DIRECT ELASTIC IMAGING OF A SMALL INCLUSION

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Abstract. In this paper we consider the problem of locating a small three-dimensional elastic inclusion, using arrays of elastic source transmitters and receivers. This procedure yields the multi-static response matrix that is characteristic of the elastic inclusion. We show how the eigenvalue structure of this matrix can be employed within the framework of a non-iterative method of MUSIC (standing for MUltiple Signal Classification)-type in order to retrieve the elastic inclusion. We illustrate our reconstruction procedure with a variety of computational examples from synthetic, noiseless, and severely noisy data.

Key words. elasticity, small inclusion, detection, asymptotic formula, reconstruction, MUSIC-type algorithm

AMS subject classifications. 35R30

1. Introduction. In this paper we consider the localization of a small, three-dimensional bounded homogeneous inclusion via time-harmonic elastic means, typically using arrays of elastic source transmitters and receivers with given direction(s) at some distance from the inclusion, possibly also lying in the far field. The problem is formulated as an inverse problem for the 3D elasticity system and it involves a robust asymptotic modeling of the multistatic response matrix.

Non Destructive Testing (NDT) of elastic structures is the main motivation of this study. NDT is widely used to examine materials during the manufacturing or in process. Inclusions of small size are believed to be the starting point of crack development in elastic bodies. The aim of this work is to enhance the ability of the NDT methods not only to detect small inclusions, but also to characterize them. It is in connection with the CIVA simulation software developed by the CEA (www-civa.cea.fr).

The elastic inclusion is illuminated by an array of elastic sources with any given direction(s) of incidence, located at some, not necessarily large, distance from it. The resulting displacement field is collected by another array, possibly but not necessarily the same as the transmitter one—in effect, one is always dealing with strongly aspect-limited data, and no full coverage is assumed. This procedure yields the multi-static data response (MSR) matrix that is characteristic of the elastic inclusion for given sets (e.g., arrays) of transmitters and receivers at the (single) frequency of operation.

The proposed imaging approach uses a MUSIC (multiple signal classification)-type algorithm, and yields accurate localization, and estimation of the elastic moment tensor (EMT) of the inclusion.

Our MUSIC-type method makes use of asymptotic expansions of the displacement field. Starting from application of Green's theorems to the elasticity system, we formally derive asymptotic formulations of the displacement field in terms of the order of magnitude of the diameter of the inclusion.

Extending previous works in electromagnetics [2, 3, 5] to full three-dimensional vector elasticity, we show how the eigenvalue structure of the MSR matrix can be employed within the framework of the MUSIC method in order to retrieve a 3D elastic inclusion.

Then, the viability of this algorithm—which would be easily extended to layered environments by introduction of their Green's functions—is documented by a variety of numerical results from synthetic, noiseless, and severely noisy displacement data.

To put the present work in proper context, we will now emphasize some significant differences with prior work. Nearly all of the existing algorithms to solve three-dimensional inverse problems in elasticity are iterative and are based on regularization techniques [9]. Recently, non-iterative methods such as linear

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sampling methods and factorization methods have been developed which overcome the issue of nonlinearity but not ill-posedness [16]. In order to handle this challenging issue it is generally advisable to incorporate all a priori knowledge about the unknown parameter and to try to determine very specific features. In this context, the method of small volume expansions provides a useful framework to accurately reconstruct the location and geometric features of the inclusions in a stable way, even for moderately noisy data. For comprehensive study on the development in this direction, we refer to recent texts [6, 8]. We particularly mention that in [14, 15] a direct numerical reconstruction of a single or a cluster of elastic inclusions is developed in the context of static elasticity. Among all the direct reconstruction methods that have been developed during the last years, MUSIC-type algorithms seem to be particularly accurate and stable. In [2,3,5], we have designed robust numerical methods of MUSIC type for efficiently determining the locations and/or shapes of a collection of small acoustical and electromagnetic inclusions in both two-dimensional scalar scattering situations and three-dimensional vector situations.

To the best of our knowledge, the present paper is the first attempt to design and analyze a MUSIC-type algorithm for imaging a small volumetric elastic inclusion of arbitrary shape and nature for fully general three-dimensional geometrical configurations and illuminations.

The paper is organized as follows. In section 2, we introduce the notation used throughout this paper and review some basic facts on elastic waves. In section 3, an asymptotic formula for the perturbations in the displacement field due to the presence of a small inclusion is provided. The concept of elastic moment tensor (EMT) is the basic building block for this asymptotic expansion. Because it is important from an imaging point of view to precisely characterize this concept, some of the properties of the EMT, such as symmetry and positivity are reviewed. Explicit formulae of the EMTs associated with spherical inclusions are given. Based on the asymptotic expansion of the displacement field, the MSR matrix is formulated and its spectral structure is analyzed. In section 4, we show how the eigenvalue structure of the MSR matrix can be employed within the framework of the MUSIC method in order to retrieve a small elastic inclusion. In practice this means carrying out two steps: first, the calculation of singular values (and the corresponding eigenvectors), the number of nonzero values depending upon the elastic nature of the inclusion and upon the transmitters and/or receivers' geometrical arrangement; second, the orthogonal projection of a properly built vector propagator onto the null space of the MSR matrix, coincidence with an inclusion being then associated to a peak of the inverse norm of the projection. In the final section, several computations are performed to illustrate relevant features of our direct imaging method.

2. Notation and Preliminaries. We begin with introducing the notation used throughout this paper. All vectors (first-rank tensors) and vector fields will be denoted either in bold script ($\mathbf{u}$), all second-rank tensors with single underscore ($\underline{\mathbf{A}}$), all third-rank tensor with double underscore ($\underline{\underline{\mathbf{A}}}$), $\ldots$, while for simplify the notations in matrix analysis we will tend to write matrices using boldface capital letters without underscore ($\mathbf{A}$).

The propagation of time-harmonic waves in an isotropic homogeneous medium with Lamé constants λ and μ ($\mu > 0,\ 2\mu + 3\lambda > 0$) and mass density ρ is governed by the equation

$$\nabla \cdot \underline{\sigma}(\mathbf{x}) + \omega^2 \rho \mathbf{u}(\mathbf{x}) = 0, \tag{1}$$

together with the constitutive relation (Hookes' law)

$$\underline{\sigma}(\mathbf{x}) = \underline{\underline{\mathcal{C}}} : \underline{\varepsilon}(\mathbf{x}) = \underline{\underline{\mathcal{C}}} : \nabla \mathbf{u}(\mathbf{x}), \tag{2}$$

where $\mathbf{u}$ denotes the *displacement field*, ω is a given frequency of operation (positive circular frequency, the time-harmonic dependence $e^{-i\omega t}$ being implied), $\underline{\varepsilon}(\mathbf{x}) = \dfrac{1}{2}\big(\nabla \mathbf{u} + \nabla \mathbf{u}^t\big)(\mathbf{x})$ denotes a symmetric second-rank *strain* tensor, $\underline{\sigma}(\mathbf{x})$ is a symmetric second-rank *stress* tensor and $\underline{\underline{\mathcal{C}}} = \lambda \underline{\underline{\mathbf{I}}}^{+} + 2\mu \underline{\underline{\mathbf{I}}}^{\text{sym}}$ is a symmetric fourth-rank *isotropic elasticity tensor*. Here, the upper index t denotes the transpose, $\underline{\underline{\underline{\mathbf{I}}}}^{+} = \underline{\mathbf{I}} \otimes \underline{\mathbf{I}}$ and $\underline{\underline{\underline{\mathbf{I}}}}^{\text{sym}}$ are fourth-rank tensors satisfying the following property:

$$\underline{\underline{\underline{\mathbf{I}}}}^{+} : \underline{\mathbf{A}} = \underline{\mathbf{A}} : \underline{\underline{\underline{\mathbf{I}}}}^{+} = \underline{\mathbf{I}}\,\mathrm{tr}(\underline{\mathbf{A}}) \qquad \text{and} \qquad \underline{\underline{\underline{\mathbf{I}}}}^{\text{sym}} : \underline{\mathbf{A}} = \underline{\mathbf{A}} : \underline{\underline{\underline{\mathbf{I}}}}^{\text{sym}} = \frac{1}{2}(\underline{\mathbf{A}} + \underline{\mathbf{A}}^{t}),$$

under double contraction : with any second-rank tensor $\underline{\mathbf{A}}$ and $\underline{\mathbf{I}}$ is a second-rank identity tensor. Therefore the elasticity tensor has the following symmetries,

$$\mathcal{C}_{ijkl} = \mathcal{C}_{jikl} = \mathcal{C}_{ijlk} = \mathcal{C}_{klij}, \qquad 1 \leq i, j, k, l \leq 3.$$

The above symmetry relations reduce the number of independent components of $\underline{\underline{\underline{\mathcal{C}}}}$ to the number of independent elements of a symmetric 6×6 matrix $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{3}$$

Using the summation convention the relation between the stress and the displacement given by (2) can by written simply as $\sigma_{ij} = \mathcal{C}_{ijkl}\varepsilon_{kl}$, where the summation is understood to be over the repeated indexes k and l. By defining the vectors $\boldsymbol{s}$ and $\boldsymbol{e}$ as

$$\boldsymbol{s} = \left[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\right]^{t}, \tag{4}$$

$$\boldsymbol{e} = \left[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}\right]^{t}, \tag{5}$$

we can write the following relationship between the six independent components of stress and strain tensors

$$\boldsymbol{s} = \mathbf{C} \cdot \boldsymbol{e}. \tag{6}$$

We also need to define the surface traction: on any surface area dS with a normal $\hat{\mathbf{n}}$ we define the *surface traction* $T\mathbf{u}$ by

$$T\mathbf{u}(\mathbf{x}) = \hat{\mathbf{n}} \cdot \underline{\sigma}(\mathbf{x}).$$

