Gamma}^\omega(\cdot - z)(x) &= -\partial_z^\alpha\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega\big)^{-1}\mathbf{\Gamma}^\omega(\cdot - z)(x),\\
&= \sum_{j=1}^{+\infty}\frac{1}{\kappa_j - \omega^2}\mathbf{u}_j(x)\partial^\alpha\mathbf{u}_j(z)^t. \qquad (4.16)
\end{aligned}$$

We also know from (3.14) that

$$\partial^\alpha \mathcal{D}_\Omega^\omega(\mathbf{u}_k)(z) = \partial^\alpha \mathbf{u}_k(z) + (\kappa_k - \omega^2)\int_\Omega \partial^\alpha \mathbf{\Gamma}^\omega(z-y)\mathbf{u}_k(y)dy. \tag{4.17}$$

From (4.16) and (4.17) it follows that

$$\frac{1}{2\pi i}\int_{\partial V_k} \partial^\alpha \mathcal{D}_\Omega^\omega(\mathbf{T}_3(\cdot))(z)d\omega = -\frac{1}{2\sqrt{\kappa_k}}\sum_{|\beta|=1}\partial^\alpha \mathbf{u}_k(z)\partial^\beta \mathbf{u}_k(z)^t \int_{\partial B} y^\beta \cdot d\sigma(y). \tag{4.18}$$

Combining (4.15) and (4.18), we have

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_k}(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_1^\omega(\mathcal{A}_0^\omega)^{-1}\mathcal{A}_2^\omega\omega^3 d\omega \\
&= -\frac{1}{2\sqrt{\kappa_k}}\operatorname{tr}\sum_{|\alpha|=|\beta|=1}\Big(\mathbf{A}_1(x^\alpha \mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha \mathbf{I})}{\partial\nu}\Big)\Big)\partial^\alpha \mathbf{u}_k(z)\partial^\beta \mathbf{u}_k(z)^t\int_{\partial B} y^\beta \cdot d\sigma(y) \\
&= -\frac{1}{2\sqrt{\kappa_k}}\operatorname{tr}\sum_{|\alpha|=|\beta|=1}\partial^\alpha \mathbf{u}_k(z)\partial^\beta \mathbf{u}_k(z)^t\int_{\partial B} y^\beta\Big(\mathbf{A}_1(x^\alpha \mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha \mathbf{I})}{\partial\nu}\Big)\Big)d\sigma(y) \\
&= -\frac{1}{2\sqrt{\kappa_k}}\sum_{|\alpha|=|\beta|=1}\partial^\beta \mathbf{u}_k(z)^t\Big[\int_{\partial B} y^\beta\Big(\mathbf{A}_1(x^\alpha \mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha \mathbf{I})}{\partial\nu}\Big)\Big)d\sigma(y)\Big]\partial^\alpha \mathbf{u}_k(z).
\end{aligned} \tag{4.19}$$

But, by the definition of $\mathbf{A}_1$ and $\mathbf{A}_2$, the (i,j)-component of

$$\int_{\partial B} y^\beta\Big(\mathbf{A}_1(x^\alpha \mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha \mathbf{I})}{\partial\nu}\Big)\Big)d\sigma(y)$$

is equal to $-m_{\beta i}^{\alpha j}$. Now, plugging (4.19) into (4.13) we arrive at

$$\omega_\epsilon - \omega_0 = \frac{\epsilon^3}{2\sqrt{\kappa_k}}\sum_{i,j,\alpha,\beta=1}^{3} m_{\beta i}^{\alpha j}\partial_\beta(\mathbf{u}_k)_i(z)\partial_\alpha(\mathbf{u}_k)_j(z) + O(\epsilon^4),$$

as desired. □

Proof of (4.14). We can use the similar procedure as in the proof of the above theorem to obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial V_k}\mathbf{T}_1\mathbf{T}_5 d\omega \\
&= \frac{1}{2\sqrt{\kappa_k}}\sum_{|\alpha|=2}\frac{1}{\alpha!}\mathbf{u}_k(z)^t\Big[\int_{\partial B}\Big(\mathbf{A}_1(x^\alpha \mathbf{I}) + \mathbf{A}_2\Big(\frac{\partial(x^\alpha \mathbf{I})}{\partial\nu}\Big)\Big)d\sigma(y)\Big]\partial^\alpha \mathbf{u}_k(z) \\
&= \frac{1}{2\sqrt{\kappa_k}}\mathbf{u}_k(z)^t\int_{\partial B}\mathbf{A}_2\Big(\frac{\partial \mathbf{u}_k^{(2)}}{\partial\nu}\Big)d\sigma(y),
\end{aligned}$$

where $\mathbf{u}_k^{(2)}(x) = \sum_{|\alpha|=2} \frac{1}{\alpha!} x^\alpha \partial^\alpha \mathbf{u}_k(z)$. Using (3.15), we also know

$$\begin{aligned}
&\frac{1}{2\pi i} \operatorname{tr} \int_{\partial V_k} \omega^2 \mathbf{T}_2 (\mathbf{A}_1 \tilde{\mathbf{S}}_2 + \mathbf{A}_2 \tilde{\mathbf{K}}_2) d\omega \\
&= -\frac{1}{2\pi i} \operatorname{tr} \int_{\partial V_k} \omega^2 \mathcal{D}_\Omega^\omega \Big[\big(\frac{1}{2}\mathbf{I} - \mathcal{K}_\Omega^\omega \big)^{-1} \mathbf{\Gamma}^\omega (x - z) \Big](z) \int_{\partial B} \mathbf{A}_2 \tilde{\mathbf{K}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} d\sigma(y) d\omega \\
&= \frac{\sqrt{\kappa_k}}{2} \operatorname{tr} \mathbf{u}_k(z) \mathbf{u}_k(z)^t \int_{\partial B} \mathbf{A}_2 \tilde{\mathbf{K}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} d\sigma(y) \\
&= \frac{\sqrt{\kappa_k}}{2} \mathbf{u}_k(z)^t \int_{\partial B} \mathbf{A}_2 \tilde{\mathbf{K}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} \left(\mathbf{u}_k(z) \right) d\sigma(y).
\end{aligned}$$

Inserting the Taylor's expansion of $\mathbf{u}_k$ at z into $(\mathcal{L}^{\lambda,\mu} + \kappa_k)\mathbf{u}_k = 0$, we have

$$\mathcal{L}^{\lambda,\mu} \mathbf{u}_k^{(2)} + \kappa_k \mathbf{u}_k(z) = 0. \tag{4.20}$$

Since $\tilde{\mathcal{S}}_B^\omega = \tilde{\mathcal{S}}_B^0 + \sum_{n=1}^\infty \omega^n \tilde{\mathbf{S}}_n$, we have

$$\mathcal{L}^{\tilde{\lambda},\tilde{\mu}} \tilde{\mathbf{S}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} \left(\mathbf{u}_k(z) \right) + \mathbf{u}_k(z) = 0. \tag{4.21}$$

By the definition of $\mathbf{A}_n$ and the jump relation of single layer, we have

$$\mathbf{A}_2(\mathbf{f}) = \frac{\partial}{\partial \nu} \mathcal{S}_B^0 \mathbf{A}_2(\mathbf{f}) \Big|_+ - \frac{\partial}{\partial \nu} \mathcal{S}_B^0 \mathbf{A}_2(\mathbf{f}) \Big|_- = \mathbf{f} + \frac{\partial}{\partial \tilde{\nu}} \tilde{\mathcal{S}}_B^0 \mathbf{A}_4(\mathbf{f}) \Big|_- - \frac{\partial}{\partial \nu} \mathcal{S}_B^0 \mathbf{A}_2(\mathbf{f}) \Big|_-,$$

and hence

$$\int_{\partial B} \mathbf{A}_2(\mathbf{f}) d\sigma = \int_{\partial B} \mathbf{f} d\sigma. \tag{4.22}$$

From (4.20), (4.21), and (4.22), we have

$$\begin{aligned}
&\int_{\partial B} \mathbf{A}_2 \Big(\frac{\partial \mathbf{u}_k^{(2)}}{\partial \nu} \Big) - \kappa_k \mathbf{A}_2 \tilde{\mathbf{K}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} \left(\mathbf{u}_k(z) \right) d\sigma(y) \\
&= \int_{\partial B} \frac{\partial \mathbf{u}_k^{(2)}}{\partial \nu} - \kappa_k \tilde{\mathbf{K}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} \left(\mathbf{u}_k(z) \right) d\sigma(y) \\
&= \int_B \mathcal{L}^{\lambda,\mu} \mathbf{u}_k^{(2)} - \kappa_k \mathcal{L}^{\tilde{\lambda},\tilde{\mu}} \tilde{\mathbf{S}}_2 (\tilde{\mathcal{S}}_B^0)^{-1} \left(\mathbf{u}_k(z) \right) d\sigma(y) = 0,
\end{aligned}$$

which completes the proof. □

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