The differential operator Δ^{e} in equation (1), which is given by

$$\Delta^{e}\mathbf{u} = (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla\times\nabla\times\mathbf{u} = (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^{2}\mathbf{u},$$

is the *Lamé operator* of linear elasticity. Therefore, the equation for the complex displacement field given by (1) takes the following form

$$(\Delta^{e} + \omega^{2}\rho)\mathbf{u} = 0. \tag{7}$$

We refer to (7) as the spectral or stationary or time-independent Navier equation. It is well-known that any solution $\mathbf{u}$ to the above equation can be decomposed as

$$\mathbf{u} = \mathbf{u}_{p} + \mathbf{u}_{s}, \tag{8}$$

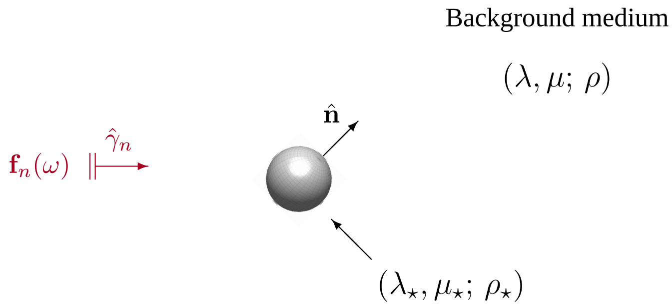

FIG. 3.1. *Configuration under study.*

where $\mathbf{u}_p = -\frac{1}{k_p^2}\nabla\nabla\cdot\mathbf{u}$ is known as the *compressional part* of $\mathbf{u}$ and $\mathbf{u}_s = \frac{1}{k_s^2}\nabla\times\nabla\times\mathbf{u}$ as a *shear part* of $\mathbf{u}$. There holds

$$\nabla^2\mathbf{u}_p + k_p^2\mathbf{u}_p = 0, \qquad \nabla\times\mathbf{u}_p = 0,$$
$$\nabla^2\mathbf{u}_s + k_s^2\mathbf{u}_s = 0, \qquad \nabla\cdot\mathbf{u}_s = 0,$$

where the wavenumbers of compressional and shear waves k_p and k_s, respectively, are given by

$$k_p = \omega\sqrt{\frac{\rho}{\lambda+2\mu}} \qquad \text{and} \qquad k_s = \omega\sqrt{\frac{\rho}{\mu}}.$$

We impose on $\mathbf{u}_p$ and $\mathbf{u}_s$ the *Kupradze's radiation conditions* for solutions of (7) by requiring that

$$\lim_{x\to\infty}\mathbf{u}_p(\mathbf{x}) = 0, \qquad \lim_{x\to\infty} x(\partial_x - ik_p)\mathbf{u}_p(\mathbf{x}) = 0, \tag{9}$$
$$\lim_{x\to\infty}\mathbf{u}_s(\mathbf{x}) = 0, \qquad \lim_{x\to\infty} x(\partial_x - ik_s)\mathbf{u}_s(\mathbf{x}) = 0, \tag{10}$$

where $x = |\mathbf{x}|$ and all limits are to hold uniformly over all directions. We say that $\mathbf{u}$ satisfies the radiation condition if it allows the decomposition (8) with $\mathbf{u}_p$ and $\mathbf{u}_s$ satisfying (9) and (10).

Finally, we will use the following uniqueness results from [17] for the exterior problem.

LEMMA 2.1. *Let $\mathbf{u}(\mathbf{x})$ be a solution to (7) in $\mathbb{R}^3\setminus\overline{D}$, where D is bounded domain with a connected Lipschitz boundary, satisfying the Kupradze's radiation condition. If either $\mathbf{u} = 0$ or $T\mathbf{u} = 0$ on ∂D, $\mathbf{u}$ is identically zero in $\mathbb{R}^3\setminus\overline{D}$.*

3. Perturbations in the Displacement Field Due to a Small Penetrable Inclusion.

3.1. Problem Formulation. Suppose that the elastic inclusion D is of the form $D = \mathbf{x}_\star + \alpha B$, where B is bounded Lipschitz domain in $\mathbb{R}^3$ containing the origin. The small parameter α represents the order of magnitude of D and $\mathbf{x}_\star$ is its location. Suppose that D has the pair of Lamé constants $(\lambda_\star, \mu_\star)$ and density $\rho_\star$ which are different from those of the background elastic medium, (λ, μ) and ρ. Using this notation, we introduce the constants

$$(\lambda_\alpha, \mu_\alpha; \rho_\alpha)(\mathbf{x}) := \begin{cases} (\lambda_\star, \mu_\star; \rho_\star) & \text{in } D, \\ (\lambda, \mu; \rho) & \text{in } \mathbb{R}^3\setminus\overline{D}. \end{cases}$$

It is always assumed that $\mu_\alpha > 0$ and $3\lambda_\alpha + 2\mu_\alpha > 0$. We also define the perturbations $\delta\lambda$, $\delta\mu$ and $\delta\rho$ as

$$\delta\lambda = \lambda - \lambda_\star, \qquad \delta\mu = \mu - \mu_\star \qquad \text{and} \qquad \delta\rho = \rho - \rho_\star. \tag{11}$$

We assume that the inclusion D is illuminated by a time-harmonic point source $\mathbf{f}_n(\mathbf{x},\omega) = \delta(\mathbf{x}-\mathbf{x}_n)\hat{\gamma}_n$ acting at the point $\mathbf{x}_n \in \mathbb{R}^3 \setminus \overline{D}$ in the direction $\hat{\gamma}_n \in \mathbb{R}^3$ with frequency ω. Then the incident displacement field in a homogeneous medium due to the point source $\mathbf{x}_n$ in the direction $\hat{\gamma}_n$ is given by

$$\mathbf{u}_n^{\text{inc}}(\mathbf{x}) = \underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_n)\cdot\hat{\gamma}_n,$$

where $\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_n)$, denotes the outgoing Green's function for an unbounded, homogeneous medium solution to

$$(\Delta^e + \omega^2\rho)\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')\underline{\mathbf{I}}.$$

In other words the Green's function $\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_n)$ gives the displacement field at location $\mathbf{x}$ generated by a point force at $\mathbf{x}_n$ in a certain direction. Explicitly, it is given by

$$\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \underline{\mathbf{G}}_P(\mathbf{x},\mathbf{x}') + \underline{\mathbf{G}}_S(\mathbf{x},\mathbf{x}'),$$

where

$$\underline{\mathbf{G}}_P(\mathbf{x},\mathbf{x}') = -\frac{1}{\mu}\frac{\nabla\nabla}{k_s^2}\frac{e^{ik_p|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|},$$

$$\underline{\mathbf{G}}_S(\mathbf{x},\mathbf{x}') = \frac{1}{\mu}\left(\underline{\mathbf{I}} + \frac{\nabla\nabla}{k_s^2}\right)\frac{e^{ik_s|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$

The following reciprocity relationship satisfied by the Green's function holds:

$$\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = [\underline{\mathbf{G}}(\mathbf{x}',\mathbf{x})]^t.$$

The displacement field $\mathbf{u}(\mathbf{x})$ in the presence of the elastic inclusion D satisfies the elastic wave equation

$$(\Delta_\alpha^e + \omega^2\rho_\alpha)\,\mathbf{u} = -\mathbf{f}_n \quad \text{in } \mathbb{R}^3 \tag{12}$$

with transmission conditions

$$\mathbf{u}|_+ = \mathbf{u}|_- \quad \text{and} \quad T_\alpha\,\mathbf{u}|_+ = T_\alpha\,\mathbf{u}|_- \quad \text{on } \partial D, \tag{13}$$

where the Lamé operator Δ_α^e is defined by $\Delta_\alpha^e\mathbf{u} = \nabla\cdot\big(\underline{\underline{\underline{\mathcal{C}_\alpha}}} : \nabla\mathbf{u}\big)$ with symmetric fourth rank elasticity tensor $\underline{\underline{\underline{\mathcal{C}_\alpha}}} = \lambda_\alpha\underline{\underline{\underline{\mathbf{I}}}}^+ + 2\mu_\alpha\underline{\underline{\underline{\mathbf{I}}}}^{\text{sym}}$; the surface traction $T_\alpha\mathbf{u}$ is defined by

$$T_\alpha\,\mathbf{u} = \hat{\mathbf{n}}\cdot\big(\underline{\underline{\underline{\mathcal{C}_\alpha}}} : \nabla\mathbf{u}\big),$$

where $\hat{\mathbf{n}}$ denotes the outward unit normal to ∂D; superscript $+$ and $-$ indicate the limiting values as we approach ∂D from outside D, and from inside D, respectively.

Outside the source region, the right-hand-side of (12) is zero.

Appropriate use of Green's theorem straightforwardly yields the Lipppman-Schwinger integral formulation for the scattered displacement field $\mathbf{u}_n^{\text{sc}} = \mathbf{u} - \mathbf{u}_n^{\text{inc}}$ in the presence of the inclusion

$$\mathbf{u}_n^{\text{sc}}(\mathbf{x}) = \int_D d\mathbf{x}'\left[-\nabla'\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}') : \delta\underline{\underline{\underline{\mathcal{C}}}} : \nabla\mathbf{u}(\mathbf{x}') - \omega^2\delta\rho\,\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}')\cdot\mathbf{u}(\mathbf{x}')\right], \tag{14}$$

where $\delta\underline{\underline{\underline{\mathcal{C}}}} = \delta\lambda\underline{\underline{\underline{\mathbf{I}}}}^+ + 2\delta\mu\underline{\underline{\underline{\mathbf{I}}}}^{\text{sym}}$ and ∇' denotes the differentiation with respect to the second variable $\mathbf{x}'$ of the Green's function $\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}')$.

3.2. Asymptotic Formula. Full asymptotic formula can be derived by using the method of matched asymptotic expansions. The displacement field differs little from $\mathbf{u}^{\rm inc} = \mathbf{u}_n^{\rm inc}$ for $\mathbf{x}$ far from $\mathbf{x}_\star$ but differs appreciably from $\mathbf{u}^{\rm inc}$ for $\mathbf{x}$ near $\mathbf{x}_\star$. In particular, $\nabla \mathbf{u}(\mathbf{x})$ can not be approximated by $\nabla \mathbf{u}^{\rm inc}(\mathbf{x}_\star)$ for $\mathbf{x}$ near $\mathbf{x}_\star$, which shows that Born-type approximations are not valid in this case. This is due to the singular aspect of the perturbation due to the inclusion. We have as $\alpha\omega$ goes to zero:

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}^{\rm inc}(\mathbf{x}_\star) + o(\alpha\omega), \tag{15}$$

$$\nabla\mathbf{u}(\mathbf{x}) \approx \nabla\widehat{\mathbf{u}}(\mathbf{x}) + \nabla\mathbf{u}^{\rm inc}(\mathbf{x}_\star) + o(\alpha\omega), \qquad \mathbf{x} \in D, \tag{16}$$

where $\widehat{\mathbf{u}}(\mathbf{x})$ is the solution to

$$\begin{cases} \Delta_\alpha^e \widehat{\mathbf{u}} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \text{ and in } D, \\ \widehat{\mathbf{u}}|_+ = \widehat{\mathbf{u}}|_- \quad \text{on} \quad \partial D, \\ T_\alpha \widehat{\mathbf{u}}|_+ - T_\alpha \widehat{\mathbf{u}}|_- = -\hat{\mathbf{n}} \cdot \delta \underline{\underline{\mathcal{C}}} : \nabla\mathbf{u}^{\rm inc}(\mathbf{x}_\star) \quad \text{on} \quad \partial D, \\ \widehat{\mathbf{u}}(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to +\infty. \end{cases} \tag{17}$$

Since the boundary traction $T_\alpha \widehat{\mathbf{u}}$ is defined in terms of a constant stress tensor, the solution to (17) can be conveniently recast as

$$\widehat{\mathbf{u}}(\mathbf{x}) = \underline{\underline{\widehat{v}}}(\mathbf{x}) : \nabla\mathbf{u}^{\rm inc}(\mathbf{x}_\star) \qquad (\widehat{v}_{jkl} = \widehat{\mathbf{v}}_j^{kl},\ j,k,l = 1,2,3),$$

in terms of the solutions $\widehat{\mathbf{v}}^{kl} = \widehat{\mathbf{v}}^{lk}$, $\widehat{\mathbf{v}}^{kl} \in \mathbb{C}^3$ to six canonical problems

$$\begin{cases} \Delta_\alpha^e \widehat{\mathbf{v}}^{kl} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \text{ and in } D, \\ \widehat{\mathbf{v}}^{kl}|_+ = \widehat{\mathbf{v}}^{kl}|_- \quad \text{on} \quad \partial D, \\ T_\alpha \widehat{\mathbf{v}}^{kl}|_+ - T_\alpha \widehat{\mathbf{v}}^{kl}|_- = -\hat{\mathbf{n}} \cdot \delta \underline{\underline{\mathcal{C}}} : \hat{\mathbf{e}}_k \hat{\mathbf{e}}_l \quad \text{on} \quad \partial D, \\ \widehat{\mathbf{v}}^{kl}(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to +\infty. \end{cases} \tag{18}$$

By letting $\nabla\mathbf{v}^{kl} = \nabla\widehat{\mathbf{v}}^{kl} + \hat{\mathbf{e}}_k\hat{\mathbf{e}}_l$, where $\mathbf{v}^{kl}$ is the solution to

$$\begin{cases} \Delta_\alpha^e \mathbf{v}^{kl} = 0 \quad \text{in} \quad \mathbb{R}^3, \\ \mathbf{v}^{kl}(\mathbf{x}) - x_k \hat{\mathbf{e}}_l \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \quad k,l = 1,2,3, \end{cases} \tag{19}$$

the second approximation in (16) can be rewritten as

$$\nabla\mathbf{u}(\mathbf{x}) = \underline{\underline{\mathcal{V}}}(\mathbf{x}) : \nabla\mathbf{u}^{\rm inc}(\mathbf{x}_\star), \tag{20}$$

where

$$\mathcal{V}_{ijkl} = \partial_{x_i} \mathbf{v}_j^{kl}, \qquad i,j,k,l = 1,2,3.$$

Taking the Taylor expansions of $\nabla' \underline{\mathbf{G}}(\mathbf{x},\mathbf{x}')$, $\mathbf{x}' \in D$ at the point $\mathbf{x}_\star$ and using (15) and (20), the integral representation formula for the scattered displacement field given by (14) can be rewritten as

$$\mathbf{u}_n^{\rm sc}(\mathbf{x}) = -\nabla' \underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_\star) : \left[\int_D d\mathbf{x}'\, \delta\underline{\underline{\mathcal{C}}} : \underline{\underline{\mathcal{V}}}(\mathbf{x}')\right] : \nabla\mathbf{u}_n^{\rm inc}(\mathbf{x}_\star) \\ -\omega^2 \delta\rho\, |D| \underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_\star) \cdot \mathbf{u}_n^{\rm inc}(\mathbf{x}_\star) + o(\alpha^3\omega^3). \tag{21}$$

Define the fourth-rank Elastic Moment Tensor (EMT) $\underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda, \delta\mu; D)$ associated with the domain D and Lamé parameters (λ, μ) for the background and $(\lambda_\star, \mu_\star)$ for D by

$$\begin{aligned} \mathcal{M}_{ijkl} &= \int_D d\mathbf{x}\, \nabla x_i \hat{\mathbf{e}}_j : \delta\underline{\underline{\mathcal{C}}} : \nabla \mathbf{v}^{kl}(\mathbf{x}) \\ &= \int_D d\mathbf{x}\, \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j : \delta\underline{\underline{\mathcal{C}}} : \nabla \mathbf{v}^{kl}(\mathbf{x}), \qquad i,j,k,l = 1,2,3. \end{aligned} \tag{22}$$

Using the summation convention, this can be written as

$$\mathcal{M}_{ijkl} = \int_D d\mathbf{x}\, \delta\mathcal{C}_{ijpq}\, \partial_{x_p} \mathbf{v}_q^{kl}(\mathbf{x}), \tag{23}$$

where the summation is understood to be over the indexes p and q, $p, q = 1, 2, 3$.

The following theorem can be proved in the exactly same manner as Theorem 8.3 in [6].

THEOREM 3.1. *The following asymptotic expansion holds for* $\mathbf{x}$ *away from* $\mathbf{x}_\star$*:*

$$\begin{aligned} \mathbf{u}_n^{\rm sc}(\mathbf{x}) = &-\nabla' \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_\star) : \underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda, \delta\mu; D) : \nabla \mathbf{u}_n^{\rm inc}(\mathbf{x}_\star) - \omega^2 \delta\rho\, |D| \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_\star) \cdot \mathbf{u}_n^{\rm inc}(\mathbf{x}_\star) \\ &+ o(\alpha^3 \omega^3). \end{aligned} \tag{24}$$

3.3. Properties of EMT. We now recall some important properties of the EMT and give explicit formulae for the EMT associated with spherical inclusions, with the limiting soft inclusion or rigid inclusion encompassed as well.

The following properties of EMT hold [6]:

THEOREM 3.2.

(i) *[Symmetry] For* $i, j, k, l = 1, 2, 3$, *the following holds*

$$\mathcal{M}_{ijkl} = \mathcal{M}_{jikl} = \mathcal{M}_{ijlk} = \mathcal{M}_{klij}. \tag{25}$$

(ii) *[Positive-definiteness] If* $\mu_\star > \mu$ *(*$\mu_\star < \mu$*), then* $\underline{\underline{\underline{\mathcal{M}}}}$ *is positive (negative) definite on the space of symmetric matrices.*

(iii) *Let* B *be a bounded domain in* $\mathbb{R}^3$ *and* $\underline{\underline{\underline{\mathcal{M}}}}(B)$ *denote the EMT associated with* B*. Then*

$$\underline{\underline{\underline{\mathcal{M}}}}(\alpha B) = \alpha^3\, \underline{\underline{\underline{\mathcal{M}}}}(B).$$

Let us only check point (iii). Since $D = \mathbf{x}_\star + \alpha B$, $\mathbf{x} = \mathbf{x}_\star + \alpha\mathbf{y}$, $\mathbf{x} \in D$ and $\mathbf{y} \in B$, from (22) it follows

$$\underline{\underline{\underline{\mathcal{M}}}}(D) = \int_D d\mathbf{x}\, \delta\underline{\underline{\mathcal{C}}} : \underline{\underline{\mathcal{V}}}(\mathbf{x}) =$$

$$\underline{\underline{\underline{\mathcal{M}}}}(\alpha B) = \int_{\alpha B} d\alpha\mathbf{y}\, \delta\underline{\underline{\mathcal{C}}} : \underline{\underline{\mathcal{V}}}(\alpha\mathbf{y}) =$$

$$\alpha^3 \underline{\underline{\underline{\mathcal{M}}}}(B) = \alpha^3 \int_B d\mathbf{y}\, \delta\underline{\underline{\mathcal{C}}} : \underline{\underline{\mathcal{V}}}(\mathbf{y}),$$

where $\mathcal{V}_{ijkl}(\mathbf{y}) = \partial_{y_i} \mathbf{v}_j^{kl}(\mathbf{y})$ and $\mathbf{v}^{kl}(\mathbf{y})$ is the solution to

$$\begin{cases} \Delta_\alpha^e \mathbf{v}^{kl} = 0 \quad \text{in} \quad \mathbb{R}^3, \\ \mathbf{v}^{kl}(\mathbf{x}) - x_k \hat{\mathbf{e}}_l \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \quad k, l = 1, 2, 3, \end{cases} \tag{26}$$

with elasticity tensor $\underline{\underline{\underline{\mathcal{C}}}}_\alpha = \underline{\underline{\mathcal{C}}}$ in $\mathbb{R}^3 \setminus \overline{B}$ and $\underline{\underline{\underline{\mathcal{C}}}}_\alpha = \underline{\underline{\mathcal{C}}} - \delta\underline{\underline{\mathcal{C}}}$ in B.

In the case when D is a unit sphere, fourth-rank Elastic Moment Tensor $\underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda, \delta\mu; D)$ has the same form as the isotropic elasticity tensor $\underline{\underline{\mathcal{C}}}$. From [7], we have

$$\underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda\,, \delta\mu\,; D) = m_1(\delta\mu\,)\left(m_2(\delta\lambda\,, \delta\mu\,)\underline{\underline{\mathbf{I}}}^{+} + 2\underline{\underline{\mathbf{I}}}^{\mathrm{sym}}\right)|D|,$$

where

$$m_1(\delta\mu\,) = \frac{15\mu\delta\mu\,(\nu-1)}{15\mu(1-\nu) + 2\delta\mu\,(5\nu-4)}, \tag{27}$$

$$m_2(\delta\lambda\,, \delta\mu\,) = \frac{\delta\lambda\left(15\mu\lambda(1-\nu) + 2\lambda\delta\mu\,(5\nu-4)\right) - 2\delta\mu\,(\lambda\delta\mu\, - 5\mu\nu\delta\lambda\,)}{5\delta\mu\left(3\lambda\mu(1-\nu) - 3\mu\nu\delta\lambda\, - \lambda\delta\mu\,(1-2\nu)\right)} \tag{28}$$

and $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ denotes the Poisson ratio.

It is worth mentioning that in the limiting cases of *soft* ($\lambda_\star$, $\mu_\star$ and $\rho_\star$ go to 0) and *rigid* ($\lambda_\star$, $\mu_\star$ and $\rho_\star$ go to $+\infty$, with $\mu_\star/\lambda_\star \to c$, for some positive constant c) sphere the asymptotic formula (24) has a limit as $\lambda_\star$ and $\mu_\star$ go to $+\infty$ or 0. However, the situation is slightly different if we are interested in the case $\rho_\star$ goes to $+\infty$. We can not take this limit independently from $|D|$. What we should take is $\rho_\star|D| \to c'$, for some positive constant c'. Doing so, the asymptotic formula still makes sense in the limiting case $\rho_\star \to +\infty$.

Since the *soft* sphere corresponds to the situation where $\lambda_\star$, $\mu_\star$ and $\rho_\star$ go to 0, then we have

$$\underline{\underline{\underline{\mathcal{M}}}}^o = \lim_{\substack{\delta\lambda\,\to\lambda \\ \delta\mu\,\to\mu}} \underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda\,, \delta\mu\,; D) = m_1^o\left(m_2^o\underline{\underline{\mathbf{I}}}^{+} + 2\underline{\underline{\mathbf{I}}}^{\mathrm{sym}}\right)|D|, \tag{29}$$

where

$$m_1^o = \lim_{\delta\mu\,\to\mu} m_1(\delta\mu\,) = \frac{15\mu(\nu-1)}{7-5\nu},$$

and

$$m_2^o = \lim_{\substack{\delta\lambda\,\to\lambda \\ \delta\mu\,\to\mu}} m_2(\delta\lambda\,, \delta\mu\,) = \frac{7\lambda - 2\mu - 5\nu(\lambda-2\mu)}{10\mu(1-2\nu)}.$$

Hence, in the case of a *soft* sphere the asymptotic formula (24) becomes

$$\begin{aligned}\mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}) = &-\nabla'\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_\star) : \underline{\underline{\underline{\mathcal{M}}}}^o : \nabla\mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star) - \omega^2\rho\,|D|\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_\star)\cdot\mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star) \\ &+o(\alpha^3\omega^3),\end{aligned} \tag{30}$$

where the EMT $\underline{\underline{\underline{\mathcal{M}}}}^o$ is given by formula (29)

Since the *rigid* sphere has the properties $\lambda_\star$ and $\mu_\star$ go to $+\infty$, with $\lambda_\star/\mu_\star$ staying bounded, then the corresponding EMT can be computed as follows:

$$\underline{\underline{\underline{\mathcal{M}}}}^\infty = \lim_{\substack{\delta\lambda\,\to-\infty \\ \delta\mu\,\to-\infty}} \underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda\,, \delta\mu\,; D) = m_1^\infty\left(m_2^\infty\underline{\underline{\mathbf{I}}}^{+} + 2\underline{\underline{\mathbf{I}}}^{\mathrm{sym}}\right)|D|, \tag{31}$$

with

$$m_1^\infty = \lim_{\delta\mu \to -\infty} m_1(\delta\mu) = \frac{15\mu(\nu-1)}{2(5\nu-4)},$$

and

$$m_2^\infty = \lim_{\substack{\delta\lambda \to -\infty \\ \delta\mu \to -\infty}} m_2(\delta\lambda, \delta\mu) = \frac{2\lambda(4-5\nu) - 2(5\mu\nu - \lambda c)}{5(3\mu\nu + \lambda c(1-2\nu))},$$

where $\lim \delta\mu/\delta\lambda = c$ for some positive constant c as $\lambda_\star$ and $\mu_\star$ go to $+\infty$.

Therefore, formally, the asymptotic formula (24) becomes

$$\mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}) = -\nabla' \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_\star) : \underline{\underline{\underline{\mathcal{M}}}}^\infty : \nabla \mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star) - \omega^2(\rho|D| - c')\underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_\star) \cdot \mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star) + o(\alpha^3\omega^3), \tag{32}$$

where the EMT $\underline{\underline{\underline{\mathcal{M}}}}^\infty$ is given by formula (31).

4. The Multi-Static Response Matrix .

4.1. Preliminaries. Recall that the incident field at the point $\mathbf{x}$ due to the point source at $\mathbf{x}_n$ in the $\hat{\gamma}_n$ direction is given by

$$\mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}) = \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_n) \cdot \hat{\gamma}_n,$$

and the corresponding second rank strain tensor $\underline{\varepsilon}_n(\mathbf{x})$ by

$$\underline{\varepsilon}_n(\mathbf{x}) = \frac{1}{2}\left(\nabla \mathbf{u}_n^{\mathrm{inc}} + \nabla(\mathbf{u}_n^{\mathrm{inc}})^t\right)(\mathbf{x}).$$

Then for any observation point $\mathbf{x}_m$ away from source point $\mathbf{x}_n$ and from the inclusion center $\mathbf{x}_\star$, the asymptotic expansion of the displacement field as a function of the size of the inclusion given by (24) can be rewritten as

$$\hat{\gamma}_m \cdot \mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}_m) = \underline{\varepsilon}_m(\mathbf{x}_\star) : \underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda, \delta\mu; D) : \underline{\varepsilon}_n(\mathbf{x}_\star) + \mathbf{u}_m^{\mathrm{inc}}(\mathbf{x}_\star) \cdot c(\delta\rho, \omega)\underline{\mathbf{I}} \cdot \mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star), \tag{33}$$

where $\hat{\gamma}_m \in \mathbb{R}^3$ is the corresponding direction of observation and

$$c(\delta\rho, \omega) = \omega^2 \delta\rho\, |D|. \tag{34}$$

Here, we have neglected the asymptotic term of order $o(\alpha^3\omega^3)$.

The symmetry relation (25) for the EMT $\underline{\underline{\underline{\mathcal{M}}}}$ reduce the number of independent components of $\underline{\underline{\underline{\mathcal{M}}}}$ (or $\mathcal{M}_{ijkl}$) to the number of independent elements of a symmetric 6×6 matrix $\widetilde{\mathbf{M}}$:

$$\widetilde{\mathbf{M}} = \begin{bmatrix} \mathcal{M}_{1111} & \mathcal{M}_{1122} & \mathcal{M}_{1133} & \mathcal{M}_{1123} & \mathcal{M}_{1131} & \mathcal{M}_{1112} \\ \mathcal{M}_{2211} & \mathcal{M}_{2222} & \mathcal{M}_{2233} & \mathcal{M}_{2223} & \mathcal{M}_{2231} & \mathcal{M}_{2212} \\ \mathcal{M}_{3311} & \mathcal{M}_{3322} & \mathcal{M}_{3333} & \mathcal{M}_{3323} & \mathcal{M}_{3331} & \mathcal{M}_{3312} \\ \mathcal{M}_{2311} & \mathcal{M}_{2322} & \mathcal{M}_{2333} & \mathcal{M}_{2323} & \mathcal{M}_{2331} & \mathcal{M}_{2312} \\ \mathcal{M}_{3111} & \mathcal{M}_{3122} & \mathcal{M}_{3133} & \mathcal{M}_{3123} & \mathcal{M}_{3131} & \mathcal{M}_{3112} \\ \mathcal{M}_{1211} & \mathcal{M}_{1222} & \mathcal{M}_{1233} & \mathcal{M}_{1223} & \mathcal{M}_{1231} & \mathcal{M}_{1212} \end{bmatrix}. \tag{35}$$

From the above and the symmetry of the strain tensor $\underline{\varepsilon}$, the first term in (33) can be rewritten as

$$\underline{\varepsilon}_m(\mathbf{x}_\star) : \underline{\underline{\underline{\mathcal{M}}}}(\delta\lambda, \delta\mu; D) : \underline{\varepsilon}_n(\mathbf{x}_\star) = \boldsymbol{e}_m(\mathbf{x}_\star) \cdot \widetilde{\mathbf{M}}(\delta\lambda, \delta\mu; D) \cdot \boldsymbol{e}_n(\mathbf{x}_\star),$$

where the vector $\boldsymbol{e}(\mathbf{x})$ of six independent elements of the strain tensor $\underline{\varepsilon}$ is given by (5). Therefore, the leading-order term in our asymptotic formula (33) can be rewritten in terms of the incident displacement field and the corresponding strain as follows

$$\hat{\gamma}_m \cdot \mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}_m) = \boldsymbol{e}_m(\mathbf{x}_\star) \cdot \widetilde{\mathbf{M}}(\delta\lambda, \delta\mu; D) \cdot \boldsymbol{e}_n(\mathbf{x}_\star) + \mathbf{u}_m^{\mathrm{inc}}(\mathbf{x}_\star) \cdot c(\rho, \omega)\underline{\mathbf{I}} \cdot \mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star). \tag{36}$$

From the Hookes' law (6) the above asymptotic formula admits en equivalent form in terms of the incident displacement field and corresponding stress:

$$\hat{\gamma}_m \cdot \mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}_m) = \boldsymbol{s}_m(\mathbf{x}_\star) \cdot \mathbf{M}(\delta\lambda, \delta\mu; D) \cdot \boldsymbol{s}_n(\mathbf{x}_\star) + \mathbf{u}_m^{\mathrm{inc}}(\mathbf{x}_\star) \cdot c(\rho, \omega)\underline{\mathbf{I}} \cdot \mathbf{u}_n^{\mathrm{inc}}(\mathbf{x}_\star). \tag{37}$$

Here,

$$\mathbf{M}(\delta\lambda, \delta\mu; D) = \mathbf{C}^{-1}\, \widetilde{\mathbf{M}}(\delta\lambda, \delta\mu; D)\, \mathbf{C}^{-1}, \tag{38}$$

the vector $\boldsymbol{s}(\mathbf{x})$ of six independent elements of the stress tensor $\underline{\sigma}$ is given by (4) and the matrix $\mathbf{C}$ is given by (3). Henceforth we will work with asymptotic formula (37).

Note that from positive (or negative) definiteness of the EMT $\underline{\underline{\underline{\mathcal{M}}}}$ (Theorem 3.2) and symmetry of the elasticity tensor $\underline{\underline{\underline{\mathcal{C}}}}$ it follows immediately that the matrix $\mathbf{M}$ is symmetric, positive (or negative) definite on the space of symmetric matrices.

4.2. Decomposition of the MSR Matrix. For the sake of simplicity, let us consider two coincident transmitter and receiver arrays $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ of N elements, used to detect the elastic inclusion located at $\mathbf{x}_\star$. Let $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_N\}$ be the corresponding (unit) directions of incident fields/observation directions.

Any transmit-receive process performed from an array of N transmitters to an array of N receivers is described in the frequency domain by the $N \times N$ inter-element responses $A_{mn}(\omega) = A(\omega, \mathbf{x}_m, \mathbf{x}_n)$, $m, n = 1, \ldots, N$, which is the MSR matrix. In the presence of a small elastic inclusion, the scattering displacement field induced on the mth receiving element in frequency domain can be expressed as follows:

$$A_{mn}(\omega) = \hat{\gamma}_m \cdot \mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}_m),$$

where the perturbation $\hat{\gamma}_m \cdot \mathbf{u}_n^{\mathrm{sc}}(\mathbf{x}_m)$ is given by (37).

For any point $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, let us introduce the $N \times 3$ matrix of the incident field emitted by the array of N transmitters $\mathbf{G}(\mathbf{x}, \omega)$, which will be called the Green's matrix, and the $N \times 6$ matrix of the corresponding independent components of the stress tensors $\mathbf{S}(\mathbf{x}, \omega)$, which will be called the stress matrix:

$$\mathbf{G}(\mathbf{x}, \omega) = \left[\mathbf{u}_1^{\mathrm{inc}}(\mathbf{x}), \ldots, \mathbf{u}_N^{\mathrm{inc}}(\mathbf{x})\right]^t, \tag{39}$$

$$\mathbf{S}(\mathbf{x}, \omega) = \left[\boldsymbol{s}_1(\mathbf{x}), \ldots, \boldsymbol{s}_N(\mathbf{x})\right]^t. \tag{40}$$

Therefore, the MSR matrix $\mathbf{A}(\omega)$ is formed, and it is factorized as follows:

$$\mathbf{A}(\omega) = \mathbf{H}(\mathbf{x}_\star, \omega)\, \mathbf{D}(\omega)\, \mathbf{H}^t(\mathbf{x}_\star, \omega), \tag{41}$$

where

$$\mathbf{H}(\mathbf{x},\omega) = [\mathbf{S}(\mathbf{x},\omega),\ \mathbf{G}(\mathbf{x},\omega)] \tag{42}$$

and $\mathbf{D}(\omega)$ is a symmetric 9×9 matrix given by

$$\mathbf{D}(\omega) = \mathrm{diag}\big(\mathbf{M}, c(\rho,\omega)\underline{\mathbf{I}}\big). \tag{43}$$

Here, $\mathbf{M}$ and $c(\rho,\omega)$ are given by (38) and (34), respectively.

Consequently, the MSR matrix $\mathbf{A}(\omega)$ is the product of three matrices $\mathbf{H}^t(\mathbf{x}_\star,\omega)$, $\mathbf{D}(\omega)$ and $\mathbf{H}(\mathbf{x}_\star,\omega)$. The physical meaning of the above factorization is the following: the matrix $\mathbf{H}^t(\mathbf{x}_\star,\omega)$ is the propagation matrix from the transmit array toward the inclusion located at the point $\mathbf{x}_\star$, the matrix $\mathbf{D}(\omega)$ is the scattering matrix and $\mathbf{H}(\mathbf{x}_\star,\omega)$ is the propagation matrix from the inclusion toward the receive array, see [18].

Note that from (36) the above factorization of $\mathbf{A}(\omega)$ can be written in the following equivalent form:

$$\mathbf{A}(\omega) = \widetilde{\mathbf{H}}(\mathbf{x}_\star,\omega)\,\widetilde{\mathbf{D}}(\omega)\,\widetilde{\mathbf{H}}^t(\mathbf{x}_\star,\omega),$$

where $\widetilde{\mathbf{H}}(\mathbf{x},\omega) = [\mathbf{E}(\mathbf{x},\omega),\ \mathbf{G}(\mathbf{x},\omega)]$, $\mathbf{E}(\mathbf{x},\omega) = [\boldsymbol{e}_1(\mathbf{x}),\dots,\boldsymbol{e}_N(\mathbf{x})]^t$ is a $N\times 6$ matrix of the independent components of the strain tensors, $\widetilde{\mathbf{D}}(\omega) = \mathrm{diag}\big(\widetilde{\mathbf{M}}, c(\rho,\omega)\underline{\mathbf{I}}\big)$ and $\widetilde{\mathbf{M}}$ is given by (35). Since the matrices $\widetilde{\mathbf{M}}$ and $\mathbf{M}$ are equivalent and nonsingular, it turns out that the matrices $\widetilde{\mathbf{D}}(\omega)$ and $\mathbf{D}(\omega)$ are also equivalent and nonsingular. Moreover, the definition of the matrices $\widetilde{\mathbf{H}}(\mathbf{x},\omega)$ and $\mathbf{H}(\mathbf{x},\omega)$ yields that these matrices are similar, and so they have the same rank.

5. MUSIC Characterization of the MSR Matrix. Recall that MUSIC is essentially based on characterizing the range of the multi-static response matrix $\mathbf{A} = \mathbf{A}(\omega)$, so-called *signal space*, forming projections onto its null (*noise*) spaces, and computing its singular value decomposition.

From the factorization (41) of $\mathbf{A}$ and the fact that the scattering matrix $\mathbf{D}$ is nonsingular (so, it has rank 9), the standard argument from linear algebra yields that, if $N\geq 9$ and if the propagation matrix $\mathbf{H}(\mathbf{x}_\star)$ has maximal rank 9 then the ranges $\mathcal{R}(\mathbf{H}(\mathbf{x}_\star))$ and $\mathcal{R}(\mathbf{A})$ coincide.

Let us suppose that the number of the receivers and transmitters elements are sufficiently large. Following exactly the same lines as in [4, 13, 16], we introduce the operators $\lambda:\mathbb{C}^3\to\mathbb{C}^3$ as $(\lambda\varphi)(\mathbf{x}) = \underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_\star)\cdot\varphi$ and $\Lambda:\mathbb{C}^{3\times 3}\to\mathbb{C}^3$ as $(\Lambda\underline{\phi})(\mathbf{x}) = \frac{1}{2}(\nabla\underline{\mathbf{G}}+\nabla\underline{\mathbf{G}}^t)(\mathbf{x},\mathbf{x}_\star):\underline{\phi}$ for $\mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{x}_\star\}$, where φ is a vector in $\mathbb{C}^3$ and $\underline{\phi}$ (or $\phi_{ij},\ i,j=1,2,3$) is a symmetric second-rank tensor. Invoking the uniqueness result in Lemma 2.1 together with unique continuation for the elasticity system it can be shown that the both of the operators λ and Λ are one-to-one. Noticing that the singularity of $\underline{\mathbf{G}}(\cdot,\mathbf{x}_\star)$ and $\nabla\underline{\mathbf{G}}(\cdot,\mathbf{x}_\star)$ in a vicinity of $\mathbf{x}_\star$ are not of the same order, it follows that the range of $\mathbf{H}(\mathbf{x}_\star)$ has dimension 9. Consequently, the rank of the propagation matrix $\mathbf{H}(\mathbf{x}_\star)$, and so the rank of the MSR matrix $\mathbf{A}$, does not depend upon the location $\mathbf{x}_\star$ of the inclusion. Indeed, if for $\mathbf{a}\in\mathbb{C}^9\setminus\{0\}$, $\mathbf{H}(\mathbf{x})\cdot\mathbf{a}\in\mathcal{R}(\mathbf{A})$ then, provided the numbers of the receivers and transmitters elements are sufficiently large, we can prove that there exists a linear combination of $\underline{\mathbf{G}}(\cdot,\mathbf{x})$ and $\nabla\underline{\mathbf{G}}(\cdot,\mathbf{x})+\nabla\underline{\mathbf{G}}^t(\cdot,\mathbf{x})$, say $\underline{\mathbf{G}}(\cdot,\mathbf{x})\cdot\mathbf{d}_1 + (\nabla\underline{\mathbf{G}}(\cdot,\mathbf{x})+\nabla\underline{\mathbf{G}}^t(\cdot,\mathbf{x})):\mathbf{d}_2$, such that

$$\underline{\mathbf{G}}(\cdot,\mathbf{x})\cdot\mathbf{d}_1 + (\nabla\underline{\mathbf{G}}(\cdot,\mathbf{x})+\nabla\underline{\mathbf{G}}^t(\cdot,\mathbf{x})):\mathbf{d}_2 = \underline{\mathbf{G}}(\cdot,\mathbf{x}_\star)\cdot\mathbf{d}_1 + (\nabla\underline{\mathbf{G}}(\cdot,\mathbf{x}_\star)+\nabla\underline{\mathbf{G}}^t(\cdot,\mathbf{x}_\star)):\mathbf{d}_2$$

in $\mathbb{R}^3\setminus\{\mathbf{x},\mathbf{x}_\star\}$, which yields $\mathbf{x}=\mathbf{x}_\star$.

From the above argument, the following is a MUSIC characterization of the location of the elastic inclusion and is valid if the numbers of the receivers and transmitters elements are sufficiently large.

PROPOSITION 5.1. *Suppose* $N\geq 9$. *Let* $\mathbf{a}\in\mathbb{C}^9\setminus\{0\}$, *then*

$$\mathbf{H}(\mathbf{x})\cdot\mathbf{a}\in\mathcal{R}(\mathbf{A}) \qquad \textit{if and only if} \qquad \mathbf{x}=\mathbf{x}_\star.$$

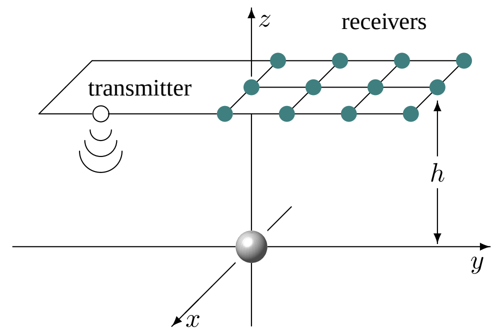

FIG. 6.1. *Sketch of the configuration under study. (Only a single transmitter in the array and a few receivers are shown for readability.)*

In other words, any linear combination of the vector columns of the propagation matrix $\mathbf{H}(\mathbf{x})$ defined by (42) belongs to range of $\mathbf{A}$ (signal space) if and only if the points $\mathbf{x}$ and $\mathbf{x}_\star$ coincide.

If the dimension of the signal space, s (=9), is known or is estimated from the singular value decomposition of $\mathbf{A}$, defined by $\mathbf{A} = \mathbf{V\Sigma U}^*$, then the MUSIC algorithm applies [10, 12]. Furthermore, if $\mathbf{v}_i$ denote the column vectors of the matrix $\mathbf{V}$ then for any vector $\mathbf{a} \in \mathbb{C}^9 \setminus \{0\}$ and for any space point $\mathbf{x}$ within the search domain, a map of the estimator $W(\mathbf{x})$ defined as the inverse of the squared Euclidean distance from the vector $\mathbf{H}(\mathbf{x}) \cdot \mathbf{a}$ to the signal space by

$$W(\mathbf{x}) = \frac{1}{\sum_{i=s+1}^{N} |\langle \mathbf{v}_i, \mathbf{H}(\mathbf{x}) \cdot \mathbf{a} \rangle|^2} \tag{44}$$

peak (to infinity, in theory) at the center $\mathbf{x}_\star$ of the inclusion. The visual aspect of the peak of W at $\mathbf{x}_\star$ depends upon the choice of the vector $\mathbf{a}$. A common choice which means that we are working with all the significant singular vectors is $\mathbf{a} = (1, 1, \ldots, 1)^t$. However, we shall emphasize the fact that a choice of the vector $\mathbf{a}$ in (44) with dimension (number of nonzero components) much lower than 9 still permits one to image the elastic inclusion with our MUSIC-type algorithm. See the numerical results below. We will say that the estimator $W(\mathbf{x})$ is obtained via the projection of the linear combination of the vector columns of the Green's matrix $\mathbf{G}(\mathbf{x})$ onto the noise subspace of the $\mathbf{A}$ for a signal space of dimension l if the dimension of $\mathbf{a}$ is l.

Let us also point out here that the function $W(\mathbf{x})$ does not contain any information about the shape and the orientation of the inclusion. Yet, if the position of the inclusion is found (approximately at least) via observation of the map of W, then one could attempt, using the decomposition (41), to retrieve the EMT of the inclusion (which is of order α^3).

Some words about the rank of the singular vectors of the matrix $\mathbf{A}$ are required. As it has been shown above, the column vectors of the propagation matrix $\mathbf{H}$ are linearly independent. However, they are not orthogonal, which implies that each singular vector of $\mathbf{A}$ corresponding to the response of one inclusion located at $\mathbf{x}_\star$ is a linear combination of the column vectors of $\mathbf{H}(\mathbf{x}_\star)$, *i.e.*, a linear combination of the column vectors of the Green's matrix $\mathbf{G}(\mathbf{x}_\star)$ and the stress matrix $\mathbf{S}(\mathbf{x}_\star)$.

6. Imaging a Spherical Inclusion from Synthetic Data. In order to illustrate relevant features of the method reviewed in the previous section, several computations have been performed, and typical results acquired from synthetically generated data are presented herein.

The configuration of interest is schematized in Fig. 6.1. A planar transmitter/receiver array, taken symmetric about the axis z, is made of 12×12 elements, all with same single direction (orthogonal to the array), distributed at the nodes of a regular mesh with a half-a-wavelength step size (λ_p - wavelength of P waves). Frequency of operation is $f = 2$MHz. The array is placed at $h = 10\lambda_p$ in the far field of one elastic sphere with size parameter $\alpha = \lambda_p/5$ and centered at the origin. The cases of a rigid background medium and a soft one are both considered here. The first case is illustrated by an epoxy sphere in a steel medium, while a soft background is exemplified by a steel sphere in an epoxy medium. Wave speeds and mass densities for these materials are taken as $\rho = 7.54\,$g/cm^3, $c_p = 5.80 \times 10^5$ cm/s and $c_s = 3.10 \times 10^5$ cm/s for steel; and $\rho = 1.18\,$g/cm^3, $c_p = 2.54 \times 10^5$ cm/s and $c_s = 1.16 \times 10^5$ cm/s for epoxy.

Within the above setting, the retrieval of the inclusion involves the calculation of the singular value decomposition of the MSR matrix $\mathbf{A} \in \mathbb{C}^{12\times12}$. The distribution of singular values of $\mathbf{A}$ in the absence of noise and in the case of noisy data with 30dB signal-to-noise ratio is exhibited in Fig. 6.2 (rigid background case) and in Fig. 6.6 (soft background case). The 3D plots of the MUSIC estimator $W(\mathbf{x})$, in the case of noisy data with 30dB signal-to-noise ratio, computed within a cubical box $\Omega = [-1,1]^3 \subset \mathbb{R}^3$, are shown in Figs. 6.3-6.4 (rigid background case) and in Figs. 6.7-6.8 (soft background case). The surface displayed is associated to values of half the peak magnitudes. In both cases the MUSIC criterion was computed via the orthogonal projection of the linear combination of the vector columns of the Green's matrix $\mathbf{G}(\mathbf{x})$ (Figs. 6.3 and 6.7) and the stress matrix $\mathbf{S}(\mathbf{x})$ (Figs. 6.4 and 6.8) onto the noise space of the MSR matrix $\mathbf{A}$ for a signal subspace of dimension 1 and 9 (rigid background case) and 3 and 9 (soft background case).

From the distribution of the singular values of $\mathbf{A}$ we see that, in both cases, the nine singular values associated to the inclusion appear well discriminated from the other (linked to the noise space) in the absence of noise, but in the case of noisy data six singular values are but lost. This result shows also that the magnitudes of the singular values associated to the inclusion in the soft background case are larger than those of the singular values of $\mathbf{A}$ in the rigid background case.

In the case of the rigid background the inclusion is imaged with MUSIC for a signal subspace of dimension one. But in the case of the soft background the image is obtained for a signal subspace of dimension three (in this case the results obtained for a signal subspace of dimension one do not produce a satisfactory image of the inclusion). In both cases, the best images are obtained with MUSIC for a signal subspace of dimension nine. As it has been indicated in the above, the vector columns of the matrix $\mathbf{G}(\mathbf{x})$ and the stress matrix $\mathbf{S}(\mathbf{x})$ are linearly independent but they are not orthogonal, which is why we are able to image the inclusion via both matrices for a signal dimension which is smaller than nine.

We give in Figs. 6.5(a)–6.5(c) a map of $W(x)$ corresponding to MUSIC reconstruction via Green's matrix for a signal space of dimension 9 on some cross sections passing through the origin.

7. Conclusion. In this paper, a direct imaging approach of MUSIC-type for imaging 3D elastic inclusions has been proposed and tested. The knowledge of the Green's matrix is sufficient to fully image the inclusion. The method makes use of an asymptotic expansion of the MSR matrix. The asymptotic MSR matrix used as synthetic data is quite accurate, but for larger inclusions discrepancies between asymptotic and exact MSR matrices could not be reproduced by addition of noise to asymptotic data, and it is one of the topics of current interest to the authors. One should attempt in a forthcoming work to characterize the physical nature of the inclusion from the SVD of the MSR matrix.

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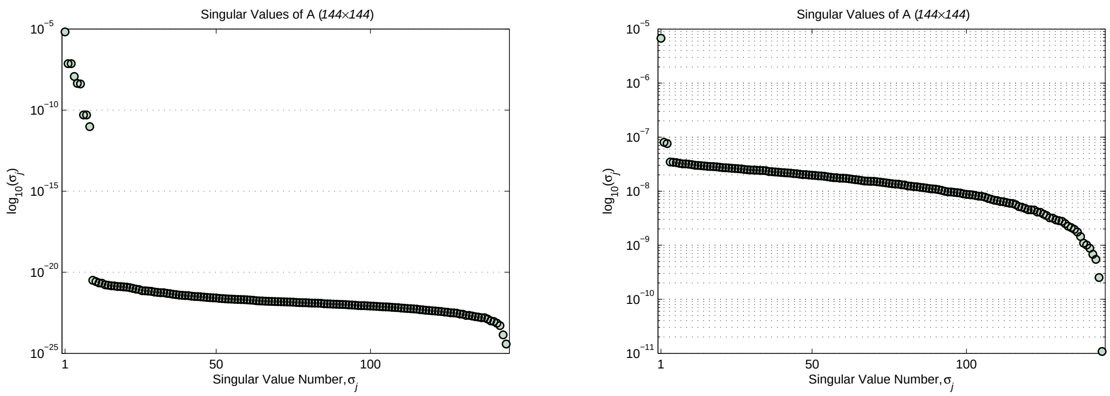
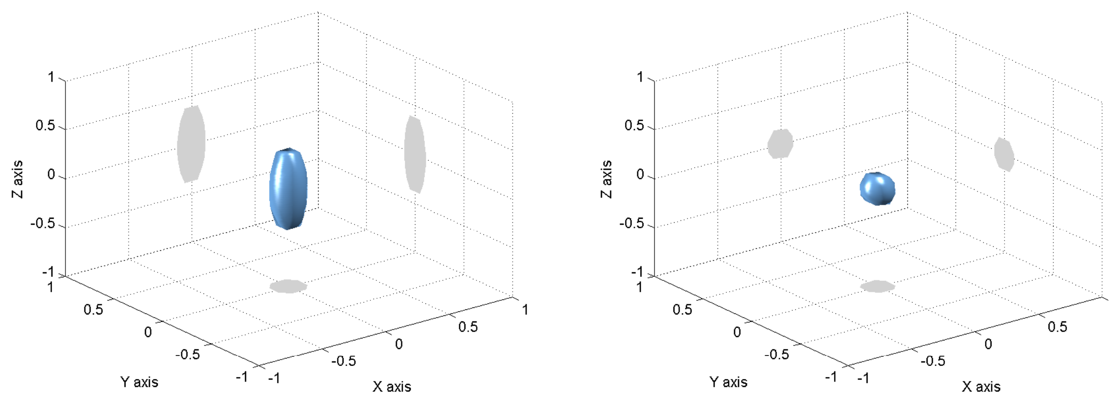
REFERENCES



(a) (b)

FIG. 6.2. *(rigid background case) Distribution of the singular values of* $\mathbf{A}$ *for* 144×144 *singly-polarized transmitters and receivers in the absence of noise (a); distribution of the full set in the case of noisy data with 30dB signal-to-noise ratio (b).*



(a) (b)

FIG. 6.3. *(rigid background case) 3D plots of the MUSIC estimator* $W(\mathbf{x})$ *obtained via the projection of the linear combination of the vector columns of the Green's matrix* $\mathbf{G}(\mathbf{x})$ *onto the noise subspace of the* $\mathbf{A}$ *for a signal space of dimension* 1 *(a) and* 9 *(b).*

[1] J.F. AHNER AND G.C. HSIAO, *On the two-dimensional exteriour boundary-value problems of elastisity*, SIAM J. Appl. Math. 31(1976), 677-685.

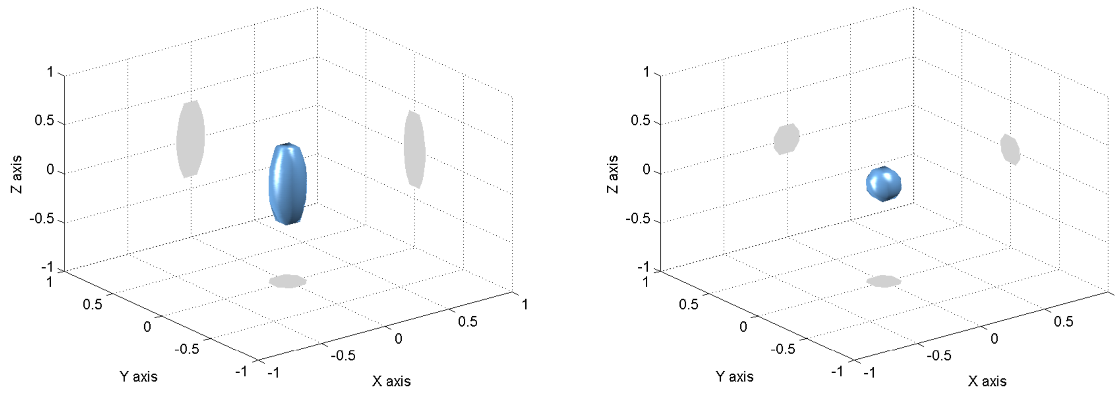
[2] H. AMMARI, E. IAKOVLEVA, AND D. LESSELIER, *A MUSIC algorithm for locating small inclusions buried in a half-space from the scattering amplitude at a fixed frequency,* SIAM Mult. Model. Simu., 3 (2005), pp. 597–628.

[3] H. AMMARI, E. IAKOVLEVA, AND D. LESSELIER, *Two numerical methods for recovering small inclusions from the scattering amplitude at a fixed frequency*, SIAM J. Sci. Comput., 27 (2005), pp. 130-158.

[4] H. AMMARI, E. IAKOVLEVA, D. LESSELIER, AND G. PERRUSSON, *A MUSIC-type electromagnetic imaging of a collection of small three-dimensional inclusions*, SIAM J. Sci. Comput., 29 (2007), pp. 674–709.

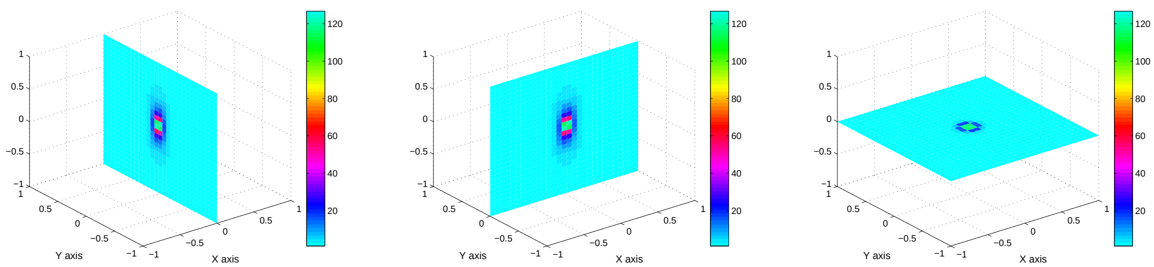
[5] H. AMMARI, E. IAKOVLEVA, AND H. KANG, *Reconstruction of a small inclusion in a two-dimensional open waveguide,* SIAM J. Appl. Math., 65 (2005), pp. 2107–2127.

[6] H. AMMARI AND H. KANG, Reconstruction of Small Inhomogeneities from Boundary Measurements, Lecture Notes in Math-



(a) (b)

FIG. 6.4. *(rigid background case) 3D plots of the MUSIC estimator $W(\mathbf{x})$ obtained via the projection of the linear combination of the vector columns of the stress matrix $\mathbf{S}(\mathbf{x})$ onto the noise subspace of the $\mathbf{A}$ for a signal space of dimension* 1 *(a) and* 9 *(b).*



(a) (b) (c)

FIG. 6.5. *(rigid background case) color maps of $W(\mathbf{x})$ in cross-sectional planes $x = 0$ (a), $y = 0$ (b) and $z = 0$ (c) (refer to Fig. 6.3 b)).*

ematics, Vol. 1846, Springer-Verlag, Berlin, 2004.

[7] H. AMMARI, H. KANG AND H. LEE, *A boundary integral method foe computing elastic moment tensor for ellipses and ellipsoids*, J. Comput. Math., 25 (2007), pp. 2–12.

[8] H. AMMARI AND H. KANG, Polarization and Moment Tensors: with Applications to Inverse Problems and Effective Medium Theory, Applied Mathematical Sciences, Vol. 162, Springer-Verlag, New York, 2007.

[9] M. BONNET AND A. CONSTANTINESCU, *Inverse problems in elasticity*, Inverse Problems 21 (2005), R1–R50.

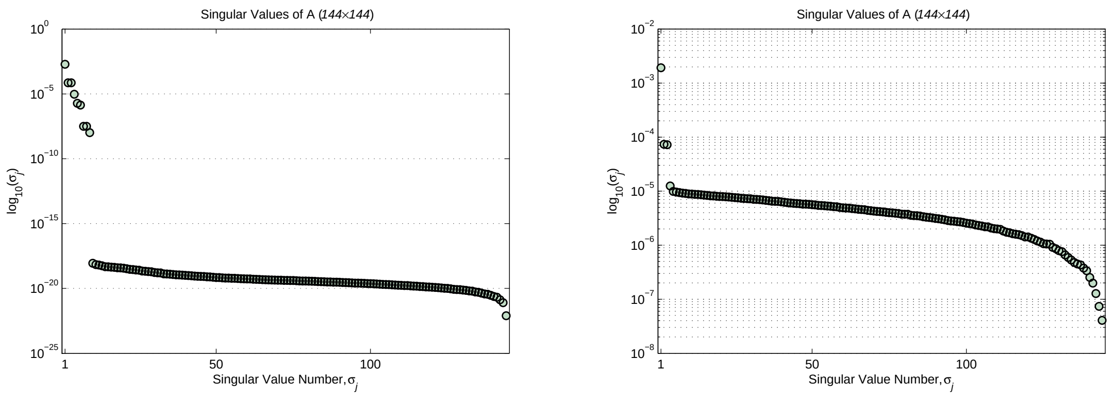
[10] M. CHENEY, *The linear sampling method and the MUSIC algorithm*, Inverse Problems, 17 (2001), pp. 591–595.

[11] G. DASSIOS AND R.E. KLEINMAN, Low Frequency Scattering, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.

[12] A.J. DEVANEY, *Time reversal imaging of obscured targets from multistatic data*, IEEE Trans. Antennas Propagat., 523 (2005), pp. 1600–1610.

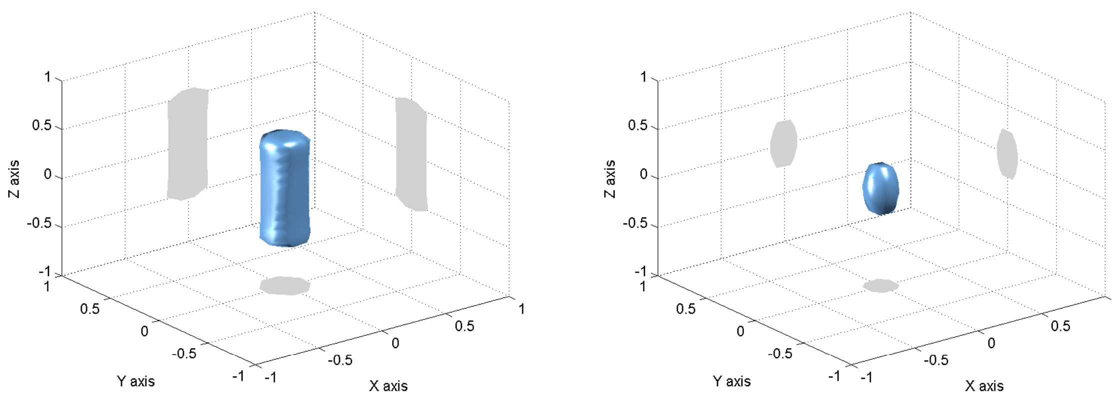
[13] R. GRIESMAIER, *An asymptotic factorization method for detecting small objects using electromagnetic scattering*, preprint, 2007.

[14] H. KANG, E. KIM, AND J.-Y. LEE, *Identification of elastic inclusions and elastic moment tensors by boundary measurements*, Inverse Problems, 19 (2003), pp. 703–724.



(a) (b)

FIG. 6.6. *(soft background case) Distribution of the singular values of* $\mathbf{A}$ *for* 144×144 *singly-polarized transmitters and receivers in the absence of noise (a); distribution of the full set in the case of noisy data with 30dB signal-to-noise ratio (b).*



(a) (b)

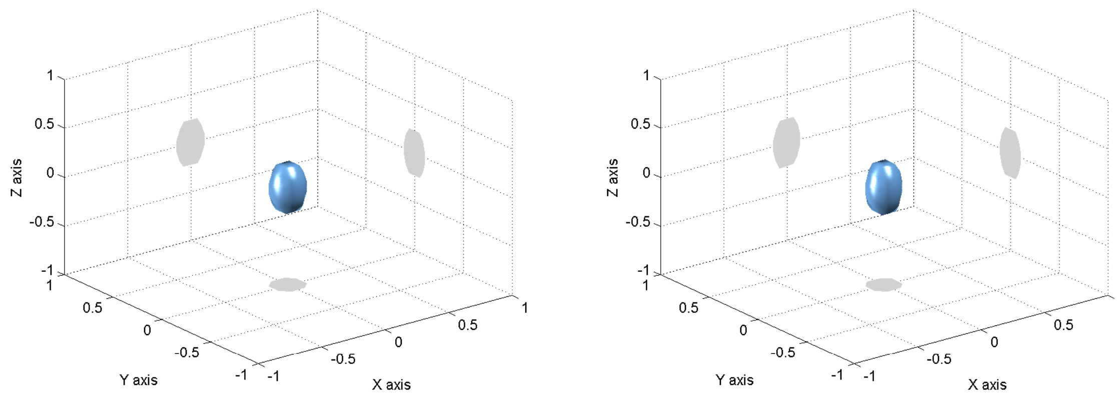
FIG. 6.7. *(soft background case) 3D plots of the MUSIC estimator* $W(\mathbf{x})$ *obtained via the projection of the linear combination of the vector columns of the Green's matrix* $\mathbf{G}(\mathbf{x})$ *onto the noise subspace of the* $\mathbf{A}$ *for a signal space of dimension* 3 *(a) and* 9 *(b).*

[15] H. KANG, E. KIM, AND J.-Y. LEE, *Numerical reconstruction of a cluster of small elastic inclusions*, Inverse Problems, 23 (2007), 2311–2324.

[16] A. KIRSCH, *The MUSIC algorithm and the factorisation method in inverse scattering theory for inhomogeneous media*, Inverse Problems, 18 (2002), pp. 1025–1040.

[17] V.D. KUPRADZE, T.G. GEGELIA, M.O. BASHELEISHVILI AND T.V. BURCHULADZE, *Three-Dimantional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, 1979.

[18] C. PRADA AND J-L. THOMAS, *Experimental subwavelength localization of the scatterers by decomposition of the time reversal operator interpreted as a covariance matrix*, J. Acoust. Soc. Am., 114 (2003), 235–243.



(a) (b)

FIG. 6.8. *(soft background case) 3D plots of the MUSIC estimator* $W(\mathbf{x})$ *obtained via the projection of the linear combination of the vector columns of the stress matrix* $\mathbf{S}(\mathbf{x})$ *onto the noise subspace of the* $\mathbf{A}$ *for a signal space of dimension* 3 *(a) and* 9 *(b).*